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SOME BOUNDARY VALUE PROBLEMS OF THE THEORY OF CONSOLIDATION WITH DOUBLE POROSITY

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Abstract. In this paper, the boundary value problems of statics of consolidation theory with double porosity for a half-space are considered. On the boundary of the half-space, the limiting values of normal component of the displacement vector, the tangent components of the stress vector, and the normal derivatives of the fluid pressure in the primary and secondary pores, or the limiting values of the normal component of the stress vector, the tangent components of the displacement vector, and the fluid pressure in the primary and secondary pores are given. Using general formulas for representing the solution of a system of differential equations of consolidation theory and the Fourier transform, the solutions of the boundary value problems are constructed in explicit form in quadratures.

1. Introduction

The physical and mathematical foundations of the consolidation theory with double porosity were proposed in [6,12,17].

In [4], the two-dimensional version of the quasi-static Aifantis equation of the consolidation theory with double porosity is consider and the uniqueness and existence of solutions of basic boundary value problems are studied.

In [3], the first and second boundary value problems of the consolidation theory with double porosity for the half-space are explicitly solved by using the potential method and the theory of integral equations.

More detailed information on the theory of porous media and the results obtained in this direction can be found in [1,2,7,9,10,13,16].

The present paper deals with two boundary value problems of statics of the consolidation theory with double porosity for a half-space. In the case of problem (A), the limiting values of the tangent components of the stress vector, the normal component of displacement vector, and the normal derivatives of the fluid pressure in the primary and secondary pores are specified on the boundary of the half-space. In the case of problem (B), the tangent components of the displacement vector, the limiting values of the normal component of the stress vector and the fluid pressure in primary and secondary pores are specified. It should be noted that in solving these problems, the formulas for representing the general solution of the system of homogeneous differential equations of the consolidation theory play an important role. As in [8,11,15], in this work, a general solution of the system of equations, containing four harmonic and one metaharmonic functions, is constructed. We look for these functions in the form of certain potentials with unknown densities. Using the Fourier transform, we obtain a system of algebraic equations with respect to the densities, and than, applying the inverse Fourier transform, we construct explicit solutions of the problems.

2. Basic Differential Equations and Boundary Value Problems

Let Ω^- be a half-space, $\Omega^- := \{\, x : x \in \mathbb{R}^3,\ x_3 > 0 \,\}$, whose boundary $\partial\Omega^-$ is a plane $\partial\Omega^- = \{\, x : x \in \mathbb{R}^3,\ x_3 = 0 \,\}$. By $n = (0,0,1)^\top$ we denote the unit normal vector to $\partial\Omega^-$. The symbol $(\cdot)^\top$ denotes transposition operation.

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The homogeneous system of partial differential equations of statics of the consolidation theory with double porosity reads as [6, 17]:

$$\mu\Delta u(x) + (\lambda+\mu)\operatorname{grad}\operatorname{div} u(x) - \operatorname{grad}(\beta_1 p_1(x) + \beta_2 p_2(x)) = 0, \tag{2.1}$$

$$(m_1\Delta - \varkappa)\, p_1(x) + \varkappa\, p_2(x) = 0, \tag{2.2}$$

$$\varkappa\, p_1(x) + (m_2\Delta - \varkappa)\, p_2(x) = 0, \tag{2.3}$$

where Δ is the tree-dimensional Laplace operator, $u = (u_1, u_2, u_3)^\top$ is the displacement vector; p_1 is the fluid pressure within the primary pores, and p_2 is the fluid pressure in the secondary pores; λ and μ are the Lamé constants; β_j, $j = 1, 2$, are measures of the change of porosities due to an applied volumetric strain; $m_j = k_j/\mu^*$, $j = 1, 2$, where k_1 and k_2 are the permeabilities of the primary and secondary systems of pores, μ^* denotes the viscosity of the pore fluid; $\varkappa$ measures the transfer of fluid from the secondary pores to the primary pores. The material parameter $\lambda, \mu, \varkappa, \beta_j, k_j$, $j = 1, 2$, and μ^* are the positive constants.

Definition 2.1. A vector-function $U = (u,\, p_1,\, p_2)^\top$ is said to be regular in a domain Ω^-, if $U \in C^2(\Omega^-) \cap C^1(\overline{\Omega^-})$ and satisfies at infinity the following conditions:

$$\begin{aligned}
&U_j(x) = O(|x|^{-1}), \quad \frac{\partial U_j}{\partial x_i} = O(|x|^{-2}), \ \ x_3 > 0, \ \ |x| \to \infty,\\
&U_j(x) = o(1), \quad \frac{\partial U_j}{\partial x_i} = O(|x|^{-1}), \ \ x_3 = 0, \ \ |x| \to \infty,\\
&j = 1, 2, \dots, 5, \ \ i = 1, 2, 3; \quad |x|^2 = x_1^2 + x_2^2 + x_3^2.
\end{aligned}$$

Here, we assume that $U_j = u_j$, $j = 1, 2, 3$, $U_4 = p_1$, $U_5 = p_2$.

For the system of equations (2.1)–(2.3), we consider the following boundary value problems.

Problem (A). Find a regular vector-function $U = (u,\, p_1,\, p_2)^\top$ satisfying the system of differential equations (2.1)–(2.3) in Ω^- and the boundary conditions

$$\{[P(\partial, n)U(z)]_j\}^- = f_j(z), \ \ j = 1, 2, \quad \{u_3(z)\}^- = f_3(z), \tag{2.4}$$

$$\left\{\frac{\partial p_1(z)}{\partial n(z)}\right\}^- = f_4(z), \ \ \left\{\frac{\partial p_2(z)}{\partial n(z)}\right\}^- = f_5(z), \ \ z = (z_1, z_2) \in \partial\,\Omega^-, \tag{2.5}$$

where $P(\partial, n)U$ is the stress vector having the form [17]

$$P(\partial, n)U(x) = 2\mu\frac{\partial u}{\partial n} + \lambda\, n \operatorname{div} u + \mu[n \times \operatorname{rot} u] - n\,(\beta_1\, p_1(x) + \beta_2 p_2(x)), \quad \frac{\partial}{\partial n} = \sum_{j=1}^{3} n_j \frac{\partial}{\partial x_j};$$

$n = (n_1, n_2, n_3)^\top$ is a unit vector; the functions f_j, $j = 1, 2, \dots, 5$, are given on the boundary $\partial\,\Omega^-$, $f_j \in C^{\,1,\delta}(\partial\,\Omega^-)$, $j = 1, 2, 4, 5$, $f_3 \in C^{\,0,\delta}(\partial\,\Omega^-)$, $0 < \delta < 1$, and satisfy at infinity the following decay conditions:

$$|f_j(z)| < \frac{A}{1 + |z|^2}, \ \ j = 1, 2, 4, 5, \ \ |f_3(z)| < \frac{A}{1 + |z|}, \quad z \in \partial\,\Omega^-, \ \ A = \text{const} > 0.$$

Problem (B). Find a regular vector-function $U = (u,\, p_1,\, p_2)^\top$ satisfying the system of differential equations (2.1)–(2.3) in Ω^- and the boundary conditions

$$\{u_j(z)\}^- = F_j(z), \ \ j = 1, 2, \ \ \{[P(\partial, n)U(z)]_3\}^- = F_3(z), \tag{2.6}$$

$$\{p_1(z)\}^- = F_4(z), \ \ \{p_2(z)\}^- = F_5(z), \ \ z \in \partial\,\Omega^-, \tag{2.7}$$

where the functions F_j, $j = 1, 2, \dots, 5$, are given on the boundary $\partial\,\Omega^-$, $F_j \in C^{\,0,\delta}(\partial\,\Omega^-)$, $j = 1, 2, 4, 5$, $F_3 \in C^{\,1,\delta}(\partial\,\Omega^-)$, $0 < \delta < 1$, and satisfying at infinity the decay conditions:

$$|F_j(z)| < \frac{A}{1 + |z|}, \ \ j = 1, 2, 4, 5, \ \ |F_3(z)| < \frac{A}{1 + |z|^2}, \ \ z \in \partial\,\Omega^-, \ \ A = \text{const} > 0.$$

Here and in what follows, the symbol $[a \times b]$ denotes the cross product of two vectors in $\mathbb{R}^3$, while the central dot denotes the real scalar product $a \cdot b = \sum_{k=1}^{3} a_k b_k$ for $a, b \in \mathbb{R}^3$, the symbol $\{\cdot\}^-$ denotes the limiting value on the boundary $\partial\Omega^-$ from Ω^-,

$$\{w(z)\}^- = \lim_{\Omega^- \ni x \to z \in \partial\,\Omega^-} w(x), \quad \left\{\frac{\partial w(z)}{\partial x_j}\right\}^- = \lim_{\Omega^- \ni x \to z \in \partial\,\Omega^-} \frac{\partial w(x)}{\partial x_j}, \quad j = 1, 2, 3.$$

The uniqueness of the solution to the above-formulated problems can be easily proved using Green's formula for the system of differential equations of the theory of consolidation [5].

3. Solution of Problem (A)

First, we prove the following assertion.

Theorem 3.1. *A vector $U = (u_1, u_2, u_3, p_1, p_2)^\top \in C^2(\Omega^-)$ is a general solution of the homogeneous system (2.1)–(2.3) in a domain $\Omega^- \subset \mathbb{R}^3$ if and only if it is representable in the form*

$$\begin{aligned} u(x) = {} & \operatorname{grad} \Phi_1(x) + \alpha\, x_3 \operatorname{grad} \Phi_2(x) - e_3\, \Phi_2(x) + \operatorname{rot}(e_3\, \Phi_3(x)) \\ & + \alpha_1 \operatorname{grad}(x_3\, \Phi_4(x)) + \alpha_2 \operatorname{grad} \Phi_5(x), \end{aligned} \tag{3.1}$$

$$p_1(x) = \frac{\partial \Phi_4(x)}{\partial x_3} + m_2 \Phi_5(x), \quad p_2(x) = \frac{\partial \Phi_4(x)}{\partial x_3} - m_1 \Phi_5(x), \tag{3.2}$$

where

$$\Delta \Phi_j(x) = 0, \ j = 1, 2, 3, 4, \ \ (\Delta - \lambda_0^2)\Phi_5(x) = 0, \ \ \lambda_0^2 = \frac{m_1 + m_2}{m_1 m_2}\varkappa,$$

$$\alpha = \frac{\lambda + \mu}{\lambda + 3\mu}, \ \ \alpha_1 = \frac{\beta_1 + \beta_2}{2(\lambda + 2\mu)}, \ \ \alpha_2 = \frac{1}{\lambda_0^2(\lambda + 2\mu)}\,(\beta_1 m_2 - \beta_2 m_1), \quad e_3 = (0, 0, 1)^\top.$$

Proof. Assume that a vector $U = (u_1, u_2, u_3, p_1, p_2)^\top \in C^2(\Omega^-)$, is a solution of the homogeneous system (2.1)–(2.3). Than from equations (2.2)–(2.3), we get

$$\Delta(\Delta - \lambda_0^2)P(x) = 0, \quad P = (p_1, p_2)^\top,$$

which means that the functions $p_j(x)$, $j = 1, 2$, can be represented as a linear combination of harmonic and metaharmonic functions

$$p_1(x) = A\frac{\partial \Phi_4(x)}{\partial x_3} + B\Phi_5(x), \quad p_2(x) = C\frac{\partial \Phi_4(x)}{\partial x_3} + D\Phi_5(x).$$

Here, A, B, C, D are arbitrary constants. In particular, if $A = C = 1$, $B = m_2$, and $D = -m_1$, we get equalities (3.1), which satisfy equations (2.2)–(2.3). Substituting the expessions of the functions $p_j(x)$, $j = 1, 2$ from (3.2) into (2.1), we obtain

$$\mu\Delta u(x) + (\lambda + \mu) \operatorname{grad} \operatorname{div} u(x) = \operatorname{grad}\left((\beta_1 + \beta_2)\frac{\partial \Phi_4(x)}{\partial x_3} + (\beta_1 m_2 - \beta_2 m_1))\Phi_5(x)\right). \tag{3.3}$$

Denote by $u^{(0)}(x)$ a particular solution of equation (3.3). Then a general solution of the homogeneous equation (3.3) we can represent in the form $u(x) = u'(x) + u^{(0)}(x)$, where $u'(x)$ is a general solution of the homogeneous equation of the classical theory of elasticity for isotropic homogeneous bodies

$$\mu\Delta u'(x) + (\lambda + \mu) \operatorname{grad} \operatorname{div} u'(x) = 0. \tag{3.4}$$

The particular solution of equation (3.3) we can construct explicitly:

$$u^{(0)}(x) = \alpha_1 \operatorname{grad}\big(x_3 \Phi_4(x)\big) + \alpha_2 \operatorname{grad} \Phi_5(x).$$

As it is known, the general solution of equation (3.4) has the following form [8]:

$$u'(x) = \operatorname{grad} \Phi_1(x) + \alpha\, x_3 \operatorname{grad} \Phi_2(x) - e_3\, \Phi_2(x) + \operatorname{rot}\big(e_3\, \Phi_3(x)\big),$$

where $\Delta \Phi_j(x) = 0$, $j = 1, 2, 3$.

The sum of the vectors $u^{(0)}(x)$ and $u'(x)$ gives us the vector $u(x)$, which is defined by formula (3.1), and this proves the first part of the theorem.

The second part of the theorem can be proved by substituting the vector $U=(u,\,p_1,\,p_2)^\top$ represented by (3.1)–(3.2) into the homogeneous system (2.1)–(2.3). $\square$

According to formulas (3.1)–(3.2), the stress vector has the form

$$\begin{aligned}P(\partial,n)U(x)&=2\mu\,\mathrm{grad}\,\frac{\partial\Phi_1(x)}{\partial x_3}+2\mu\,\alpha\,x_3\,\mathrm{grad}\,\frac{\partial\Phi_2(x)}{\partial x_3}-2\mu\,\alpha\,e_3\,\frac{\partial\Phi_2(x)}{\partial x_3}+\mu(\alpha-1)\,\mathrm{grad}\,\Phi_2(x)\\&\quad+\mu\,\mathrm{rot}\left(e_3\,\frac{\partial\Phi_3(x)}{\partial x_3}\right)+2\mu\,\alpha_1\,x_3\,\mathrm{grad}\,\frac{\partial\Phi_4(x)}{\partial x_3}+2\mu\,\alpha_1\,\mathrm{grad}\,\Phi_4(x)\\&\quad+\alpha\,(\beta_1+\beta_2)\,e_3\frac{\partial\Phi_4(x)}{\partial x_3}+2\mu\,\alpha_2\left(\mathrm{grad}\,\frac{\partial}{\partial x_3}-\lambda_0^2\,e_3\right)\Phi_5(x),\quad x\in\Omega^-.\end{aligned}\tag{3.5}$$

We look for the functions $\Phi_j(x)$, $j=1,2,3,4,5$ in the following form:

$$\Phi_j(x)=\frac{1}{2\pi}\int\limits_{\partial\Omega^-}e^{-i\,\widetilde{x}\cdot\xi-x_3|\xi|}\,g_j(\xi)\,d\xi_1d\xi_2,\ \ j=1,2,3,4,\tag{3.6}$$

$$\begin{gathered}\Phi_5(x)=\frac{1}{2\pi}\int\limits_{\partial\Omega^-}e^{-i\,\widetilde{x}\cdot\xi-x_3\,\sqrt{|\xi|^2+\lambda_0^2}}\,g_5(\xi)\,d\xi_1d\xi_2,\\ x\in\Omega^-,\quad\widetilde{x}=(x_1,x_2)^\top,\ \xi=(\xi_1,\xi_2)^\top,\ |\xi|=\sqrt{\xi_1^2+\xi_2^2},\end{gathered}\tag{3.7}$$

where $g_j(\xi)$, $j=1,2,3,4,5$ are sought for functions.

Substitute $\Phi_j(x)$, $j=1,2,3,4,5$, into (3.1) and (3.2), to obtain

$$\begin{aligned}u(x)=&\frac{1}{2\pi}\int\limits_{\partial\Omega^-}\Big[\Big(-e_1\,g_1(\xi)-(\,e_3+\alpha\,x_3\,e_1)\,g_2(\xi)+e_2\,g_3(\xi)+\alpha_1(e_3-x_3\,e_1)\,g_4(\xi)\Big)e^{-x_3|\xi|}\\&-\alpha_2\,e_4\,e^{-x_3\,\sqrt{|\xi|^2+\lambda_0^2}}g_5(\xi)\Big]e^{-i\,\widetilde{x}\cdot\xi}\,d\xi_1d\xi_2,\quad x\in\Omega^-,\end{aligned}\tag{3.8}$$

where $e_1=(\,i\xi_1,i\xi_2,|\xi|\,)^\top$, $e_2=(\,-i\xi_2,i\xi_1,0\,)^\top$ $e_4=\Big(\,i\xi_1,i\xi_2,\sqrt{|\xi|^2+\lambda_0^2}\,\Big)^\top$.

Similarly, we have

$$\begin{pmatrix}p_1(x)\\p_2(x)\end{pmatrix}=\frac{1}{2\pi}\int\limits_{\partial\Omega^-}A(x,\xi)\,e^{-i\,\widetilde{x}\cdot\xi}\,\begin{pmatrix}g_4(\xi)\\g_5(\xi)\end{pmatrix}d\xi_1d\xi_2,\tag{3.9}$$

where

$$\begin{aligned}&A_{11}(x,\xi)=-|\xi|e^{-x_3|\xi|},\ \ A_{12}(x,\xi)=m_2\,e^{-x_3\sqrt{|\xi|^2+\lambda_0^2}},\\&A_{21}(x,\xi)=-|\xi|e^{-x_3|\xi|},\ \ A_{22}(x,\xi)=-m_1\,e^{-x_3\sqrt{|\xi|^2+\lambda_0^2}}.\end{aligned}$$

Analogously, substituting the function $\Phi_j(x)$, $j=1,2,3,4,5$, defined by (3.6) and (3.7) into the right-hand side of (3.5), we have

$$\begin{aligned}P(\partial,n)U(x)=&\frac{1}{2\pi}\int\limits_{\partial\Omega^-}\Big\{\Big[2\mu\,|\xi|\,e_1\,g_1(\xi)+\Big(2\mu\,\alpha\,x_3|\xi|e_1-\mu(\alpha-1)\,e_1+2\mu\,\alpha\,|\xi|\,e_3\Big)\,g_2(\xi)\\&-\mu|\xi|\,e_2\,g_3(\xi)+\Big(2\mu\,\alpha_1\,x_3|\xi|\,e_1-2\mu\,\alpha_1\,e_1-\alpha(\beta_1+\beta_2)|\xi|e_3\Big)g_4(\xi)\Big]e^{-x_3|\xi|}\\&+2\mu\,\alpha_2\Big(\sqrt{|\xi|^2+\lambda_0^2}\,e_4-\lambda_0^2\,e_3\Big)\,g_5(\xi)\,e^{-x_3\sqrt{|\xi|^2+\lambda_0^2}}\Big\}e^{-i\,\widetilde{x}\cdot\xi}\,d\xi_1d\xi_2.\end{aligned}\tag{3.10}$$

The normal derivatives of the fluid pressure in the primary and the secondary pores p_1 and p_2 in Ω^- have the form

$$\frac{\partial}{\partial x_3}\begin{pmatrix}p_1(x)\\p_2(x)\end{pmatrix}=\frac{1}{2\pi}\int\limits_{\partial\Omega^-}\frac{\partial}{\partial x_3}A(x,\xi)\,e^{-i\,\widetilde{x}\cdot\xi}\,\begin{pmatrix}g_4(\xi)\\g_5(\xi)\end{pmatrix}d\xi_1d\xi_2.\tag{3.11}$$

Taking into account the boundary conditions (2.4), (2.5) and passing to the limit in (3.10), (3.8), (3.11) as $\Omega^- \ni x \to z \in \partial\,\Omega^-$, and using the inverse Fourier transform, for the unknown functions $g_j(\xi)$, $j=1,2,\dots,5$, we obtain the following system of equations:

$$\begin{aligned} &2\mu\left[|\xi|\,g_1(\xi)-\alpha_1\,g_4(\xi)+\alpha_2\sqrt{|\xi|^2+\lambda_0^2}\,g_5(\xi)\right]i\xi_j-\mu\,|\xi|i\,(\delta_{2j}\xi_1-\delta_{1j}\xi_2)\,g_3(\xi)\\ &-\mu(\alpha-1)\,i\xi_j\,g_2(\xi)=\widehat{f}_j(\xi),\ \ j=1,2, \end{aligned} \tag{3.12}$$

$$|\xi|\,g_1(\xi)+g_2(\xi)-\alpha_1\,g_4(\xi)+\alpha_2\sqrt{|\xi|^2+\lambda_0^2}\,g_5(\xi)=-\widehat{f}_3(\xi), \tag{3.13}$$

$$|\xi|^2\,g_4(\xi)-m_2\sqrt{|\xi|^2+\lambda_0^2}\,g_5(\xi)=\widehat{f}_4(\xi), \tag{3.14}$$

$$|\xi|^2\,g_4(\xi)+m_1\sqrt{|\xi|^2+\lambda_0^2}\,g_5(\xi)=\widehat{f}_5(\xi),\ \ \xi\in\partial\Omega^-, \tag{3.15}$$

where

$$\widehat{f}_j(\xi)=\frac{1}{2\pi}\int\limits_{\partial\Omega^-}e^{i\,z\cdot\xi}f_j(z)dz_1dz_2,\ \ j=1,2,\dots,5. \tag{3.16}$$

Using the equalitis

$$\begin{gathered} \widetilde{e}_1\cdot\widetilde{e}_1=-|\xi|^2,\quad \widetilde{e}_1\cdot e_1=-|\xi|^2,\quad \widetilde{e}_1\cdot e_2=0,\quad \widetilde{e}_1\cdot e_3=0,\quad \widetilde{e}_1\cdot e_4=-|\xi|^2,\\ e_1\cdot e_1=0,\quad e_1\cdot e_2=0,\quad e_1\cdot e_3=|\xi|,\quad e_1\cdot e_4=|\xi|\left(\sqrt{|\xi|^2+\lambda_0^2}-|\xi|\right),\\ e_2\cdot e_2=-|\xi|^2,\quad e_2\cdot e_3=0,\quad e_2\cdot e_4=0,\quad e_3\cdot e_4=\sqrt{|\xi|^2+\lambda_0^2}, \end{gathered}$$

where $\widetilde{e}_1=(i\xi_1,i\xi_2,0)^\top$, from equations (3.12)–(3.15), we obtain

$$\begin{aligned} g_1(\xi)=&-\frac{1}{\mu(\alpha+1)}\left(\frac{i\xi_1}{|\xi|^3}\,\widehat{f}_1(\xi)+\frac{i\xi_2}{|\xi|^3}\,\widehat{f}_2(\xi)\right)+\frac{1-\alpha}{1+\alpha}\,\frac{1}{|\xi|}\,\widehat{f}_3(\xi)\\ &+\left(\frac{\alpha_1\,m_1}{m_1+m_2}\,\frac{1}{|\xi|^3}+\frac{\alpha_2}{m_1+m_2}\,\frac{1}{|\xi|}\right)\widehat{f}_4(\xi)+\left(\frac{\alpha_1\,m_2}{m_1+m_2}\,\frac{1}{|\xi|^3}-\frac{\alpha_2}{m_1+m_2}\,\frac{1}{|\xi|}\right)\widehat{f}_5(\xi),\\ g_2(\xi)=&\frac{1}{\mu(\alpha+1)}\left(\frac{i\xi_1}{|\xi|^2}\,\widehat{f}_1(\xi)+\frac{i\xi_2}{|\xi|^2}\,\widehat{f}_2(\xi)\right)-\frac{2}{\alpha+1}\,\widehat{f}_3(\xi),\\ g_3(\xi)=&-\frac{i\xi_2}{\mu|\xi|^3}\,\widehat{f}_1(\xi)+\frac{i\xi_1}{\mu|\xi|^3}\,\widehat{f}_2(\xi),\\ g_4(\xi)=&\frac{m_1}{m_1+m_2}\,\frac{1}{|\xi|^2}\,\widehat{f}_4(\xi)+\frac{m_2}{m_1+m_2}\,\frac{1}{|\xi|^2}\,\widehat{f}_5(\xi),\\ g_5(\xi)=&-\frac{1}{(m_1+m_2)\sqrt{|\xi|^2+\lambda_0^2}}\,\widehat{f}_4(\xi)+\frac{1}{(m_1+m_2)\sqrt{|\xi|^2+\lambda_0^2}}\,\widehat{f}_5(\xi). \end{aligned}$$

Taking into account the above equalities in (3.8), we get

$$u(x)=\frac{1}{2\pi}\int\limits_{\partial\Omega^-}\left\{N^{(1)}(x,\xi)\,e^{-x_3|\xi|}+N^{(2)}(\xi)\,e^{-x_3\sqrt{|\xi|^2+\lambda_0^2}}\right\}e^{-i\widetilde{x}\cdot\xi}\,\widehat{f}(\xi)\,d\xi_1d\xi_2, \tag{3.17}$$

where

$$N^{(1)}(x,\xi)=\left[N^{(1)}_{kj}(x,\xi)\right]_{3\times5},\ \ N^{(2)}(\xi)=\left[N^{(2)}_{kj}(\xi)\right]_{3\times5},\ \ \widehat{f}=(\widehat{f}_1,\widehat{f}_2,\widehat{f}_3,\widehat{f}_4,\widehat{f}_5)^\top,$$

$$\begin{aligned} N^{(1)}_{11}(x,\xi)&=-\frac{1}{\mu|\xi|}+\frac{\alpha}{\mu(\alpha+1)}\frac{\xi_1^2}{|\xi|^3}+\frac{\alpha\,x_3}{\mu(\alpha+1)}\,\frac{\xi_1^2}{|\xi|^2},\\ N^{(1)}_{21}(x,\xi)&=\frac{\alpha}{\mu(\alpha+1)}\frac{\xi_1\xi_2}{|\xi|^3}+\frac{\alpha\,x_3}{\mu(\alpha+1)}\,\frac{\xi_1\xi_2}{|\xi|^2},\\ N^{(1)}_{31}(x,\xi)&=-\frac{\alpha\,x_3}{\mu(\alpha+1)}\,\frac{i\xi_1}{|\xi|},\ \ N^{(1)}_{12}(x,\xi)=\frac{\alpha}{\mu(\alpha+1)}\frac{\xi_1\xi_2}{|\xi|^3}+\frac{\alpha\,x_3}{\mu(\alpha+1)}\,\frac{\xi_1\xi_2}{|\xi|^2}, \end{aligned}$$

$$
\begin{aligned}
&N^{(1)}_{22}(x,\xi)=-\frac{1}{\mu}\frac{1}{|\xi|}+\frac{\alpha}{\mu(\alpha+1)}\frac{\xi_2^2}{|\xi|^3}+\frac{\alpha\,x_3}{\mu(\alpha+1)}\frac{\xi_2^2}{|\xi|^2},\\
&N^{(1)}_{32}(x,\xi)=-\frac{\alpha}{\mu(\alpha+1)}\frac{i\xi_2}{|\xi|},\ \ N^{(1)}_{13}(x,\xi)=\frac{\alpha-1}{\alpha+1}\frac{i\xi_1}{|\xi|}+\frac{2\alpha\,x_3}{\alpha+1}\,i\xi_1,\\
&N^{(1)}_{23}(x,\xi)=\frac{\alpha-1}{\alpha+1}\frac{i\xi_2}{|\xi|}+\frac{2\alpha\,x_3}{\alpha+1}\,i\xi_2,\quad N^{(1)}_{33}(x,\xi)=1+\frac{2\alpha\,x_3}{\alpha+1}\,|\xi|,\\
&N^{(1)}_{14}(x,\xi)=-\frac{\alpha_2}{m_1+m_2}\frac{i\xi_1}{|\xi|}-\frac{\alpha_1\,m_1}{m_1+m_2}\frac{i\xi_1}{|\xi|^3}-\frac{\alpha_1\,m_1\,x_3}{m_1+m_2}\frac{i\xi_1}{|\xi|^2},\\
&N^{(1)}_{24}(x,\xi)=-\frac{\alpha_2}{m_1+m_2}\frac{i\xi_2}{|\xi|}-\frac{\alpha_1\,m_1}{m_1+m_2}\frac{i\xi_2}{|\xi|^3}-\frac{\alpha_1\,m_1\,x_3}{m_1+m_2}\frac{i\xi_2}{|\xi|^2},\\
&N^{(1)}_{34}(x,\xi)=-\frac{\alpha_2}{m_1+m_2}-\frac{\alpha_1\,m_1\,x_3}{m_1+m_2}\frac{1}{|\xi|},\\
&N^{(1)}_{15}(x,\xi)=\frac{\alpha_2}{m_1+m_2}\frac{i\xi_1}{|\xi|}-\frac{\alpha_1\,m_2}{m_1+m_2}\frac{i\xi_1}{|\xi|^3}-\frac{\alpha_1\,m_2\,x_3}{m_1+m_2}\frac{i\xi_1}{|\xi|^2},\\
&N^{(1)}_{25}(x,\xi)=\frac{\alpha_2}{m_1+m_2}\frac{i\xi_2}{|\xi|}-\frac{\alpha_1\,m_2}{m_1+m_2}\frac{i\xi_2}{|\xi|^3}-\frac{\alpha_1\,m_2\,x_3}{m_1+m_2}\frac{i\xi_2}{|\xi|^2},\\
&N^{(1)}_{35}(x,\xi)=\frac{\alpha_2}{m_1+m_2}-\frac{\alpha_1\,m_2\,x_3}{m_1+m_2}\frac{1}{|\xi|},
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
&N^{(2)}_{11}(\xi)=N^{(2)}_{21}(\xi)=N^{(2)}_{31}(\xi)=N^{(2)}_{12}(\xi)=N^{(2)}_{22}(\xi)=0,\\
&N^{(2)}_{32}(\xi)=N^{(2)}_{13}(\xi)=N^{(2)}_{23}(\xi)=N^{(2)}_{33}(\xi)=0,\\
&N^{(2)}_{14}(\xi)=\frac{\alpha_2}{m_1+m_2}\frac{i\xi_1}{\sqrt{|\xi|^2+\lambda_0^2}},\ \ N^{(2)}_{24}(\xi)=\frac{\alpha_2}{m_1+m_2}\frac{i\xi_2}{\sqrt{|\xi|^2+\lambda_0^2}},\\
&\qquad\qquad N^{(2)}_{34}(\xi)=\frac{\alpha_2}{m_1+m_2},\\
&N^{(2)}_{15}(\xi)=-\frac{\alpha_2}{m_1+m_2}\frac{i\xi_1}{\sqrt{|\xi|^2+\lambda_0^2}},\ N^{(2)}_{25}(\xi)=-\frac{\alpha_2}{m_1+m_2}\frac{i\xi_2}{\sqrt{|\xi|^2+\lambda_0^2}},\\
&\qquad\qquad N^{(2)}_{35}(\xi)=-\frac{\alpha_2}{m_1+m_2}\,.
\end{aligned}
\tag{3.19}
$$

Further, we use the following equalities [14]:

$$
\begin{aligned}
&\frac{1}{2\pi}\int\limits_{\partial\Omega^-}\frac{1}{|\xi|}e^{-x_3\,|\xi|-i(\widetilde{x}-y)\cdot\xi}d\xi_1d\xi_2=\frac{1}{r},\\
&\frac{1}{2\pi}\int\limits_{\partial\Omega^-}\frac{1}{|\xi|^2}e^{-x_3\,|\xi|-i(\widetilde{x}-y)\cdot\xi}d\xi_1d\xi_2=-\frac{\partial\Phi(x,y)}{\partial x_3},\\
&\frac{1}{2\pi}\int\limits_{\partial\Omega^-}\frac{i\xi_k}{|\xi|^2}e^{-x_3\,|\xi|-i(\widetilde{x}-y)\cdot\xi}d\xi_1d\xi_2=\frac{\partial^2\Phi(x,y)}{\partial x_k\partial x_3},\\
&\frac{1}{2\pi}\int\limits_{\partial\Omega^-}\frac{i\xi_k}{|\xi|^3}e^{-x_3\,|\xi|-i(\widetilde{x}-y)\cdot\xi}d\xi_1d\xi_2=-\frac{\partial\Phi(x,y)}{\partial x_k},\\
&\frac{1}{2\pi}\int\limits_{\partial\Omega^-}\frac{\xi_k\,\xi_j}{|\xi|^3}e^{-x_3\,|\xi|-i(\widetilde{x}-y)\cdot\xi}d\xi_1d\xi_2=-\frac{\partial^2\Phi(x,y)}{\partial x_k\,\partial x_j},\\
&\frac{1}{2\pi}\int\limits_{\partial\Omega^-}\frac{1}{\sqrt{|\xi|^2+\lambda_0^2}}e^{-x_3\,\sqrt{|\xi|^2+\lambda_0^2}-i(\widetilde{x}-y)\cdot\xi}\,d\xi_1d\xi_2=\frac{e^{-\lambda_0\,r}}{r},\quad k=1,2,
\end{aligned}
\tag{3.20}
$$

where

$$r=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+x_3^2},\qquad \Phi(x,y)=x_3\,ln(r+x_3)-r,\qquad \frac{\partial^2\Phi(x,y)}{\partial x_3^2}=\frac{1}{r}\,.$$

Substituting $\widehat{f}_j(\xi)$, $j=1,2,\dots,5$, from (3.16) into (3.17) and employing equalities (3.18)–(3.20), for the displacement vector in the case of the boundary value problem (A), we get the following expression:

$$u(x)=\frac{1}{2\pi}\int\limits_{\partial\Omega^-}K(x,y)f(y)dy_1dy_2, \tag{3.21}$$

where

$$\begin{aligned}
&K(x,y)=[K_{kj}(x,y)]_{3\times5}\,,\qquad f=(f_1,f_2,f_3,f_4,f_5)^\top,\\
&K_{11}(x,y)=-\frac{1}{\mu\,r}+\frac{\alpha}{\mu(\alpha+1)}\Big(x_3\frac{\partial}{\partial x_3}-1\Big)\frac{\partial^2\Phi(x,y)}{\partial x_1^2},\\
&K_{12}(x,y)=-\frac{\alpha}{\mu(\alpha+1)}\Big(x_3\frac{\partial}{\partial x_3}-1\Big)\frac{\partial^2\Phi(x,y)}{\partial x_1\partial x_2},\\
&K_{13}(x,y)=\frac{\partial}{\partial x_1}\frac{1}{r}+\frac{2\alpha}{\alpha+1}\Big(x_3\frac{\partial}{\partial x_3}-1\Big)\frac{\partial}{\partial x_1}\frac{1}{r},\\
&K_{14}(x,y)=-\frac{\alpha_2}{m_1+m_2}\frac{\partial}{\partial x_1}\frac{e^{-\lambda_0\,r}-1}{r}-\frac{\alpha_1\,m_1}{m_1+m_2}\Big(x_3\,\frac{\partial}{\partial x_3}-1\Big)\frac{\partial\Phi(x,y)}{\partial x_1},\\
&K_{15}(x,y)=\frac{\alpha_2}{m_1+m_2}\frac{\partial}{\partial x_1}\frac{e^{-\lambda_0\,r}-1}{r}-\frac{\alpha_1\,m_2}{m_1+m_2}\Big(x_3\,\frac{\partial}{\partial x_3}-1\Big)\frac{\partial\Phi(x,y)}{\partial x_1},\\
&K_{21}(x,y)=\frac{\alpha}{\mu(\alpha+1)}\Big(x_3\frac{\partial}{\partial x_3}-1\Big)\frac{\partial^2\Phi(x,y)}{\partial x_1\partial x_2},\\
&K_{22}(x,y)=-\frac{1}{\mu\,r}-\frac{\alpha}{\mu(\alpha+1)}\Big(x_3\frac{\partial}{\partial x_3}-1\Big)\frac{\partial^2\Phi(x,y)}{\partial x_2^2},\\
&K_{23}(x,y)=\frac{\partial}{\partial x_2}\frac{1}{r}+\frac{2\alpha}{\alpha+1}\Big(x_3\frac{\partial}{\partial x_3}-1\Big)\frac{\partial}{\partial x_2}\frac{1}{r},\\
&K_{24}(x,y)=-\frac{\alpha_2}{m_1+m_2}\frac{\partial}{\partial x_2}\frac{e^{-\lambda_0\,r}-1}{r}+\frac{\alpha_1\,m_1}{m_1+m_2}(x_3+1)\,\frac{\partial\Phi(x,y)}{\partial x_2},\\
&K_{25}(x,y)=\frac{\alpha_2}{m_1+m_2}\frac{\partial}{\partial x_2}\frac{e^{-\lambda_0\,r}-1}{r}-\frac{\alpha_1\,m_2}{m_1+m_2}\Big(x_3\,\frac{\partial}{\partial x_3}-1\Big)\frac{\partial\Phi(x,y)}{\partial x_2},\\
&K_{31}(x,y)=\frac{\alpha}{\mu(\alpha+1)}\,x_3\,\frac{\partial}{\partial x_1}\,\frac{1}{r},\qquad K_{32}(x,y)=\frac{\alpha}{\mu(\alpha+1)}\,x_3\,\frac{\partial}{\partial x_2}\,\frac{1}{r},\\
&K_{33}(x,y)=-\frac{\partial}{\partial x_3}\frac{1}{r}+\frac{2\alpha}{\alpha+1}\,x_3\,\frac{\partial^2}{\partial x_3^2}\frac{1}{r},\\
&K_{34}(x,y)=-\frac{\alpha_2}{m_1+m_2}\frac{\partial}{\partial x_3}\frac{e^{-\lambda_0\,r}-1}{r}-\frac{\alpha_1\,m_1}{m_1+m_2}\,\frac{x_3}{r},\\
&K_{35}(x,y)=\frac{\alpha_2}{m_1+m_2}\frac{\partial}{\partial x_3}\frac{e^{-\lambda_0\,r}-1}{r}-\frac{\alpha_1\,m_1}{m_1+m_2}\,\frac{x_3}{r}.
\end{aligned}$$

Note that

$$\lim_{\Omega^-\ni\,x\to z\,\in\,\partial\,\Omega^-}K_{3k}(x,y)=0,\qquad k=1,2,\dots,5. \tag{3.22}$$

Quite similarly, for the fluid pressure in the primary and secondary pores in the case of problem (A), we get the following formulas:

$$\begin{pmatrix}p_1(x)\\ p_2(x)\end{pmatrix}=\frac{1}{2\pi}\int\limits_{\partial\Omega^-}L(x,y)\begin{pmatrix}f_4(y)\\ f_5(y)\end{pmatrix}dy_1dy_2,$$

where

$$L(x,y) = [L_{kj}(x,y)]_{2\times 2}\,,$$

$$L_{11}(x,y) = -\frac{1}{r} - \frac{m_2}{m_1+m_2}\,\frac{e^{\lambda_0 r}-1}{r}, \qquad L_{12}(x,y) = \frac{m_2}{m_1+m_2}\,\frac{e^{\lambda_0 r}-1}{r},$$

$$L_{21}(x,y) = \frac{m_1}{m_1+m_2}\,\frac{e^{\lambda_0 r}-1}{r}, \qquad L_{22}(x,y) = -\frac{1}{r} - \frac{m_2}{m_1+m_2}\,\frac{e^{\lambda_0 r}-1}{r}\,.$$

Therefore, the normal derivatives of the fluid pressure in the primary and the secondary pores p_1 and p_2 in Ω^- have the form

$$\frac{\partial}{\partial x_3}\begin{pmatrix} p_1(x)\\ p_2(x)\end{pmatrix} = \frac{1}{2\pi}\int\limits_{\partial\Omega^-}\frac{\partial}{\partial x_3}L(x,y)\begin{pmatrix} f_4(y)\\ f_5(y)\end{pmatrix} dy_1dy_2. \tag{3.23}$$

Note that

$$\lim_{\Omega^-\ni x\to z\in\partial\Omega^-}\frac{\partial}{\partial x_3}L(x,y) = 0. \tag{3.24}$$

The stress vector in the case of the boundary value problem (A), according to (3.10), has the form

$$P(\partial,n)U(x) = \frac{1}{2\pi}\int\limits_{\partial\Omega^-} M(x,y)f(y)\,dy_1dy_2, \tag{3.25}$$

where

$$M(x,y) = [M_{kj}(x,y)]_{3\times 5}\,,$$

$$M_{11}(x,y) = -\frac{\partial}{\partial x_3}\,\frac{1}{r} + \frac{2\alpha}{\alpha+1}\,x_3\frac{\partial^2}{\partial x_1^2}\,\frac{1}{r}, \quad M_{21}(x,y) = \frac{2\alpha}{\alpha+1}\,x_3\frac{\partial^2}{\partial x_1\partial x_2}\,\frac{1}{r},$$

$$M_{31}(x,y) = \frac{1-\alpha}{\alpha+1}\frac{\partial}{\partial x_1}\,\frac{1}{r} + \frac{2\alpha}{\alpha+1}\,x_3\frac{\partial^2}{\partial x_1\partial x_3}\,\frac{1}{r}, \quad M_{12}(x,y) = \frac{2\alpha}{\alpha+1}\,x_3\frac{\partial^2}{\partial x_1\partial x_2}\,\frac{1}{r},$$

$$M_{22}(x,y) = -\frac{\partial}{\partial x_3}\,\frac{1}{r} + \frac{2\alpha}{\alpha+1}\,x_3\frac{\partial^2}{\partial x_2^2}\,\frac{1}{r}, \quad M_{32}(x,y) = \frac{1-\alpha}{\alpha+1}\frac{\partial}{\partial x_2}\,\frac{1}{r} + \frac{2\alpha}{\alpha+1}\,x_3\frac{\partial^2}{\partial x_2\partial x_3}\,\frac{1}{r},$$

$$M_{13}(x,y) = \frac{4\mu\alpha}{\alpha+1}\,x_3\frac{\partial^3}{\partial x_1\partial x_3^2}\,\frac{1}{r}, \quad M_{23}(x,y) = \frac{4\mu\alpha}{\alpha+1}\,x_3\frac{\partial^3}{\partial x_2\partial x_3^2}\,\frac{1}{r},$$

$$M_{33}(x,y) = \frac{4\mu\alpha}{\alpha+1}\frac{\partial}{\partial x_3}\left(x_3\frac{\partial^2}{\partial x_3^2}\right)\frac{1}{r}, \quad M_{14}(x,y) = -\frac{2\mu\alpha_2}{m_1+m_2}\frac{\partial^2}{\partial x_1\partial x_3}\frac{e^{-\lambda_0 r}-1}{r}$$

$$-\frac{2\mu\alpha_1\,m_1}{m_1+m_2}\,x_3\frac{\partial}{\partial x_1}\,\frac{1}{r}, \quad M_{24}(x,y) = -\frac{2\mu\alpha_2}{m_1+m_2}\frac{\partial^2}{\partial x_2\partial x_3}\frac{e^{-\lambda_0 r}-1}{r} - \frac{2\mu\alpha_1\,m_1}{m_1+m_2}\,x_3\frac{\partial}{\partial x_2}\,\frac{1}{r},$$

$$M_{34}(x,y) = -\frac{2\mu\alpha_2}{m_1+m_2}\frac{\partial^2}{\partial x_3^2}\frac{e^{-\lambda_0 r}-1}{r} + \frac{2\mu\alpha_2\lambda_0^2}{m_1+m_2}\,\frac{e^{-\lambda_0 r}}{r} - \frac{2\mu\alpha_1\,m_1}{m_1+m_2}\,x_3\frac{\partial}{\partial x_3}\,\frac{1}{r}$$

$$-\frac{\alpha\,(\beta_1+\beta_2)\,m_1}{m_1+m_2}\,\frac{1}{r}, \quad M_{15}(x,y) = \frac{2\mu\alpha_2}{m_1+m_2}\frac{\partial^2}{\partial x_1\partial x_3}\frac{e^{-\lambda_0 r}-1}{r} - \frac{2\mu\alpha_1\,m_2}{m_1+m_2}\,x_3\frac{\partial}{\partial x_1}\,\frac{1}{r},$$

$$M_{25}(x,y) = \frac{2\mu\alpha_2}{m_1+m_2}\frac{\partial^2}{\partial x_2\partial x_3}\frac{e^{-\lambda_0 r}-1}{r} - \frac{2\mu\alpha_1\,m_2}{m_1+m_2}\,x_3\frac{\partial}{\partial x_2}\,\frac{1}{r},$$

$$M_{35}(x,y) = \frac{2\mu\alpha_2}{m_1+m_2}\frac{\partial^2}{\partial x_3^2}\frac{e^{-\lambda_0 r}-1}{r} - \frac{2\mu\alpha_2\lambda_0^2}{m_1+m_2}\,\frac{e^{-\lambda_0 r}}{r} - \frac{2\mu\alpha_1\,m_2}{m_1+m_2}\,x_3\frac{\partial}{\partial x_3}\,\frac{1}{r}$$

$$-\frac{\alpha\,(\beta_1+\beta_2)\,m_2}{m_1+m_2}\,\frac{1}{r}\,.$$

Note that

$$\lim_{\Omega^-\ni x\to z\in\partial\Omega^-} M_{jk}(x,y) = 0, \quad j=1,2, \quad k=1,2,\dots,5. \tag{3.26}$$

Taking into account formulas (3.22), (3.24), and (3.26), we can show that the tangent components of vector (3.25), the normal component of vector (3.21), and vector (3.23) satisfy the boundary conditions of problem(A):

$$\{[P(\partial,n)U(x)]_j\}^- = f_j(z),\ \ j=1,2,\ \ \{u_3(z)\}^- = f_3(z),$$
$$\left\{\frac{\partial p_1(z)}{\partial x_3}\right\}^- = f_4(z),\ \ \left\{\frac{\partial p_2(z)}{\partial x_3}\right\}^- = f_5(z),\ \ z\in\partial\Omega^-.$$

4. Solution of Problem (B)

According to formulas (3.8), (3.9), and (3.10) in the case of BVP (B) we get the following system of algebraic equations with respect to unknown functions $g_j(\xi)$, $j=1,2,\dots,5$:

$$\begin{aligned}
&i\xi_1\big(g_1(\xi)+\alpha_2 g_5(\xi)\big)+i\xi_2\, g_3(\xi) = -\widehat{F_1}(\xi),\\
&i\xi_2\big(g_1(\xi)+\alpha_2 g_5(\xi)\big)-i\xi_1\, g_3(\xi) = -\widehat{F_2}(\xi),\\
&2\mu\,|\xi|^2\big(g_1(\xi)+\alpha_2 g_5(\xi)\big)+\mu(\alpha+1)|\xi|g_2(\xi)-\big(2\mu\alpha_1+\alpha(\beta_1+\beta_2)\big)|\xi|g_4(\xi)=\widehat{F_3}(\xi),\\
&-|\xi|g_4(\xi)+m_2 g_5(\xi)=\widehat{F_4}(\xi),\\
&-|\xi|g_4(\xi)-m_1 g_5(\xi)=\widehat{F_5}(\xi).
\end{aligned}$$

The solution of the system is given by the formulas:

$$\begin{aligned}
g_1(\xi) &= \frac{i\xi_1}{|\xi|^2}\widehat{F_1}(\xi)+\frac{i\xi_2}{|\xi|^2}\widehat{F_2}(\xi)-\frac{\alpha_2}{m_1+m_2}\widehat{F_4}(\xi)+\frac{\alpha_2}{m_1+m_2}\widehat{F_5}(\xi),\\
g_2(\xi) &= -\frac{2}{\alpha+1}\frac{i\xi_1}{|\xi|}\widehat{F_1}(\xi)-\frac{2}{\alpha+1}\frac{i\xi_2}{|\xi|}\widehat{F_2}(\xi)+\frac{1}{\mu(\alpha+1)\,|\xi|}\widehat{F_3}(\xi)\\
&\quad-\frac{m_1\,\gamma}{(m_1+m_2)\,|\xi|}\widehat{F_4}(\xi)-\frac{m_2\,\gamma}{(m_1+m_2)\,|\xi|}\widehat{F_5}(\xi),\\
g_3(\xi) &= \frac{i\xi_2}{|\xi|^2}\widehat{F_1}(\xi)-\frac{i\xi_1}{|\xi|^2}\widehat{F_2}(\xi),\\
g_4(\xi) &= -\frac{m_1}{(m_1+m_2)|\xi|}\widehat{F_4}(\xi)-\frac{m_2}{(m_1+m_2)|\xi|}\widehat{F_5}(\xi),\\
g_5(\xi) &= \frac{1}{m_1+m_2}\widehat{F_4}(\xi)-\frac{1}{m_1+m_2}\widehat{F_5}(\xi),\qquad \gamma=\frac{2\mu\alpha_1+\alpha(\beta_1+\beta_2)}{\mu(\alpha+1)}\,.
\end{aligned}$$

Substituting the functions $g_j(\xi)$, $j=1,2,\dots,5$, into formula (3.8) and applying the inverse Fourier transform, for the displacement vector, we obtain

$$u(x)=\frac{1}{2\pi}\int\limits_{\partial\Omega^-}B(x,y)F(y)dy_1dy_2,$$

where

$$B(x,y)=[B_{kj}(x,y)]_{3\times5}\,;\qquad F=(F_1,F_2,F_3,F_4,F_5)^\top;$$

$$\begin{aligned}
&B_{11}(x,y)=-\frac{\partial}{\partial x_3}\frac{1}{r}+\frac{2\alpha}{\alpha+1}\,x_3\frac{\partial^2}{\partial x_1^2}\frac{1}{r},\quad B_{21}(x,y)=\frac{2\alpha}{\alpha+1}\,x_3\frac{\partial^2}{\partial x_1\partial x_2}\frac{1}{r},\\
&B_{31}(x,y)=\frac{\alpha-1}{\alpha+1}\frac{\partial}{\partial x_1}\frac{1}{r}+\frac{2\alpha}{\alpha+1}\,x_3\frac{\partial^2}{\partial x_1\partial x_3}\frac{1}{r},\quad B_{12}(x,y)=\frac{2\alpha}{\alpha+1}\,x_3\frac{\partial^2}{\partial x_1\partial x_2}\frac{1}{r},\\
&B_{22}(x,y)=-\frac{\partial}{\partial x_3}\frac{1}{r}+\frac{2\alpha}{\alpha+1}\,x_3\frac{\partial^2}{\partial x_2^2}\frac{1}{r},\quad B_{32}(x,y)=\frac{\alpha-1}{\alpha+1}\frac{\partial}{\partial x_2}\frac{1}{r}+\frac{2\alpha}{\alpha+1}\,x_3\frac{\partial^2}{\partial x_2\partial x_3}\frac{1}{r},\\
&B_{13}(x,y)=\frac{\alpha}{\mu(\alpha+1)}\,x_3\frac{\partial}{\partial x_1}\frac{1}{r},\quad B_{23}(x,y)=\frac{\alpha}{\mu(\alpha+1)}\,x_3\frac{\partial}{\partial x_2}\frac{1}{r},\\
&B_{33}(x,y)=-\frac{1}{\mu(\alpha+1)}\frac{1}{r}+\frac{\alpha}{\mu(\alpha+1)}\,x_3\frac{\partial}{\partial x_3}\frac{1}{r},
\end{aligned}$$

$$B_{14}(x,y)=-\frac{\alpha_2}{m_1+m_2}\,\frac{\partial^2}{\partial x_1\partial x_3}\,\frac{e^{-\lambda_0\,r}-1}{r}-\frac{m_1(\alpha_1+\alpha\gamma)}{m_1+m_2}\,x_3\frac{\partial}{\partial x_1}\frac{1}{r},$$
$$B_{24}(x,y)=-\frac{\alpha_2}{m_1+m_2}\,\frac{\partial^2}{\partial x_2\partial x_3}\,\frac{e^{-\lambda_0\,r}-1}{r}-\frac{m_1(\alpha_1+\alpha\gamma)}{m_1+m_2}\,x_3\frac{\partial}{\partial x_2}\frac{1}{r},$$
$$B_{34}(x,y)=-\frac{\alpha_2}{m_1+m_2}\,\frac{\partial^2}{\partial x_3^2}\,\frac{e^{-\lambda_0\,r}-1}{r}-\frac{m_1(\alpha_1+\alpha\gamma)}{m_1+m_2}\,x_3\frac{\partial}{\partial x_3}\frac{1}{r}+\frac{m_1(\gamma-\alpha_1)}{m_1+m_2}\,\frac{1}{r},$$
$$B_{15}(x,y)=\frac{\alpha_2}{m_1+m_2}\,\frac{\partial^2}{\partial x_1\partial x_3}\,\frac{e^{-\lambda_0\,r}-1}{r}-\frac{m_2(\alpha_1+\alpha\gamma)}{m_1+m_2}\,x_3\frac{\partial}{\partial x_1}\frac{1}{r},$$
$$B_{25}(x,y)=\frac{\alpha_2}{m_1+m_2}\,\frac{\partial^2}{\partial x_2\partial x_3}\,\frac{e^{-\lambda_0\,r}-1}{r}-\frac{m_2(\alpha_1+\alpha\gamma)}{m_1+m_2}\,x_3\frac{\partial}{\partial x_2}\frac{1}{r},$$
$$B_{35}(x,y)=\frac{\alpha_2}{m_1+m_2}\,\frac{\partial^2}{\partial x_3^2}\,\frac{e^{-\lambda_0\,r}-1}{r}-\frac{m_2(\alpha_1+\alpha\gamma)}{m_1+m_2}\,x_3\frac{\partial}{\partial x_3}\frac{1}{r}+\frac{m_2(\gamma-\alpha_1)}{m_1+m_2}\,\frac{1}{r},$$
$$\lim_{\Omega^-\ni x\to z\in\partial\Omega^-}B_{jk}(x,y)=0,\quad j=1,2,\quad k=1,2,\dots,5.$$

For the pressure function, we get

$$\begin{pmatrix}p_1(x)\\p_2(x)\end{pmatrix}=\frac{1}{2\pi}\int\limits_{\partial\Omega^-}C(x,y)\,\begin{pmatrix}F_4(y)\\F_5(y)\end{pmatrix}dy_1dy_2,$$

where

$$C(x,y)=[C_{kj}(x,y)]_{2\times 2}\,,$$

$$C_{11}(x,y)=-\frac{\partial}{\partial x_3}\,\frac{1}{r}-\frac{m_2}{m_1+m_2}\frac{\partial}{\partial x_3}\,\frac{e^{\lambda_0\,r}-1}{r},\quad C_{12}(x,y)=\frac{m_2}{m_1+m_2}\frac{\partial}{\partial x_3}\,\frac{e^{\lambda_0\,r}-1}{r},$$
$$C_{21}(x,y)=\frac{m_1}{m_1+m_2}\frac{\partial}{\partial x_3}\,\frac{e^{\lambda_0\,r}-1}{r},\quad C_{22}(x,y)=-\frac{\partial}{\partial x_3}\,\frac{1}{r}-\frac{m_2}{m_1+m_2}\frac{\partial}{\partial x_3}\,\frac{e^{\lambda_0\,r}-1}{r}\,.$$

Note that

$$\lim_{\Omega^-\ni x\to z\in\partial\Omega^-}C(x,y)=0.$$

Using (3.10), for the stress vector, we have the following formula:

$$P(\partial,n)U(x)=\frac{1}{2\pi}\int\limits_{\partial\Omega^-}D(x,y)F(y)dy_1dy_2,$$

where

$$D(x,y)=[D_{kj}(x,y)]_{3\times 5}\,,\tag{4.1}$$

$$D_{11}(x,y)=-\,\mu\frac{\partial^2}{\partial x_3^2}\frac{1}{r}-\frac{\mu(1-3\alpha)}{\alpha+1}\,\frac{\partial^2}{\partial x_1^2}\frac{1}{r}+\frac{4\mu\alpha}{\alpha+1}\,x_3\frac{\partial^3}{\partial x_1^2\partial x_3}\frac{1}{r},$$
$$D_{21}(x,y)=-\,\frac{\mu(1-3\alpha)}{\alpha+1}\,\frac{\partial^2}{\partial x_1\partial x_2}\frac{1}{r}+\frac{4\mu\alpha}{\alpha+1}\,x_3\frac{\partial^3}{\partial x_1\partial x_2\partial x_3}\frac{1}{r},$$
$$D_{31}(x,y)=\frac{4\mu\alpha}{\alpha+1}\,\frac{\partial^3}{\partial x_1\partial x_3^2}\frac{1}{r},\quad D_{12}(x,y)=-\frac{\mu(1-3\alpha)}{\alpha+1}\,\frac{\partial^2}{\partial x_1\partial x_2}\frac{1}{r}+\frac{4\mu\alpha}{\alpha+1}\,x_3\frac{\partial^3}{\partial x_1\partial x_2\partial x_3}\frac{1}{r},$$
$$D_{22}(x,y)=-\,\mu\frac{\partial^2}{\partial x_3^2}\frac{1}{r}-\frac{\mu(1-3\alpha)}{\alpha+1}\,\frac{\partial^2}{\partial x_2^2}\frac{1}{r}+\frac{4\mu\alpha}{\alpha+1}\,x_3\frac{\partial^3}{\partial x_2^2\partial x_3}\frac{1}{r},$$
$$D_{32}(x,y)=\frac{4\mu\alpha}{\alpha+1}\,x_3\frac{\partial^3}{\partial x_2\partial x_3^2}\frac{1}{r},\quad D_{13}(x,y)=\frac{\alpha-1}{\alpha+1}\frac{\partial}{\partial x_1}\frac{1}{r}+\frac{2\alpha}{\alpha+1}\,x_3\frac{\partial^2}{\partial x_1\partial x_3}\frac{1}{r},$$
$$D_{23}(x,y)=\frac{\alpha-1}{\alpha+1}\frac{\partial}{\partial x_2}\frac{1}{r}+\frac{2\alpha}{\alpha+1}\,x_3\frac{\partial^2}{\partial x_2\partial x_3}\frac{1}{r},\quad D_{33}(x,y)=-\frac{\partial}{\partial x_3}\frac{1}{r}+\frac{2\alpha}{\alpha+1}\,x_3\frac{\partial^2}{\partial x_3^2}\frac{1}{r},$$
$$D_{14}(x,y)=-\,\frac{2\mu\alpha_2}{m_1+m_2}\,\frac{\partial^3}{\partial x_1\partial x_3^2}\frac{e^{-\lambda_0\,r}-1}{r}-\frac{2\mu m_1(\alpha_1+\alpha\gamma)}{m_1+m_2}\,x_3\frac{\partial^2}{\partial x_1\partial x_3}\frac{1}{r}$$

$$
\begin{aligned}
&\quad -\frac{\mu\, m_1(2\alpha_1+(\alpha-1)\gamma)}{m_1+m_2}\frac{\partial}{\partial x_1}\frac{1}{r}, \qquad D_{24}(x,y)=-\frac{2\mu\alpha_2}{m_1+m_2}\frac{\partial^3}{\partial x_2\partial x_3^2}\frac{e^{-\lambda_0 r}-1}{r}\\
&\quad -\frac{2\mu m_1(\alpha_1+\alpha\gamma)}{m_1+m_2}x_3\frac{\partial^2}{\partial x_2\partial x_3}\frac{1}{r}-\frac{\mu\, m_1(2\alpha_1+(\alpha-1)\gamma)}{m_1+m_2}\frac{\partial}{\partial x_2}\frac{1}{r},\\
&D_{34}(x,y)=-\frac{2\mu\alpha_2}{m_1+m_2}\frac{\partial^3}{\partial x_3^3}\frac{e^{-\lambda_0 r}-1}{r}+\frac{2\mu\alpha_2\lambda_0^2}{m_1+m_2}\frac{\partial}{\partial x_3}\frac{e^{-\lambda_0 r}}{r}-\frac{2\mu m_1(\alpha_1+\alpha\gamma)}{m_1+m_2}x_3\frac{\partial^2}{\partial x_3^2}\frac{1}{r},\\
&D_{15}(x,y)=\frac{2\mu\alpha_2}{m_1+m_2}\frac{\partial^3}{\partial x_1\partial x_3^2}\frac{e^{-\lambda_0 r}-1}{r}-\frac{2\mu m_2(\alpha_1+\alpha\gamma)}{m_1+m_2}x_3\frac{\partial^2}{\partial x_1\partial x_3}\frac{1}{r}\\
&\quad -\frac{\mu\, m_2(2\alpha_1+(\alpha-1)\gamma)}{m_1+m_2}\frac{\partial}{\partial x_1}\frac{1}{r}, \qquad D_{25}(x,y)=\frac{2\mu\alpha_2}{m_1+m_2}\frac{\partial^3}{\partial x_2\partial x_3^2}\frac{e^{-\lambda_0 r}-1}{r}\\
&\quad -\frac{2\mu m_2(\alpha_1+\alpha\gamma)}{m_1+m_2}x_3\frac{\partial^2}{\partial x_2\partial x_3}\frac{1}{r}-\frac{\mu\, m_2(2\alpha_1+(\alpha-1)\gamma)}{m_1+m_2}\frac{\partial}{\partial x_2}\frac{1}{r},\\
&D_{35}(x,y)=\frac{2\mu\alpha_2}{m_1+m_2}\frac{\partial^3}{\partial x_3^3}\frac{e^{-\lambda_0 r}-1}{r}-\frac{2\mu\alpha_2\lambda_0^2}{m_1+m_2}\frac{\partial}{\partial x_3}\frac{e^{-\lambda_0 r}}{r}-\frac{2\mu m_2(\alpha_1+\alpha\gamma)}{m_1+m_2}x_3\frac{\partial^2}{\partial x_3^2}\frac{1}{r}.
\end{aligned}
$$

It can be shown that the elements of the third row of matrix (4.1) satisfy the following conditions:

$$\lim_{\Omega^-\ni x\to z\in\partial\Omega^-} D_{3k}(x,y)=0, \qquad k=1,2,\dots,5.$$

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