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SOLVABILITY AND WELL-POSEDNESS OF THE CAUCHY–NICOLETTI WEIGHTED PROBLEM FOR NONLINEAR SINGULAR FUNCTIONAL DIFFERENTIAL SYSTEMS

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Abstract. For nonlinear singular functional differential systems, the unimprovable sufficient conditions for the solvability and well-posedness of the Cauchy–Nicoletti weighted problem are established.

Let $-\infty < a < b < +\infty$, $t_i \in [a,b]$ $(i=1,\dots,n)$, $\varphi = (\varphi_i)_{i=1}^n : [a,b] \to \mathbb{R}^n$ be a continuous vector function such that

$$\varphi_i(t_i) = 0, \quad \varphi_i(t) > 0 \ \text{ for } \ t \neq t_i \ \ (i=1,\dots,n),$$

and let $C_\varphi([a,b];\mathbb{R}^n)$ be the space of continuous vector functions $x = (x_i)_{i=1}^n : [a,b] \to \mathbb{R}^n$, satisfying the condition

$$\|x\|_{C_\varphi} = \sum_{i=1}^n \|x_i\|_{C_{\varphi_i}} < +\infty,$$

where

$$\|x_i\|_{C_{\varphi_i}} = \sup\left\{ \frac{|x_i(t)|}{\varphi_i(t)} : \ \ a \le t \le b, \ \ t \neq t_i \right\} \ (i=1,\dots,n).$$

Let, moreover, $I_k = [a,b] \setminus \{t_k\}$ $(k=1,\dots,n)$ and let $L_{\rm loc}(I_k;\mathbb{R})$ be the space of functions $u : I_k \to \mathbb{R}$, Lebesgue integrable on every closed interval, contained in I_k.

We consider the nonlinear functional differential system

$$\frac{dx_i}{dt} = f_i(x_1,\dots,x_n)(t) \ \ (i=1,\dots,n) \tag{1}$$

with the weighted boundary conditions

$$\limsup_{t\to t_i} \frac{|x_i(t)|}{\varphi_i(t)} < +\infty \ \ (i=1,\dots,n). \tag{2}$$

Here, the operators $f_k : C_\varphi([a,b];\mathbb{R}^n) \to L_{\rm loc}(I_k;\mathbb{R})$ $(k=1,\dots,n)$ are such that:

(i) for any $\rho > 0$, the condition

$$f_k^*(\varphi,\rho) \in L_{\rm loc}(I_k;\mathbb{R}) \ \ (k=1,\dots,n)$$

is satisfied, where

$$f_k^*(\varphi,\rho)(t) = \sup\Big\{ |f_k(x_1,\dots,x_n)(t)| : \ \ \|(x_i)_{i=1}^n\|_{C_\varphi} \le \rho \Big\};$$

(ii) for any $\rho > 0$, $k \in \{1,\dots,n\}$, $[a_0,b_0] \subset I_k$, and for uniformly converging sequence of continuous vector functions $(x_{im})_{i=1}^n : [a,b] \to \mathbb{R}^n$ $(m=1,2,\dots)$ satisfying the conditions

$$\begin{gathered} \|(x_{im})_{i=1}^n\|_{C_\varphi} \le \rho \ \ (m=1,2,\dots), \\ \lim_{m\to\infty} (x_{im}(t))_{i=1}^n = (x_i(t))_{i=1}^n \ \text{ for } \ t \in [a,b], \end{gathered} \tag{3}$$

we have

$$\lim_{m\to\infty} \int_{a_0}^t f_k(x_{1m},\dots,x_{nm})(s)\,ds = \int_{a_0}^t f_k(x_1,\dots,x_n)(s)\,ds \ \text{ uniformly on } [a_0,b_0].$$

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From (2), we have the conditions

$$x_i(t_i) = 0 \ \ (i = 1, \dots, n).$$

Problem (1), (2) is known as the Cauchy–Nicoletti problem (see, e.g., [1, 2, 4, 6, 7]).

Thus we call (1), (2) the Cauchy–Nicoletti weighted problem.

Everywhere below, in addition to the above-introduced notation, we use the following notation:

$X = (x_{ik})_{i,k=1}^n$ is the $n \times n$ matrix with components x_{ik} $(i, k = 1, \dots, n)$;

$r(x)$ is the spectral radius of an $n \times n$ matrix;

$[x]_+$ is the positive part of a real number x, i.e.,

$$[x]_+ = \frac{x + |x|}{2}\,.$$

Theorem 1. *Let there exist a positive number ρ such that for any $\lambda \in [0, 1]$ and $\delta > 0$ every solution of problem (1), (2) admits estimate (3). Then problem (1), (2) has at least one solution.*

Theorem 2. *Let there exist nonnegative numbers p_{ik} $(i, k = 1, \dots, n)$ and a function $q : [0, +\infty[\to [0, +\infty[$ such that*

$$r(p) < 1, \quad \text{where } p = (p_{ik})_{i,k=1}^n, \tag{4}$$

$$\lim_{s \to +\infty} \frac{q(s)}{s} = 0,$$

and for any $(x_i)_{i=1}^n \in C_\varphi([a, b]; \mathbb{R}^n)$, the inequalities

$$\bigg| \int\limits_{t_i}^{t} \Big[f_i(x_1, \dots, x_n)(s) \operatorname{sgn}((s - t_i)x_i(s)) \Big]_+ ds \bigg|$$
$$\leq \varphi_i(t) \Big(\sum_{k=1}^n p_{ik} \|x_k\|_{G_{\varphi_k}} + q\Big(\sum_{k=1}^n \|x_k\|_{C_{\varphi_k}} \Big) \Big) \ \ (i = 1, \dots, n) \tag{5}$$

are satisfied in the interval $[a, b]$. Then problem (1), (2) has at least one solution.

Definition. Problem (1), (2) is said to be well-posed if:

(i) it has a unique solution $(x_i)_{i=1}^n$;

(ii) there exits a positive constant ρ such that for any integrable functions $h_i :]a, b[\to \mathbb{R}$ $(i = 1, \dots, n)$, satisfying the conditions

$$\nu_k(h_k) = \sup \bigg\{ \frac{1}{\varphi_k(t)} \bigg| \int\limits_{t_k}^{t} |h_k(s)| \, ds \bigg| : \ \ a \leq t \leq b, \ \ t \neq t_k \bigg\} < +\infty,$$

the problem

$$\frac{dy_i(t)}{dt} = f_i(y_1, \dots, y_n)(t) + h_i(t) \ \ (i = 1, \dots, n),$$

$$\limsup_{t \to t_i} \frac{|y_i(t)|}{\varphi_i(t)} < +\infty \ \ (i = 1, \dots, n)$$

has at least one solution and every such solution admits the estimate

$$\|(y_i - x_i)_{i=1}^n\|_{C_\varphi} \leq \rho \sum_{k=1}^n \nu_k(h_k).$$

Theorem 3. *Let there exist nonnegative numbers p_{ik} $(i, k = 1, \dots, n)$ satisfying conditions (4) such that for any $(x_i)_{i=1}^n \in C_\varphi([a, b]; \mathbb{R}^n)$, in the interval $[a, b]$, the inequalities*

$$\bigg| \int\limits_{t_i}^{t} \Big[f_i(x_1, \dots, x_n)(s) \operatorname{sgn}((s - t_i)x_i(s)) \Big]_+ ds \bigg| \leq \varphi_i(t) \sum_{k=1}^n p_{ik} \|x_k\|_{G_{\varphi_k}} \ \ (i = 1, \dots, n)$$

are fulfilled. Then problem (1), (2) is well-posed.

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