Transactions of A. Razmadze
Mathematical Institute
Vol. **179** (2025), issue 3, 411–417

CRITERION FOR BASICITY PROPERTIES OF THE WEIGHTED EXPONENTIAL SYSTEM WITH EXCESS IN THE GRAND LEBESGUE SPACES

MIGDAD I. ISMAILOV[1], KADER SIMSIR ACAR[2*] AND YUSUF ZEREN[3]

Abstract. In this work, we consider the exponential system $\{\rho(t)e^{\mathrm{int}}\}_{n\in Z}$ with degenerate coefficient $\rho(\cdot)$ in the grand Lebesgue space $L_{p)}(-\pi,\pi), 1<p<+\infty$. This space is non-separable and therefore, we define the subspace $G_{p)}(-\pi,\pi)\subset L_{p)}(-\pi,\pi)$ in which the infinitely differentiable functions are densely located. The basicity of such systems in classical Lebesgue spaces has been studied quite well in the works on various mathematics. We establish criteria for the approximative properties (completeness, minimality, basicity) of this system in $G_{p)}(-\pi,\pi), 1<p<+\infty$. Moreover, we consider the defective system $\{\rho(t)e^{\mathrm{int}}\}_{n\neq n_0}$ and obtain similar results regarding this system in the same space.

1. Introduction

Let $L_{p)}(-\pi,\pi), 1<p<+\infty$, be the grand Lebesgue space, which is a collection of all measurable functions f defined on $[-\pi,\pi]$ and endowed with the norm

$$\|f\|_{p)}=\sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi}|f(t)|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}.$$

It is known that the space $L_{p)}(-\pi,\pi)$ is a non-separable, non-reflexive and Banach functional space [9, 24].

The grand Lebesgue spaces were first introduced in 1992 by T. Iwaniec and C. Sbordone [22] in connection with the study of Jacobian integrability properties. As for the issues of harmonic analysis, approximation and differential equations in these spaces, one can consider, for example, [6–13, 16, 17, 20, 23]. In [16] a separable subspace of the grand Lebesgue space is considered and the density of the set of all infinitely differentiable functions with compact support in this subspace is proved. The closure of the set of all infinitely differentiable functions with compact support on the interval $[-\pi,\pi]$ consists of the function satisfying the condition

$$\lim_{\varepsilon\to+0}\varepsilon\int_{-\pi}^{\pi}|f(t)|^{p-\varepsilon}dt=0.$$

In [29,30], the subspaces $G_{p)}(-\pi,\pi)$ of the grand Lebesgue space $L_{p)}(-\pi,\pi)$ are defined as a subspace of functions $f\in L_{p)}(-\pi,\pi)$ satisfying

$$T_\delta f\to f \ at \ \delta\to 0,$$

where T_δ is the shift operator defined by the formula

$$(T_\delta f)(x)=\begin{cases} f(x+\delta), & x+\delta\in(-\pi,\pi),\\ 0, & x+\delta\notin(-\pi,\pi),\end{cases}$$

and the density of the set of all infinitely differentiable functions with compact support on the interval of $[-\pi,\pi]$ in this subspace is proved.

2020 *Mathematics Subject Classification.* 35A01, 35J05, 40A05.
Key words and phrases. Weighted exponential system; Completeness; Minimality; Basicity property; Grand Lebesgue space; Muckenhoupt condition.

*Corresponding author.

In this work, we consider the weighted exponential system $\{\rho(t)e^{\mathrm{int}}\}_{n\in Z}$ with degenerate coefficient $\rho(\cdot)$ in the grand Lebesgue space $L_{p)}(-\pi,\pi)$, $1<p<+\infty$. This space is non-separable and therefore, we define the subspace $G_{p)}(-\pi,\pi)\subset L_{p)}(-\pi,\pi)$ in which the infinitely differentiable functions are densely located. The basicity of such systems in classical Lebesgue spaces has been studied quite well in the works on various mathematics. We established criteria for the approximative properties (completeness, minimality, basicity) of this system in $G_{p)}(-\pi,\pi)$, $1<p<+\infty$. Moreover, we consider the defective system $\{\rho(t)e^{\mathrm{int}}\}_{n\neq n_0}$ and obtain analogously the results with respect to this system in the same space.

Note that the basis property of the system $\{|t|^a e^{\mathrm{int}}\}_{n\in\mathbb{Z}}$ in the Lebesgue space $L_2(-\pi,\pi)$ was studied by K. I. Babenko in [1]. In [14], V. F. Gaposhkin studied the basis property of the system $\{\rho(t)e^{\mathrm{int}}\}_{n\in\mathbb{Z}}$ in $L_2(-\pi,\pi)$ for an arbitrary measurable function $\rho(t)$. In the case of the Lebesgue space $L_p(-\pi,\pi), 1<p<+\infty$, it follows from the results of [18, 19] that the degenerate system $\{\rho(t)e^{\mathrm{int}}\}_{n\in\mathbb{Z}}$ with a general weight function forms a basis in $L_p(-\pi,\pi), 1<p<+\infty$, when the weight function satisfies the Muckenhoupt condition, i.e., it belongs to the class A_p. In [15], the completeness and minimality of the system $\{|t|e^{\mathrm{int}}\}_{n\in\mathbb{Z}\setminus\{k_0\}}$ in the space $L_2(-\pi,\pi)$ have been considered and the impossibility of the basicity of the system $\{|t|e^{\mathrm{int}}\}_{n\in\mathbb{Z}\setminus\{k_0\}}$ in the space $L_2(-\pi,\pi)$ was established.

In the Lebesgue space $L_p(-\pi,\pi)$, for weighted exponential systems and trigonometric systems of sines and cosines with power weight, this question was studied in [2, 3, 5]. Note that similar questions for degenerate system of exponent in $L_p(-\pi,\pi), 1<p<+\infty$, and for the degenerate trigonometric system in the space $L_p(0,\pi)$ for arbitrary weight were studied in [26, 27]. Moreover, the basis property of a classical system of exponents in the grand Lebesgue spaces was studied in [21]. Similar questions were also considered in [4, 25, 28].

2. Basis Property of the Degenerate System of Exponents

Consider the system

$$\{\rho(t)e^{\mathrm{int}}\}_{n\in\mathbb{Z}}, \tag{2.1}$$

with weight coefficient of the form

$$\rho(t)=\prod_{k=0}^{r}|t-t_k|^{\alpha_k}, \quad \alpha_k\in\mathbb{R}, \ \ t_k\in[-\pi,\pi), \ \ k=\overline{0,r},$$

where $\{t_k\}_0^r\subset[-\pi,\pi)$ are distinct numbers.

The following criterion for the completeness of system (2.1) in $G_{p)}(-\pi,\pi)$, $1<p<+\infty$, is established.

Theorem 2.1. *System* (2.1) *is complete in the space* $G_{p)}(-\pi,\pi), 1<p<+\infty$, *if and only if*

$$\alpha_k>-\frac{1}{p}, \quad k=\overline{0,r}. \tag{2.2}$$

Proof. Let system (2.1) be complete in the space $G_{p)}(-\pi,\pi)$. Then it is evident that $\rho(t)\in G_{p)}(-\pi,\pi)$. It follows from the results of [20] that this relation is possible under the condition $\alpha_k\geq-\frac{1}{p}$, $k=\overline{0,r}$. Assume that $\alpha_0=-\frac{1}{p}$ for some $k=\overline{0,r}$. Therefore, for a sufficiently small $\delta>0$, we have

$$\lim_{\varepsilon\to0^+}\varepsilon\int_{-\pi}^{\pi}\rho^{p-\varepsilon}(t)dt\geq c\lim_{\varepsilon\to0^+}\varepsilon\int_0^{\delta}t^{-\frac{p-\varepsilon}{p}}dt=cp\lim_{\varepsilon\to0^+}\delta^{\frac{\varepsilon}{p}}=cp.$$

The case $\alpha_k=-\frac{1}{p}$, $k\neq0$ can be shown in a similar way. It follows that $\rho(t)\notin G_{p)}(-\pi,\pi)$. Thus, $\alpha_k>-\frac{1}{p}$, $k=\overline{0,r}$.

Conversely, let (2.2) be satisfied. Then $\rho(t)\in L_p(-\pi,\pi)$ and hence system (2.1) is complete in the space $L_p(-\pi,\pi)$. Due to the density $L_p(-\pi,\pi)$ in $G_{p)}(-\pi,\pi)$, we obtain the completeness of system (2.1) in $G_{p)}(-\pi,\pi)$. □

Let us derive a minimality criterion for system (2.1).

Theorem 2.2. *System* (2.1) *is minimal in the space* $G_{p)}(-\pi,\pi)$, $1<p<+\infty$, *if and only if*

$$-\frac{1}{p}<\alpha_k<\frac{1}{q},\quad k=\overline{0,r},\quad \frac{1}{p}+\frac{1}{q}=1. \tag{2.3}$$

Proof. First we prove the necessity of condition (2.3). Let system (2.1) be minimal in $G_{p)}(-\pi,\pi)$. Since $\rho(t)\in L_p(-\pi,\pi)$, due to the continuous embedding of $L_p(-\pi,\pi)$ in $G_{p)}(-\pi,\pi)$, we get the minimality of system (2.1) in $L_p(-\pi,\pi)$. Then it is known that condition (2.3) is valid in this case.

Conversely, let the condition (2.3) hold. Consider the system of linear functionals

$$g_n(f)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\rho^{-1}(t)e^{-\,\mathrm{int}}dt,\quad f\in G_{p)}(-\pi,\pi),\ \ n\in\mathbb{Z}.$$

Let us show that the functional g_n is bounded in $G_{p)}(-\pi,\pi)$. We choose $\varepsilon\in(0,p-1)$ so that

$$\alpha_k<\frac{1}{(p-\varepsilon)'}\,,\quad k=\overline{0,r},\quad \frac{1}{(p-\varepsilon)'}=1-\frac{1}{p-\varepsilon}\,.$$

This is possible, because if the relation $\alpha_k\ge\frac{p-\varepsilon-1}{p-\varepsilon}$ is valid for any $\varepsilon\in(0,p-1)$, then, passing to the limit at $\varepsilon\to0$ in the last relation, we get $\alpha_k\ge\frac{1}{q}$. And this contradicts the condition $\alpha_k<\frac{1}{q}$. Therefore, $\rho^{-1}(t)\in L_{(p-\varepsilon)'}(-\pi,\pi)$ and we have

$$\begin{aligned}|g_n(f)|&=\frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f(t)\rho^{-1}(t)e^{-\,\mathrm{int}}dt\right|\le\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)|\rho^{-1}(t)dt\\&\le\frac{1}{2\pi}\left(\int_{-\pi}^{\pi}|f(t)|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}\left(\int_{-\pi}^{\pi}\rho^{-(p-\varepsilon)'}(t)dt\right)^{\frac{1}{(p-\varepsilon)'}}\\&\le c_1\|f\|_{p)},\end{aligned}$$

i.e., the functional g_n is bounded. On the other hand, we have

$$g_n(\rho e^{imt})=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-i(n-m)t}dt=\delta_{nm}.$$

So, system (2.1) has a biorthogonal system $\{g_n\}_{n\in\mathbb{Z}}$, and thus it is minimal in $G_{p)}(-\pi,\pi)$. □

We now give a criterion for the basis property of system (2.1) in the space $G_{p)}(-\pi,\pi)$, $1<p<+\infty$.

Theorem 2.3. *System* (2.1) *forms a basis in* $G_{p)}(-\pi,\pi)$, $1<p<+\infty$, *if and only if condition* (2.3) *is satisfied.*

Proof. Let system (2.1) be a basis in $G_{p)}(-\pi,\pi)$. Then it is complete and minimal in the space $G_{p)}(-\pi,\pi)$. By Theorem 2.2, condition (2.3) is true.

Conversely, let condition (2.3) be satisfied. Then, according to Theorems 2.1 and 2.2, system (2.1) is complete and minimal in $G_{p)}(-\pi,\pi)$. Consider the following projectors:

$$S_m(f)(t)=\sum_{n=-m}^{m}g_n(f)\rho(t)e^{\mathrm{int}},\quad f\in G_{p)}(-\pi,\pi),\quad m\in\mathbb{Z}_+,$$

where $\mathbb{Z}_+$ is the set of non-negative integers. We show that the system of projectors S_m is uniformly bounded in $G_{p)}(-\pi,\pi)$. Since condition (2.3) holds, as is known, the system $\{\rho(t)e^{\mathrm{int}}\}_{n\in\mathbb{Z}}$ forms a basis in $L_p(-\pi,\pi)$. Let $\varepsilon\in(0,p-1)$ be such that the following inequalities:

$$-\frac{1}{p-\varepsilon}<\alpha_k<\frac{1}{(p-\varepsilon)'}\,,\quad k=\overline{0,r},$$

hold. Then, as is known, the system $\{\rho(t)e^{\text{int}}\}_{n\in\mathbb{Z}}$ forms a basis in $L_{p-\varepsilon}(-\pi,\pi)$. Consequently, the system of projectors S_m is uniformly bounded in the spaces $L_p(-\pi,\pi)$ and $L_{p-\varepsilon}(-\pi,\pi)$, i.e., there are the numbers $c_p>0$ and $c_{p-\varepsilon}>0$ such that the following estimates:

$$\|S_m(f)\|_p \le c_p\|f\|_p$$

and

$$\|S_m(f)\|_{p-\varepsilon} \le c_{p-\varepsilon}\|f\|_{p-\varepsilon}, \quad \forall m\in\mathbb{Z}_+$$

hold. Applying the Riesz–Thorin interpolation theorem ([24]), from these estimates we find that the system $\{S_m\}$ is uniformly bounded in the spaces $L_{p-\eta}(-\pi,\pi): 0<\eta\le\varepsilon$, i.e., it is true that

$$\|S_m(f)\|_{p-\eta} \le c_1\|f\|_{p-\eta}, \quad \forall m\in\mathbb{Z}_+, \tag{2.4}$$

where $c_1>0$ is independent of the number m.

Consider the case $\varepsilon<\eta<p-1$. According to Hölder's inequality with the exponent $\frac{p-\varepsilon}{p-\eta}>1$, we get

$$\begin{aligned}\left(\int_{-\pi}^{\pi}|S_m(f)(t)|^{p-\eta}dt\right)^{\frac{1}{p-\eta}} &\le (2\pi)^{\frac{\eta-\varepsilon}{(p-\varepsilon)(p-\eta)}}\left(\int_{-\pi}^{\pi}|S_m(f)(t)|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}\\ &\le c_2\|f\|_{p-\varepsilon},\end{aligned} \tag{2.5}$$

where the constant c_2 is independent of ε. Finally, using (2.4) and (2.5), we obtain

$$\begin{aligned}\|S_m(f)\|_{p)} &\le \sup_{0<\eta\le\varepsilon}\left(\frac{\eta}{2\pi}\right)^{\frac{1}{p-\eta}}\|S_m(f)\|_{p-\eta} + \sup_{\varepsilon<\eta<p-1}\left(\frac{\eta}{2\pi}\right)^{\frac{1}{p-\eta}}\|S_m(f)\|_{p-\eta}\\ &\le c_1\|f\|_{p)} + c_2\sup_{\varepsilon<\eta<p-1}\left(\frac{\eta}{2\pi}\right)^{\frac{1}{p-\eta}}\|f\|_{p-\varepsilon} \le c\|f\|_{p)},\end{aligned}$$

i.e., $\|S_m\|\le c$.

It remains to apply the basis property criterion to systems. □

3. The Basis Property of the Degenerate Exponential System with a Removed Element

This section studies the basis properties of the system

$$\{\rho(t)e^{\text{int}}\}_{n\in\mathbb{Z}\setminus\{n_0\}}, \tag{3.1}$$

with the weight coefficient $\rho(\cdot)$ in the space $G_{p)}(-\pi,\pi)$, where the weight function $\rho(\cdot)$ is defined by

$$\rho(t)=\prod_{k=0}^{r}|t-t_k|^{\alpha_k}, \quad \alpha_k\in\mathbb{R}, \quad t_k\in[-\pi,\pi), \ \ k=\overline{0,r},$$

$\{t_k\}_0^r$ are different numbers: $-\pi=t_0<t_1<\cdots<t_r<\pi$, and $n_0\in\mathbb{Z}$ be some number.

To obtaining the main result, we need the following fact.

Lemma 3.1 ([27]). *Let n_0 be an arbitrary integer and $\omega(t)$ be an arbitrary measurable function. The system $\{\omega(t)e^{\text{int}}\}_{n\in\mathbb{Z}\setminus\{n_0\}}$ is complete and minimal in $L_p(-\pi,\pi)$, $1<p<+\infty$, if and only if $\omega(t)\in L_p(-\pi,\pi)$, $\frac{1}{\omega(t)}\notin L_q(-\pi,\pi)$, $\frac{1}{p}+\frac{1}{q}=1$, and the following item:*

1. *there is a (unique) point $s\in[-\pi,\pi]$ such that $\frac{t-s}{\omega(t)}\in L_q(-\pi,\pi)$,*

or

2. *$\frac{(t+\pi)(t-\pi)}{\omega(t)}\in L_q(-\pi,\pi)$ holds.*

In the case of minimality, the biorthogonal system to the system $\{\omega(t)e^{\text{int}}\}_{n\in\mathbb{Z}\setminus\{n_0\}}$ is represented by a system of functions

$$b_n(t)=\frac{e^{\text{int}}-\xi_n e^{in_0t}}{\omega(t)}, \quad \forall n\ne n_0,$$

where ξ_n are some complex numbers.

We present a criterion for the completeness and minimality of system (3.1) in the space $G_{p)}(-\pi,\pi)$, $1<p<+\infty$.

Theorem 3.1. *System* (3.1) *is complete and minimal in* $G_{p)}(-\pi,\pi)$, $1<p<+\infty$, *if and only if*

1) *there is a* (*unique*) $\alpha_{k_0}\in\left[\frac{1}{q},1+\frac{1}{q}\right)$, $\alpha_k\in\left(-\frac{1}{p},\frac{1}{q}\right)$, $k=\overline{0,r}$, $k\neq k_0$;

or

2) $\alpha_0,\alpha_r\in\left(-\frac{1}{p},1+\frac{1}{q}\right)$, $\{\alpha_0,\alpha_r\}\cap\left[\frac{1}{q},1+\frac{1}{q}\right)\neq\emptyset$, $\alpha_k\in\left(-\frac{1}{p},\frac{1}{q}\right)$, $k=\overline{1,r-1}$.

Proof. Let system (3.1) be complete and minimal in $G_{p)}(-\pi,\pi)$. The relation $\rho(t)\in G_{p)}(-\pi,\pi)$ is equivalent to condition (2.2). Obviously, $\rho(t)\in L_p(-\pi,\pi)$. If $\alpha_k\in\left(-\frac{1}{p},\frac{1}{q}\right)$, $k=\overline{0,r}$, then system (3.1) is minimal in the space $G_{p)}(-\pi,\pi)$. This contradicts the completeness of system (3.1) in $G_{p)}(-\pi,\pi)$. Consequently, there is $k_0\in\{0,\dots,r\}$ such that $\alpha_{k_0}\in\left[\frac{1}{q},+\infty\right)$ and therefore $\rho^{-1}(t)\notin L_q(-\pi,\pi)$.

According to the continuous embedding $L_p(-\pi,\pi)\subset G_{p)}(-\pi,\pi)$, it is evident that system (3.1) is complete and minimal in $L_p(-\pi,\pi)$. Therefore, by Lemma 3.1, there is a (unique) point $t_0\in[-\pi,\pi]$ such that

$$(t-t_0)\rho^{-1}(t)\in L_q(-\pi,\pi), \tag{3.2}$$

or

$$(\pi-t)(\pi+t)\rho^{-1}(t)\in L_q(-\pi,\pi). \tag{3.3}$$

Relation (3.2) is possible only if

$$|t-t_0|\rho^{-1}(t)=|t-t_{k_0}|^{1-\alpha_{k_0}}\prod_{k=0,k\neq k_0}^{r}|t-t_k|^{-\alpha_k}\in L_q(-\pi,\pi),$$

and together with the condition $\rho^{-1}(t)\notin L_q(-\pi,\pi)$ we obtain the validity of condition 1). Condition (3.3) is equivalent

$$|\pi+t|^{1-\alpha_0}|\pi-t|^{1-\alpha_r}\prod_{k=1}^{r-1}|t-t_k|^{-\alpha_k}\in L_q(-\pi,\pi).$$

From here, taking into account $\rho^{-1}(t)\notin L_q(-\pi,\pi)$ we obtain the validity of condition 2).

Conversely, let conditions 1) or 2) be satisfied. Then either

$$\rho(t)\in L_p(-\pi,\pi),\quad \rho^{-1}(t)\notin L_q(-\pi,\pi),$$

or

$$(t-t_{\alpha_{k_0}})\rho^{-1}(t)\in L_q(-\pi,\pi)\quad \textit{or}\quad (\pi-t)(\pi+t)\rho^{-1}(t)\in L_q(-\pi,\pi)$$

holds. By Lemma 3.1, the system (3.1) is minimal in $L_p(-\pi,\pi)$ and in case 1) this system has a biorthogonal system (see [27])

$$g_n(t)=\frac{e^{\mathrm{int}}-e^{i(n-n_0)t_{k_0}}e^{in_0t}}{\rho(t)},\quad \forall n\neq n_0. \tag{3.4}$$

This implies that

$$g_n(t)=\frac{e^{\mathrm{int}}-e^{i(n-n_0)t_{k_0}}e^{in_0t}}{t-t_{k_0}}\cdot\frac{t-t_{k_0}}{\rho(t)}\sim|t-t_{k_0}|^{1-\alpha_{k_0}}\prod_{k=0,k\neq k_0}^{r}|t-t_k|^{-\alpha_k}.$$

Then, taking into account condition 1), we obtain that the functional generated by the function $g_n(t)$ is bounded in $G_{p)}(-\pi,\pi)$ (see [20, Lemma 2]), i.e., the sequence of functions $g_n(t)$ forms a biorthogonal system to system (3.1). In case of the condition 2), system

$$g_n(t)=\frac{e^{int}-e^{in_0t}}{\rho(t)},\quad \forall n\neq n_0, \tag{3.5}$$

is a biorthogonal system to system (3.1). Thus, system (3.1) is minimal in $G_{p)}(-\pi,\pi)$. Finally, due to the density of $L_p(-\pi,\pi)$ in $G_{p)}(-\pi,\pi)$, we obtain the completeness of system (3.1) in $G_{p)}(-\pi,\pi)$. □

The following assertion proves that it is impossible for system (3.1) to have a basis property in the space $G_{p)}(-\pi,\pi)$, $1<p<+\infty$.

Theorem 3.2. *System* (3.1) *is not a basis in* $G_{p)}(-\pi,\pi)$, $1<p<+\infty$.

Proof. Assume the opposite, i.e., let system (3.1) form a basis in $G_{p)}(-\pi,\pi)$. Then it is complete and minimal in $G_{p)}(-\pi,\pi)$ with the biorthogonal system (3.4) or (3.5). We expand $\rho(t)e^{in_0t}\in G_{p)}(-\pi,\pi)$ in $G_{p)}(-\pi,\pi)$ according to the basis (3.1). Let

$$\rho(t)e^{in_0t}=\sum_{n\neq n_0}c_n\rho(t)e^{\mathrm{int}}. \tag{3.6}$$

Taking into account that the biorthogonal system to system (3.1) is a system (3.5) from here, we obtain

$$c_n=-e^{i(n-n_0)t_{k_0}} \quad \text{or} \quad c_n=-1,\ \ \forall n\neq n_0. \tag{3.7}$$

Since the series (3.6) converges, according to the necessary condition for the convergence of the series, we get

$$0=\lim_{n\to\infty}\|c_n\rho(t)e^{\mathrm{int}}\|_{p)}=\|\rho(t)\|_{p)}\lim_{n\to\infty}|c_n|=\|\rho(t)\|_{p)}.$$

Hence, taking into account (3.7), it follows that $\|\rho(t)\|_{p)}=0$. We get a contradiction. Thus, the system (3.1) cannot form a basis in $G_{p)}(-\pi,\pi)$. □

Acknowledgement

This work was supported by the Azerbaijan Science Foundation-Grant no: AEF-MGC-2025-1(54)-20/03/1-M-03.

References

1. K. I. Babenko, On conjugate functions. (Russian) *Doklady Akad. Nauk SSSR (N.S.)* **62** (1948), 157–160.
2. B. T. Bilalov, *Some problems of Approximation.* Baku, Elm, Azerbaijan, 2016.
3. B. T. Bilalov, F. Guliyeva, On the frame properties of degenerate system of sines. *J. Funct. Spaces Appl.* **2012**, 12 pp.
4. B. T. Bilalov, A. A. Huseynli, S. R. El-Shabrawy, Basis properties of trigonometric systems in weighted Morrey spaces. *Azerb. J. Math.* **9** (2019), no. 2, 166–192.
5. B. T. Bilalov, Z. V. Mamedova, On the frame properties of some degenerate trigonometric systems. (Russian) *Dokl. Nats. Akad. Nauk Azerb.* **68** (2012), no. 5, 14–18.
6. B. T. Bilalov, S. R. Sadigova, On solvability in the small of higher order elliptic equations in grand-Sobolev spaces. *Complex Var. Elliptic Equ.* **66** (2021), no. 12, 2117–2130.
7. B. T. Bilalov, S. R. Sadigova, Interior Schauder-type estimates for higher-order elliptic operators in grand-Sobolev spaces. *Sahand Communications in Mathematical Analysis* **18** (2021), no. 2, 129–148.
8. B. T. Bilalov, S. R. Sadigova, On the fredholmness of the Dirichlet problem for a second-order elliptic equation in grand-Sobolev spaces. *Ric. Mat.* **73** (2024), no. 1, 283–322.
9. R. E. Castilo, H. Rafeiro, *An Introductory Course in Lebesgue Spaces.* Springer, Switzerland, 2016.
10. N. Danelia, V. Kokilashvili, Approximation by trigonometric polynomials in subspace of weighted grand Lebesgue spaces. *Bull. Georgian Natl. Acad. Sci. (N.S.)* **7** (2013), no. 1, 11–15.
11. L. D'onofrio, C. Sbordone, R. Schiattarella, Grand Sobolev spaces and their applications in geometric function theory and PDEs. *J. Fixed Point Theory Appl.* **13** (2013), no. 2, 309–340.
12. A. Fiorenza, Duality and reflexivity in grand Lebesgue spaces. *Collect. Mat*h. **51** (2000), no. 2, 131–148.
13. A. Fiorenza, G. E. Karadzhov, Grand and small Lebesgue spaces and their analogs. *Z. Anal. Anwendungen* **23** (2004), no. 4, 657–681.
14. V. F. Gaposhkin, A generalization of the theorem of M. Riesz on conjugate functions. (Russian) *Mat. Sb. (N.S.)* **46(88)** (1958), 359–372.
15. E. S. Golubeva, The system of weighted exponentials with power weights. (Russian) *Vestnik Sam. GU Estestvenno-Nauchnaya Ser.* **2** (2011), 15–25.
16. L. Greco, A remark on the equality $\det Df = \operatorname{Det} Df$. *Differential Integral Equations* **6** (1993), no. 5, 1089–1100.
17. B. Gupta, A. Fiorenza, P. Jain, The maximal theorem for weighted grand Lebesgue spaces. *Studia Math.* **188** (2008), no. 2, 123–133.
18. R. A. Hunt, B. Muckenhoupt, R. L. Wheeden, Weighted norm inequalities for the conjugate function and Hilbert transform. *Trans. Amer. Math. Soc.* **176** (1973), 227–251.
19. R. A. Hunt, W. S. Young, A weighted norm inequality for Fourier series. *Bull. Amer. Math. Soc.* **80** (1974), 274–277.
20. M. I. Ismailov, On the Solvability of Riemann Problems in Grand Hardy Classes. *Mathematical Notes* **108** (2020), 523–537.
21. M. I. Ismailov, Y. Zeren, K. Simsir Acar, I. F. Aliyarova, On basicity of exponential and trigonometric systems in grand Lebesgue spaces. *Hacettepe Journal of Mathematics & Statistics*, **51** (2022), 1577–1587.

22. T. Iwaniec, C. Sbordone, On the integrability of the Jacobian under minimal hypotheses. *Arch. Rational Mech. Anal.* **119** (1992), 129–143.
23. V. M. Kokilashvili, A. Meskhi, V. Paatashvili, The Riemann–Hilbert problem in the class of Cauchy type integrals with densities of grand Lebesgue spaces. *Trans. of A. Razmadze Math. Inst.* **170** (2016), 208–211.
24. V. M. Kokilashvili, A. Meskhi, H. Rafeiro, S. Samko, *Integral Operators in Non-Standart Function Spacces.* vol. 2, Variable exponent Holder, Morrey-Companato and Grand spaces. Springer, Switzerland, 2016.
25. L. V. Kritskov, Necessary condition for the uniform minimality of Kostyuchenko type systems. *Azerbaijan Journal of Mathematics* **5**1 (2015), 97–103.
26. A. Sh. Shukurov, On basicity of the degenerate trigonometric system with excess. *Acta Comment. Univ. Tartu. Math.* **21** (2017), 249–257.
27. A. Sh. Shukurov, Basis of the properties of weighted exponential systems with excess. *Bulletin of the Samara University. Vestn. SamGU. Natural Science ser.* **24** (2018), 14–19.
28. A. Sh. Shukurov, N. A. Ismailov, On the Minimality of double exponential system in weighted Lebesgue space. *Azerbaijan Journal of Mathematics* **7**1 (2017), 125–134.
29. Y. Zeren, M. I. Ismailov, C. Karacam, Korovkin-type theorems and their statistical versions in grand Lebesgue spaces. *Turk. J. Math.* **44** (2020), 1027–1041.
30. Y. Zeren, M. Ismailov, F. Sirin, On basicity of the system of eigenfunctions of one discontinuous spectral problem for second order differential equation for grand Lebesgue space. *Turkish J. Math.* **44** (2020), 1595–1612.

(Received 24.02.2023)

[1]Department of Mathematics, Azerbaijan State Oil and Industry University, Baku, Azerbaijan

[2]Institute of Graduate School, Istanbul Ticaret University, Istanbul Commerce University, Istanbul, Türkiye

[3]Department of Mathematics, Yildiz Technical University, Istanbul, Türkiye
Email address: migdad-ismailov@rambler.ru
Email address: ksacar@ticaret.edu.tr
Email address: yzeren@yildiz.edu.tr