

ALGORITHMIC PROBLEMS IN TENSOR COMPLETIONS OF 2-NILPOTENT FINITELY GENERATED TORSION-FREE GROUPS

MIKHEIL AMAGLOBELI¹, TENGIZ BOKELAVADZE² AND ALEXEI G. MYASNIKOV³

Abstract. In this paper, we study algorithmic problems in tensor completions $G \otimes_{\mathcal{N}_{2,R}} R$ of finitely generated torsion-free nilpotent groups G of class 2 in the quasivariety $\mathcal{N}_{2,R}$ of R -exponential 2-nilpotent groups over a computable field of characteristic zero with a computable additive basis. We show that the word problem, the conjugacy problem, and the power problem are decidable in $G \otimes_{\mathcal{N}_{2,R}} R$. Note that these results do not hold for arbitrary finitely generated torsion-free 2-nilpotent R -groups. In fact, there are two generated torsion-free 2-nilpotent R -groups for which these problems are undecidable.

1. INTRODUCTION

In this paper, we investigate algorithmic problems in tensor completions $G \otimes R$ of finitely generated torsion-free nilpotent groups G of class 2, within the quasivariety $\mathcal{N}_{2,R}$ of R -exponential 2-nilpotent groups over a computable field R of characteristic zero with a computable additive basis. We prove that the word problem, the conjugacy problem, and the power problem are decidable in $G \otimes R$. Our approach relies on a recent description, given in [2], of the algebraic structure and the R -exponentiation in $G \otimes R$. Specifically, it was shown that $G \otimes R$ is isomorphic to the group $(G \otimes_{\mathcal{H}} R) \times D$, where $G \otimes_{\mathcal{H}} R$ is the classical Hall R -completion of G , D is an abelian R -group, and the direct product is taken in the category of abstract groups (not R -groups). The R -exponentiation in $G \otimes R$ is given explicitly by formula (3.2) (see Theorem 1 in Section 3). We effectively reduce the algorithmic problems for $G \otimes R$ to the corresponding problems for the Hall completions $G \otimes_{\mathcal{H}} R$, which are well studied (see, for example, [8]), and to certain algorithmic problems in the R -vector space D . When the field R is computable with a computable additive basis, these algorithmic problems in $G \otimes_{\mathcal{H}} R$ and in D are decidable.

Recall that a group G is called a *divisible group with unique roots* if, for any $g \in G$ and $0 \neq n \in \mathbb{N}$, the equation $x^n = g$ has a unique solution in G . In this case, there exists a \mathbb{Q} -exponentiation on G , that is, a mapping $G \times \mathbb{Q} \rightarrow G$ defined by $(g, \frac{m}{n}) \rightarrow (g^{\frac{1}{n}})^m$, which satisfies the following properties for any $x, y \in G$ and $\alpha, \beta \in \mathbb{Q}$:

- 1) $x^1 = x, x^0 = e, e^\alpha = e$;
- 2) $x^\alpha x^\beta = x^{\alpha+\beta}, (x^\alpha)^\beta = x^{\alpha\beta}$;
- 3) $(y^{-1}xy)^\alpha = y^{-1}x^\alpha y$.

We call such groups \mathbb{Q} -groups. Malcev proved that divisible torsion-free nilpotent groups are \mathbb{Q} -groups (see [10, 11]). A thorough study of arbitrary \mathbb{Q} -groups was carried out by Baumslag in [6]. Around the same time, while studying equations in free groups, Lyndon [9] introduced the concept of an R -group (and coined this name) for an arbitrary associative ring R with unity. By definition, these are the groups equipped with a map $G \times R \rightarrow G$, denoted $(g, \alpha) \rightarrow g^\alpha$, that exactly satisfy axioms 1)–3) above.

Unfortunately, there are abelian Lyndon R -groups that are not R -modules. Later, Myasnikov and Remeslennikov, in their paper [13], added an additional scheme of axioms (quasi-identities) to the Lyndon axioms. Namely, for any $x, y \in G$ and $\alpha \in R$ (here $[g, h] = g^{-1}h^{-1}gh$):

- 4) (MR-axiom) $[g, h] = e \rightarrow (gh)^\alpha = g^\alpha h^\alpha$.

2020 *Mathematics Subject Classification.* 20F18, 20F12.

Key words and phrases. Nilpotent groups; Tensor completions; Torsion-free groups.

Note that all \mathbb{Q} -groups satisfying axioms 1)–3) also satisfy axiom 4), as do many other natural R -groups according to Lyndon. We denote by \mathcal{M}_R the class of all R -groups satisfying axioms 1)–4) and call them as R -groups.

The tensor extension of the scalar ring plays an important role in module theory. In [13], Myasnikov and Remeslennikov defined an exact analogue of this construction, the so-called tensor completion, for an arbitrary group, and in [14] they described a concrete method for constructing tensor completions for CSA groups, in particular, for free and torsion-free hyperbolic groups (see also [4]). However, the notion of tensor completion in an arbitrary variety of groups requires a special approach, since the general construction of tensor completions is not necessarily compatible with the given variety. Amaglobeli, Myasnikov, and Nadiradze introduced the general notion of a variety of R -groups and tensor completions in such varieties in [3]. It turns out that different, although equivalent, definitions of nilpotency in the class of discrete groups can lead to formally different types of nilpotent R -groups. The question whether these classes are truly distinct remains open. However, it is known [3] that in the class of 2-nilpotent groups, different definitions of nilpotency lead to the same previously mentioned quasivariety of 2-nilpotent R -groups $\mathcal{N}_{2,R}$. It is with respect to this class that we describe the tensor completions $G \otimes_{\mathcal{N}_{2,R}} R$ of finitely generated 2-nilpotent torsion-free groups G .

In [7], Hall introduced and studied nilpotent R -groups satisfying the following additional axiom for all $x_1, \dots, x_n \in G$ and $\alpha \in R$:

$$5) \quad x_1^\alpha x_2^\alpha \cdots x_n^\alpha = (x_1 x_2 \cdots x_n)^\alpha \tau_2(\bar{x})^{\binom{\alpha}{2}} \cdots \tau_c(\bar{x})^{\binom{\alpha}{c}},$$

where τ_i are the words of the Petresco group over the alphabet x_1, \dots, x_n , c is the nilpotency class of G , and $\binom{\alpha}{i}$ are the binomial coefficients.

In this case, the ring R must be a binomial domain, i.e., an integral domain, where for each $\alpha \in R$, the binomial coefficient $\binom{\alpha}{i} = \frac{\alpha(\alpha-1)\cdots(\alpha-i+1)}{i!}$, considered as an element of the field of fractions of R , belongs to R . 5) is a more precise version of axiom 4) in the case of nilpotent groups.

2. PRELIMINARIES

2.1. R -groups. In this section, following the works of [3, 9, 13, 17], we introduce the necessary terminology and discuss some properties of R -groups that are useful for further understanding.

Let R be an associative commutative ring with identity and $G \in \mathcal{M}_R$. The standard group language L_{gr} can be extended by symbols for the unary operations f_α for each $\alpha \in R$, yielding the language of R -groups: $L_{gr}^R = L_{gr} \cup \{f_\alpha \mid \alpha \in R\}$. The operation f_α is interpreted in an R -group G as α -exponentiation $f_\alpha(g) = g^\alpha$ for $g \in G$. As usual in universal algebra, the notions of R -homomorphisms, R -subgroups, and R -generators can be defined in the category \mathcal{M}_R by viewing them as the groups in the language L_{gr}^R . Thus, a homomorphism $\phi : G \rightarrow H$ of two R -groups G and H is an R -homomorphism if $\phi(g^\alpha) = (\phi(g))^\alpha$ for all $g \in G$ and $\alpha \in R$; a subgroup $N \leq G$ is called an R -subgroup of G if it is invariant under R -exponentiation; and a subset $X \subseteq G$ R -generates an R -subgroup $\langle X \rangle_R$, which is the smallest R -subgroup of G containing X . Note that G is generated (as an R -group) by a set $A \subseteq G$ if every element of G can be represented as $w(a_1, \dots, a_n)$ for some group R -word $w(z_1, \dots, z_n)$ and $a_1, \dots, a_n \in A$. Here, if z_1, \dots, z_n are variables, then any expression $w(z_1, \dots, z_n)$ obtained from these variables by multiplication, inversion, and exponentiation by the elements of R is called a *group R -word with respect to the variables z_1, \dots, z_n* (group R -words are precisely terms in the language L_{gr}^R).

Axioms 1)–3) and 5) are identities in the language L_{gr}^R , while axiom 4) is a quasi-identity. Therefore, the class of all Lyndon R -groups (satisfying 1)–3)), \mathcal{L}_R , like the class $\mathcal{H}_{c,R}$ of all Hall groups of nilpotency class $\leq c$, is a variety in the language of L_{gr}^R , while the classes \mathcal{M}_R and $\mathcal{N}_{2,R}$ are quasivarieties. It follows from general theorems of universal algebra that these classes contain free R -groups and tensor R -completions with respect to these classes; the standard theory of R -ideals, i.e., normal subgroups whose quotient groups again belong to the same class; we can speak about the groups defined by generators and relations in these classes, etc.

2.2. Tensor completions of nilpotent R -Groups. For a natural number $c > 1$, we denote by $\mathcal{N}_{c,R}$ the quasivariety of all nilpotent R -groups satisfying the c -nilpotency identity $\forall x_1, \dots, x_{c+1} [x_1, \dots,$

$x_{c+1}] = e$, where $[x_1, \dots, x_{c+1}]$ is the left-normed commutator of weight $c + 1$. The operation of tensor completion plays a decisive role in the study of R -groups. In this section, we discuss the tensor completions in the class $\mathcal{N}_{c,R}$.

Let G be a torsion-free nilpotent group of nilpotency class $c \geq 1$, and let R be an associative commutative ring with 1 of characteristic 0.

Recall that an R -group $G \otimes_{\mathcal{N}_{c,R}} R$ is called a *tensor R -completion of G in the class $\mathcal{N}_{c,R}$* if there exists a homomorphism $\lambda : G \rightarrow G \otimes_{\mathcal{N}_{c,R}} R$ such that the following conditions are satisfied:

- (1) $\lambda(G)$ R -generates $G \otimes_{\mathcal{N}_{c,R}} R$;
- 2) For any R -group $H \in \mathcal{N}_{c,R}$ and an arbitrary homomorphism $\varphi : G \rightarrow H$, there exists an R -homomorphism $\psi : G \otimes_{\mathcal{N}_{c,R}} R \rightarrow H$ that makes the diagram commutative

$$\begin{array}{ccc} G & \xrightarrow{\lambda} & G \otimes_{\mathcal{N}_{c,R}} R \\ \varphi \downarrow & \swarrow \psi & \\ H & & \end{array} \quad (\varphi = \psi \circ \lambda).$$

It was proved in [3] that for any such group G and any such ring R , the tensor completion $G \otimes_{\mathcal{N}_{c,R}} R$ exists and is unique up to R -isomorphism. Moreover, the homomorphism λ gives a natural embedding of G into $G \otimes_{\mathcal{N}_{c,R}} R$. If G is an abelian R -group (i.e., $c = 1$), then $G \otimes_{\mathcal{N}_{c,R}} R \cong G \otimes R$ is the tensor product of the \mathbb{Z} -module G and the ring R .

The tensor R -completions $G \otimes_{\mathcal{H}_{c,R}} R$ of a group G in the class $\mathcal{H}_{c,R}$ are well known and are described by Hall in [7]. Namely, let G be a finitely generated torsion-free nilpotent group. Then G has a tuple of elements $\bar{w} = (w_1, \dots, w_n)$ (the Malcev base of G) such that every element g of G is uniquely representable as

$$g = w_1^{t_1} \cdots w_n^{t_n}, \quad (2.1)$$

for some $t_i \in \mathbb{Z}$. The set $t(g) = (t_1, \dots, t_n)$ is called the set of Malcev coordinates g . In this case, the multiplication of elements of the form (2.1) is performed using certain polynomials $f_1(\bar{x}, \bar{y}), \dots, f_n(\bar{x}, \bar{y})$, where $\bar{x} = (x_1, \dots, x_n)$, $\bar{y} = (y_1, \dots, y_n)$ such that for any $g, h \in G$

$$t(gh) = (f_1(t(g), t(h)), \dots, f_n(t(g), t(h))).$$

Similarly, raising to the power $\lambda \in \mathbb{Z}$ is introduced through the polynomials (see [16]). To construct $G \otimes_{\mathcal{H}} R$, one can take the same Malcev basis $\bar{w} = (w_1, \dots, w_n)$ of G , consider all formal products

$$w_1^{\alpha_1} \cdots w_n^{\alpha_n},$$

and introduce multiplication and exponentiation $\alpha \in R$ on them, using the same polynomials as for G . Since R is binomial, all values of these polynomials are defined in R (for details, see [16]).

3. GROUP STRUCTURE OF TENSOR COMPLETIONS OF FINITELY GENERATED 2-NILPOTENT TORSION-FREE GROUPS

In this section, following [2], we describe the algebraic structure and R -exponentiation in tensor completions $G \otimes_{\mathcal{N}_{2,R}} R$ of finitely generated 2-nilpotent torsion-free groups G over a binomial ring R of characteristic 0. For brevity, we denote $G \otimes R = G \otimes_{\mathcal{N}_{2,R}} R$ and $G \otimes_{\mathcal{H}} R = G \otimes_{\mathcal{H}_{2,R}} R$.

In [2], we defined a crucial notion of a *c -commutator* for $g, h \in G \otimes R$ and $\alpha \in R$ as follows:

$$c(g, h)_\alpha = [g, h]^{\binom{\alpha}{2}}(g, h)_\alpha.$$

Then

$$(gh)^\alpha = g^\alpha h^\alpha [g, h]^{-\binom{\alpha}{2}} c(g, h)_\alpha, \quad (3.1)$$

and hence the element $c(g, h)_\alpha$ measures how much the power $(gh)^\alpha$ in $G \otimes R$ differs from the corresponding power in $G \otimes_{\mathcal{H}} R$.

The identity map $G \rightarrow G$ extends canonically to an embedding $\eta : G \rightarrow G \otimes_{\mathcal{H}} R$. Moreover, since $G \otimes_{\mathcal{H}} R$ is an R -group, the embedding η canonically extends to an R -epimorphism $\mu : G \otimes R \rightarrow G \otimes_{\mathcal{H}} R$ that plays an essential role in our research.

Let $\bar{u} = (u_1, \dots, u_m)$ and $\bar{v} = (v_1, \dots, v_n)$ be the bases of the free abelian groups $G/Z(G)$ and $Z(G)$, respectively. Then $\bar{u} \cdot \bar{v} = (u_1, \dots, u_m, v_1, \dots, v_n)$ is a Mal'tsev basis of G . Therefore, each element $g \in G$ can be uniquely represented as a product

$$g = u_1^{\alpha_1} \cdots u_m^{\alpha_m} v_1^{\beta_1} \cdots v_n^{\beta_n} = \bar{u}^{\bar{\alpha}} \cdot \bar{v}^{\bar{\beta}},$$

where $\bar{\alpha} = (\alpha_1, \dots, \alpha_m)$, $\bar{\beta} = (\beta_1, \dots, \beta_n)$ are some tuples of integers. By construction, each element of the Hall completion $G \otimes_{\mathcal{H}} R$ has a unique decomposition:

$$g = u_1^{\alpha_1} \cdots u_m^{\alpha_m} v_1^{\beta_1} \cdots v_n^{\beta_n} = \bar{u}^{\bar{\alpha}} \cdot \bar{v}^{\bar{\beta}},$$

where $\bar{\alpha} = (\alpha_1, \dots, \alpha_m)$, $\bar{\beta} = (\beta_1, \dots, \beta_n)$ are some tuples of elements from R . Multiplication and R -exponentiation in $G \otimes_{\mathcal{H}} R$ are defined by the same polynomials as in G with respect to the basis $\bar{u} \cdot \bar{v}$.

Lemma 3.1 ([2]). *Let G be a finitely generated 2-nilpotent torsion-free group, and let R be a binomial domain of characteristic zero. Let $\bar{u} \cdot \bar{v}$ be a Mal'tsev basis of G , as defined above. Then in the group $G \otimes R$, the set of elements*

$$H_{\bar{u} \cdot \bar{v}} = \{u^{\bar{\alpha}} \cdot v^{\bar{\beta}} \mid \bar{\alpha} \in R^m, \bar{\beta} \in R^n\}$$

forms a subgroup (not necessarily an R -subgroup), that is, as an abstract group, isomorphic to the Hall completion $G \otimes_{\mathcal{H}} R$ of the group G . Moreover, $G = H_{\bar{u} \cdot \bar{v}} \times D$ where D is an abelian R -group (R -module).

In [2], it was shown that the R -exponentiation in $G \otimes R = H_{\bar{u} \cdot \bar{v}} \times D$ is defined by the following formula:

$$(u_1^{\alpha_1} \cdots u_m^{\alpha_m} v_1^{\beta_1} \cdots v_n^{\beta_n} d)^{\alpha} = u_1^{\alpha_1 \alpha} \cdots u_m^{\alpha_m \alpha} v_1^{\beta_1 \alpha + \sigma_1} \cdots v_n^{\beta_n \alpha + \sigma_n} d^{\alpha} c(u_1^{\alpha_1}, \dots, u_m^{\alpha_m})_{\alpha}, \quad (3.2)$$

where

$$v_1^{\sigma_1} \cdots v_n^{\sigma_n} = \tau_2(u_1^{\alpha_1}, \dots, u_m^{\alpha_m})^{-\binom{\alpha}{2}}, \quad (3.3)$$

$\tau_2(u_1^{\alpha_1}, \dots, u_m^{\alpha_m})$ is the second Petresco word (see [2]), and

$$c(u_1^{\alpha_1}, \dots, u_m^{\alpha_m})_{\alpha} = \prod_{i=1}^{n-1} c(u_1^{\alpha_1} \cdots u_i^{\alpha_i}, u_{i+1}^{\alpha_{i+1}})_{\alpha}. \quad (3.4)$$

Theorem 3.1 ([2]). *Let G be an arbitrary 2-nilpotent finitely generated torsion-free group and R be an arbitrary binomial domain. Then the R -group $G \otimes R$ satisfies the following conditions:*

- 1) $G \otimes R$, as an abstract group (not an R -group!), is isomorphic to $(G \otimes_{\mathcal{H}} R) \times D$, where $G \otimes_{\mathcal{H}} R$ is the Hall R -completion of G , considered as an abstract group (not an R -group), and D is the R -module generated by all c -commutators $c(x^{\alpha}, y^{\beta})_{\mu}$, $\alpha, \beta, \mu \in R$;
- 2) R -exponentiation in the group $G \otimes R$ is defined by formulas (3.2), (3.3) and (3.4);
- 3) D is a free R -module.

4. DECIDABILITY OF THE CLASSICAL ALGORITHMIC PROBLEMS

4.1. Computable fields. Recall (see [5, 12, 15]) that a ring R is called computable if it has an isomorphic copy R_0 of the form $R_0 = \langle \mathbb{N}; \oplus, \otimes \rangle$, where \mathbb{N} is the set of natural numbers and \oplus, \otimes are computable functions. In this situation, we identify R with one of its copies R_0 . We say that a computable field R of characteristic zero has a computable (recursive) additive basis (for brevity, R is a CFCB) if there exists a computable subset $B \subseteq \mathbb{N}$ that forms a basis for the additive group R^+ of R viewed as a \mathbb{Q} -vector space. A purely transcendental extension $R = \mathbb{Q}(t)$ of \mathbb{Q} is a CFCB; for instance, the set of all simple fractions

$$B = \left\{ \frac{t^k}{p(t)^m} \mid p(t) \text{ is monic irreducible over } \mathbb{Q}, m \geq 1, 0 \leq k < \deg(p) \right\}$$

can be used as an additive basis of $\mathbb{Q}(t)$. Likewise, any finite simple algebraic extension $R = \mathbb{Q}(\alpha)$ of \mathbb{Q} is CFCB. More generally, every finitely generated extension of \mathbb{Q} is CFCB. There are many more CFCBs (see, for example, [5]).

Note that if R is CFCB and B is a computable additive basis of R , then there is an algorithm that, given an element $a \in R$, computes the unique linear combination

$$a = r_1 b_1 + \cdots + r_n b_n, \quad (4.1)$$

where $r_i \in \mathbb{Q}$ and $b_i \in B$. Indeed, if $B = \{b_1, \dots, b_n\}$ is finite, the algorithm simply enumerates all tuples $(r_1, \dots, r_n) \in \mathbb{Q}^n$ and checks one by one whether (4.1) holds. In the infinite case, we effectively enumerate larger and larger finite subsets of B and, for each such subset, exhaustively test all corresponding tuples as above; in this way, the algorithm eventually finds the desired representation.

4.2. The algorithmic problems. We consider here three basic algorithmic problems for a given finitely generated R -group H with a fixed finite generating set a_1, \dots, a_n .

The word problem. *Given an R -word $w(a_1, \dots, a_n)$ in the generators a_1, \dots, a_n , decide whether this word is equal to 1 in H .*

The conjugacy problem. *Given two R -words $u(a_1, \dots, a_n)$ and $v(a_1, \dots, a_n)$, decide whether they are conjugate in H .*

The power problem. *Given two R -words $u(a_1, \dots, a_n)$ and $v(a_1, \dots, a_n)$, decide whether there is $\alpha \in R$ such that $u(a_1, \dots, a_n)^\alpha = v(a_1, \dots, a_n)$.*

Theorem 4.1. *Let G be a finitely generated, torsion-free, 2-nilpotent group, and let R be a computable field of characteristic zero with a computable additive basis. Then the word problem, the conjugacy problem, and the power problem are decidable in the tensor completion $G \otimes R$.*

ACKNOWLEDGEMENT

The work was supported by Shota Rustaveli National Foundation of Georgia, project FR-25-7138.

REFERENCES

1. M. G. Amaglobeli, T. Z. Bokelavadze, A. G. Myasnikov, Mal'tsev correspondence and bi-interpretability. *Algebra Logic* **63** (2024), no. 5, 305–322.
2. M. G. Amaglobeli, A. G. Myasnikov, Tensor completions of torsion-free 2-nilpotent groups. *Algebra and Logika* (to appear) 2025.
3. M. G. Amaglobeli, A. G. Myasnikov, T. T. Nadiradze, Varieties of exponential R -groups. *Algebra Logic* **62** (2023), no. 2, 119–136.
4. G. Baumslag, A. Myasnikov, V. Remeslennikov, Discriminating completions of hyperbolic groups. Dedicated to John Stallings on the occasion of his 65th birthday. *Geom. Dedicata* **92** (2002), 115–143.
5. Y. Ershov, *Decidability Problems and Constructible Models*. Nauka, Moscow, 1980.
6. B. Gilbert, Some aspects of groups with unique roots. *Acta Math.* **104** (1960), 217–303.
7. Ph. Hall, Nilpotent groups. In: *Canadian Mathematical Congress, Summer Seminar, University of Alberta, 12-30 August, 1957*.
8. M. I. Kargapolov, V. N. Remeslennikov, N. S. Romanovskii, V. A. Roman'kov, V. A. Čurkin, Algorithmic questions for σ -powered groups. *Algebra i Logika* **8** (1969), 643–659.
9. R. C. Lyndon, Groups with parametric exponents. *Trans. Amer. Math. Soc.* **96** (1960), 518–533.
10. A. I. Mal'cev, On a class of homogeneous spaces. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **13** (1949), 9–32.
11. A. I. Mal'cev, Nilpotent torsion-free groups. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **13** (1949), 201–212.
12. A. I. Mal'cev, Constructive algebras. I. (Russian) *Uspehi Mat. Nauk* **16** (1961), no. 3(99), 3–60.
13. A. G. Myasnikov, V. N. Remeslennikov, Degree groups. I. Foundations of the theory and tensor completions. (Russian) *translated from Sibirsk. Mat. Zh.* **35** (1994), no. 5, 1106–1118, iii *Siberian Math. J.* **35** (1994), no. 5, 986–996.
14. A. G. Myasnikov, V. N. Remeslennikov, Exponential groups. II. Extensions of centralizers and tensor completion of CSA-groups. *Internat. J. Algebra Comput.* **6** (1996), no. 6, 687–711.
15. M. O. Rabin, Computable algebra, general theory and theory of computable fields. *Trans. Amer. Math. Soc.* **95** (1960), 341–360.
16. Robert B. Jr. Warfield, *Nilpotent Groups*. Lecture Notes in Mathematics, vol. 513. Springer-Verlag, Berlin-New York, 1976.

17. M. G. Amaglobeli, T. Z. Bokelavadze, On Nilpotent Power MR-Groups. *J. Math. Sci.* **275** (2023), no. 6, 653–659.

(Received: 01.02.2026; Published online: 10.02.2026)

¹DEPARTMENT OF MATHEMATICS, IV. JAVAKHISHVILI TBILISI STATE UNIVERSITY, GEORGIA

²DEPARTMENT OF MATHEMATICS, AKAKI TSERETELI STATE UNIVERSITY, KUTAISI, GEORGIA

³STEVENS INSTITUTE OF TECHNOLOGY, HOBOKEN, NJ 07030, USA

Email address: mikheil.amaglobeli@tsu.ge

Email address: tengiz.bokelavadze@atsu.edu.ge

Email address: amiasnikov@gmail.com