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DIVERGENT FOURIER SERIES WITH RESPECT TO BIORTHONORMAL SYSTEMS IN FUNCTION SPACES NEAR L^1

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Abstract. In this paper, we generalize Bochkarev's theorem, which states that for any uniformly bounded biorthonormal system Φ, there exists a Lebesgue integrable function whose Fourier series in the system Φ diverges on a set of positive measure. We find the class of variable exponent Lebesgue spaces $L^{p(\cdot)}([0,1]^n)$, where $1 < p(x) < \infty$ almost everywhere on $[0,1]^n$, for which the aforementioned Bochkarev's theorem holds.

1. Introduction

After Kolmogorov [15,16] presented examples of functions in L^1 with almost everywhere and everywhere divergent trigonometric Fourier series, many authors have attempted to generalize these results by providing examples of functions with almost everywhere divergent trigonometric Fourier series from narrower Orlicz spaces. The most significant result in this direction is due to Konyagin [17], who achieved the same result for the space $L\varphi(L)$, provided that φ satisfies $\varphi(t) = o(\sqrt{\ln t/\ln\ln t})$.

Similar problems with respect to other orthonormal systems have been considered by various authors. One such problem was posed by Alexits (see [1, pp. 326], [2, pp. 287]) and Olevskii [24], concerning an analogue of Kolmogorov's example of a divergent trigonometric Fourier series for general uniformly bounded orthonormal systems.

The answer to this question was provided by Bochkarev [5]. He proved that for any given uniformly bounded orthonormal system, there exists a function in L^1 whose Fourier series with respect to this system diverges at every point of some set of positive measure. However, it turns out that a complete analogue of Kolmogorov's example for uniformly bounded orthonormal systems does not exist in general. This conclusion is based on Kazaryan's [13] construction of a complete orthonormal system that is uniformly bounded and for which every Fourier series converges on some set of positive measure.

Later, Bochkarev [6] extended his aforementioned result to uniformly bounded biorthonormal systems defined on a separable metric space with a Borel regular outer measure.

The authors of the paper [10] provide a different perspective on the problem of almost everywhere divergence of trigonometric Fourier series in the subspaces of L^1, specifically in terms of variable Lebesgue spaces $L^{p(\cdot)}$. They show that $L^1 = \bigcup L^{p(\cdot)}$, where the union is taken over all measurable functions $p(\cdot)$ such that $p(x) > 1$ almost everywhere. This implies that any function whose Fourier series diverges almost everywhere must belong to some variable exponent space $L^{p(\cdot)}$ with $1 < p(x) < \infty$ almost everywhere.

In [10], the authors construct a variable exponent space $L^{p(\cdot)}$, with $1 < p(x) < \infty$ almost everywhere, that shares with L^∞ the property that the space of continuous functions C is a closed linear subspace within it. Moreover, Kolmogorov's function, which has a Fourier series that diverges almost everywhere, belongs to the $L^{p'(\cdot)}$ space, where $p'(\cdot)$ is the Hölder conjugate of $p(\cdot)$. Additional results concerning the convergence of Fourier series for functions from these spaces can be found in [18] and [19]. Various other results related to these spaces are discussed in [7] and [9].

Later, in [20], the authors provided an analogue of Bochkarev's theorem for uniformly bounded orthonormal systems within a certain class of variable exponent Lebesgue spaces. They found the

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class of variable exponent Lebesgue spaces $L^{p(\cdot)}$, with $1 < p(x) < \infty$ almost everywhere, for which Bochkarev's theorem holds.

For readers who wish to gain a deeper understanding of these subjects, we kindly recommend exploring the following works related to the research topic [4, 25].

In the first of the aforementioned works, the almost everywhere divergence effect is established for any countable orthonormal system of characters of a compact group. In the second, the divergence effect is demonstrated everywhere for a wide class of character systems, including all Vilenkin systems.

Our plan for this paper is to characterize the class of variable exponent Lebesgue spaces for which an analogue of Bochkarev's theorem on the bounded biorthonormal systems is valid. To achieve this, we introduce some definitions and notations.

Definition 1.1. Let (X, S, μ) be a measurable space, where S is a σ-algebra of μ-measurable sets and $\mu(X) = 1$, and let $\{f_n, g_n\}$ be a biorthonormal system (see [14, Ch. VIII, §1]) such that $f_n, g_n \in L^\infty(X, \mu)$. For any function $F \in L^1(X, \mu)$, the system $\{f_n, g_n\}$ generates two Fourier series:

$$\sum_{n=1}^{\infty}(F, f_n)g_n(x) \quad \text{and} \quad \sum_{n=1}^{\infty}(F, g_n)f_n(x),$$

(these series are said to be conjugate).

Bochkarev [6] proved the following

Theorem 1.2. *Let X be a separable metric space with a Borel regular outer measure μ^* such that $\mu^*(X) = 1$. Then for any biorthonormal system $\{f_n, g_n\}$ satisfying the conditions*

$$\|f_n\|_\infty, \|g_n\|_\infty \le A, \ \ n \in \mathbb{N} \tag{1.1}$$

and

$$\lim_{n\to\infty} \int_E f_n(x)d\mu(x) = 0 \ \text{ and } \lim_{n\to\infty} \int_E g_n(x)d\mu(x) = 0 \tag{1.2}$$

for any measurable set $E \subset X$. Then there exist the functions $F_1, F_2 \in L^1(X, \mu)$ and a set $E \subset X$ such that

$$\mu(E) > 0$$

and for all $x \in E$, we have

$$\varlimsup_{N\to\infty} \sum_{n=1}^{N}((F_1, g_n)f_n(x) + (F_2, f_n)g_n(x)) = \infty.$$

This theorem is valid, in particular, for the space $\mathbb{R}^n$, $n \in \mathbb{N}$ and any finite Borel regular outer measure μ^*. Thus, the following assertion holds (see [6, Theorem 5]).

Theorem 1.3. *If μ is the classical Lebesgue measure on $\mathbb{R}^n$ and E is a measurable set with $\mu(E) < \infty$, then for any uniformly bounded biorthonormal system $\{f_n, g_n\}$ on E satisfying condition* (1.2)*, there exists a Fourier series divergent on a set of positive measure.*

In this paper, we are going to extend this Theorem 1.3 to the case of variable Lebesgue spaces.

Throughout the paper, let the symbol $|E|$ define the Lebesgue measure of the measurable set E. Also, $\Omega := [0; 1]^n$, and for a given $p(\cdot)$, the Hölder conjugate $p'(\cdot)$ is defined by $p'(x) := p(x)/(p(x)-1)$.

Let (X, S, μ) be a nontrivial measure space. Given a real-valued measurable function f on X, we define its decreasing permutation by

$$f^*(s) = \inf\{\alpha \ge 0 \ : \ \mu(\{x \in X : |f(x)| > \alpha\}) \le s\}, \quad s > 0.$$

Let $P_{\ln}$ be a set of all functions $p : X \to [1; \infty)$ such that

$$\limsup_{t\to 0+} \frac{(p')^*(t)}{\ln(e/t)} > 0. \tag{1.3}$$

Let $W(p)$ denote the set of all functions equimeasurable with $p(\cdot)$. Below, we will find the conditions on the function $p(\cdot)$ for which there exists $\bar{p}(\cdot) \in W(p)$ such that the space $C(\Omega)$ of continuous functions is a closed subspace in $L^{\bar{p}(\cdot)}(\Omega)$. In this paper, we generalize the result by Kopaliani, Samashvili and Zviadadze obtained in [20], from the one-variable case to the several-variables case. Moreover, we extend this generalization to uniformly bounded biorthonormal systems.

Let now state the result.

Theorem 1.4. *For any biorthonormal system $\Phi := \{f_n, g_n\}_{n\in\mathbb{N}}$ on Ω, satisfying conditions* (1.1) *and* (1.2)*, and for any $p(\cdot) \in P_{\ln}$, there exists a measure preserving transformation $\omega : \Omega \to \Omega$ such that in the corresponding $L^{p(\omega(\cdot))}(\Omega)$ space there exist the functions F_1, F_2 and a set $E \subset \Omega$ such that $|E| > 0$, and for all $x \in E$, we have*

$$\overline{\lim_{N\to\infty}} \sum_{n=1}^{N} ((F_1, g_n)f_n(x) + (F_2, f_n)g_n(x)) = \infty.$$

2. Definitions and Auxiliary Results

Let $\mathcal{M}$ denote the space of all equivalence classes of the Lebesgue measurable real-valued functions on Ω, equipped with the topology of convergence in measure, relative to each set of finite measure.

Definition 2.1. A Banach subspace X of $\mathcal{M}$ is referred to as a Banach function space (BFS) on Ω if the following conditions hold:

1) The norm $\|f\|_X$ is defined for every measurable function f, and $f \in X$ if and only if $\|f\|_X < \infty$. Also, $\|f\|_X = 0$ if and only if $f = 0$ almost everywhere.

2) $\||f|\|_X = \|f\|_X$ for all $f \in X$.

3) If $0 \le f \le g$ almost everywhere, then $\|f\|_X \le \|g\|_X$.

4) If $0 \le f_n \uparrow f$ almost everywhere, then $\|f_n\|_X \uparrow \|f\|_X$.

5) If E is a measurable subset of Ω with finite measure ($|E| < \infty$), then $\|\chi_E\|_X < \infty$, where χ_E is the characteristic function of E.

6) For every measurable set E with finite measure ($|E| < \infty$), there exists a constant $C_E < \infty$ such that $\int_E f(t)dt \le C_E\|f\|_X$.

Now, let us introduce various subspaces of a BFS X:

- A function f in X has an absolutely continuous norm in X if $\|f \cdot \chi_{E_n}\|_X \to 0$ whenever E_n is a sequence of measurable subsets of Ω such that $\chi_{E_n} \downarrow 0$ almost everywhere. The set of all such functions is denoted by X_A;
- X_B is the closure of the set of all bounded functions in X;
- A function $f \in X$ has a continuous norm in X if for every $x \in \Omega$, $\lim_{\varepsilon\to 0+} \|f\chi_{B(x,\varepsilon)}\|_X = 0$, where $B(x, \varepsilon)$ is a ball centered at x with radius ε. The set of all such functions is denoted by X_C.

The relationship between the concepts of X_A and X_B is given in [3]. Generally, the interaction between the subspaces X_A, X_B, and X_C may be intricate. For instance, there exists a BFS X in which $0 = X_A \subsetneq X_C = X$ (for example, see [23]).

Let $\mathcal{P}$ through the whole paper denote the family of all measurable functions $p : \Omega \to [1; +\infty)$. When $p(\cdot) \in \mathcal{P}$, we denote by $L^{p(\cdot)}(\Omega)$ the set of all measurable functions f on Ω such that for some $\lambda > 0$,

$$\int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

This set becomes a BFS when equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}.$$

The variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ and the corresponding variable exponent Sobolev spaces $W^{k,p(\cdot)}$ are of significant interest due to their applications in fluid dynamics, partial differential equations with non-standard growth conditions, calculus of variations, image processing, etc. (see [7,9] for more details).

For the specific case of a particular BFS $X = L^{p(\cdot)}(\Omega)$, the relationship between this space and its subspaces, namely, X_A, X_B, and X_C, has been explored in [12]. We will now present some of the key findings from that paper.

Proposition 2.2 (Edmunds, Lang, Nekvinda)**.** *Let $p(\cdot) \in \mathcal{P}$ and set $X = L^{p(\cdot)}(\Omega)$. Then*
(i) $X_A = X_C$*;*
(ii) $X_B = X$ *if and only if* $p(\cdot) \in L^\infty(\Omega)$*;*
(iii) $X_A = X_B$ *if and only if*

$$\int_0^1 c^{p^*(t)} dt < \infty, \quad \textit{for all} \;\; c > 1.$$

If ψ is an increasing convex function $\psi : [0; +\infty) \to [0; +\infty)$ such that $\psi(0) = 0$,

$$\lim_{x \to 0+} (\psi(x)/x) = 0, \quad \text{and} \quad \lim_{x \to +\infty} (\psi(x)/x) = +\infty,$$

then the Orlicz space L_ψ is defined as the set of all $f \in \mathcal{M}(\Omega)$ for which:

$$||f||_{L_\psi} = \inf \left\{ \lambda > 0 \; : \; \int_\Omega \psi\left(\frac{|f(t)|}{\lambda}\right) dt \leq 1 \right\} < +\infty.$$

Recall that a nonnegative function φ defined on $[0; +\infty)$ is called quasiconcave if it satisfies the following conditions: $\varphi(0) = 0$, $\varphi(t)$ is increasing, and $\varphi(t)/t$ is decreasing.

The Marcinkiewicz space M_φ is defined as the set of all $f \in \mathcal{M}(\Omega)$ for which:

$$||f||_{M_\varphi} = \sup_{0<t} \frac{1}{\varphi(t)} \int_0^t f^*(u) du < +\infty.$$

It is worth noting that $(M_\varphi)_A = (M_\varphi)_B$, and $(M_\varphi)_A$ can be characterized as the set of functions $f \in \mathcal{M}$ (see [22]) that satisfy:

$$\lim_{t \to 0+} \frac{1}{\varphi(t)} \int_0^t f^*(u) du = 0. \tag{2.1}$$

Additionally, when $\psi(t) = e^t - 1$ and $\varphi(t) = t \ln(e/t)$, the corresponding Orlicz and Marcinkiewicz spaces coincide (see [3]), and we denote them as e^L and $M_{\ln}$. Furthermore, it can be observed that (see [11, Corollary 3.4.28]):

$$||f||_{e^L} \asymp ||f||_{M_{\ln}} \asymp \sup_{0<t\leq 1} \frac{f^*(t)}{\ln(e/t)}. \tag{2.2}$$

The following result was initially established in [10] for the single-variable case, and our goal now is to extend it to the multi-variable scenario. Since the proof of this statement can be easily derived from the one provided in [10], we will omit it here.

Theorem 2.3. *Let X be a BFS on Ω. The space $C(\Omega)$ of continuous functions is a closed linear subspace of X if and only if there exists a positive constant c such that for every rectangle $I \subset \Omega$, we have*

$$c \leq ||\chi_I||_X.$$

Theorem 2.4. *For the existence of $\bar{p}(\cdot) \in W(p)$ for which $C(\Omega)$ is a closed subspace in $L^{\bar{p}(\cdot)}(\Omega)$, it is necessary and sufficient that*

$$\limsup_{t \to 0+} \frac{p^*(t)}{\ln(e/t)} > 0. \tag{2.3}$$

The forthcoming proof closely follows the framework presented in [21]. However, since we encounter some differences when extending the proof from the one-dimensional case to multiple dimensions, we have chosen to provide the complete proof for the sake of clarity.

Necessity. Since the space $C(\Omega)$ is closed in $L^{p(\cdot)}(\Omega)$, by Theorem 2.3, there exists positive constant d such that $d \leq ||\chi_I||_{p(\cdot)}$ for all rectangles I. This implies that $X_A \neq X_B$. Then by Proposition 2.2, there exists $c > 1$ such that

$$\int_0^1 c^{p^*(t)} dt = +\infty. \tag{2.4}$$

Consider two cases:

Case 1) $p^*(\cdot) \in e^L$. Since (2.4) holds, the function $p^*(\cdot)$ does not have absolute continuous norm, that is $p^*(\cdot) \in e^L \setminus \left(e^L\right)_A$. Then by (2.2), we get $p^*(\cdot) \in M_{\ln} \setminus (M_{\ln})_A$ and by (2.1), it is obvious that

$$\limsup_{t \to 0+} \frac{1}{t \ln(e/t)} \cdot \int_0^t p^*(u) du > 0,$$

finally, using ones more (2.2), from the last estimation, we get (2.3).

Case 2) $p^*(\cdot) \notin e^L$. Then by (2.2)

$$\sup_{0<t\leq 1} \frac{p^*(t)}{\ln(e/t)} = +\infty,$$

consequently, (2.3) holds. The necessity part of the theorem proved.

Sufficiency. Let (2.3) hold. For all $t \in [0;1]$, we define the function $h(t) = \min\{p^*(t), \ln(e/t)\}$. Obviously, in this case,

$$\limsup_{t \to 0+} \frac{h(t)}{\ln(e/t)} > 0,$$

then there exists a sequence $t_k \downarrow 0$ such that

$$\frac{h(t_k)}{\ln(e/t_k)} \geq d, \quad k \in \mathbb{N},$$

for some positive number d. Now, we choose a subsequence (t_{k_n}) such that $2t_{k_{n+1}} < t_{k_n}$, for all natural n. Since $t_k \downarrow 0$, we can always choose such a subsequence, so without loss of generality, we can assume that the sequence (t_k) is already such.

Let the given function f be defined by

$$f(t) = d \cdot \ln(e/t_k), \quad t \in (t_{k+1}; t_k], \quad k \in \mathbb{N} \quad \text{and} \quad f(t) = 1, \quad t \in (t_1; 1].$$

It is clear that $h(t) \geq f(t)$ for all $t \in [0;1]$. Now, choosing a positive number c such that $c > e^{1/d}$, we get

$$\int_0^1 c^{h(t)} dt = +\infty.$$

Indeed,

$$\int_0^1 c^{h(t)} dt \geq \int_0^1 c^{f(t)} dt > \int_{t_{k+1}}^{t_k} c^{d \cdot \ln(e/t_k)} dt$$

$$= (t_k - t_{k+1}) \cdot e^{d \cdot \ln c \cdot \ln(e/t_k)} > \frac{t_k}{2} \cdot \left(\frac{e}{t_k}\right)^{d \cdot \ln c} \to +\infty, \quad k \to +\infty.$$

Choose a decreasing sequence $\{a_k\}_{k \in \mathbb{N}}$ such that

$$\int_{a_{k+1}}^{a_k} c^{h(t)} dt = 1.$$

By (3.1), such a sequence always can be chosen. Now, let $\Delta_k = [a_{k+1}; a_k]$, and $\{r_k \,:\, k \in \mathbb{N}\}$ be a countable dense set in $[0;1]$. Define $b_k = -a_{k+1} + r_k$. Let now $A_k := \Delta_k + b_k = [r_k; r_k + a_k - a_{k+1}]$ and $g_k(t) = h(t) \cdot \chi_{\Delta_k}(t)$, $k \in \mathbb{N}$. Define the functions $p_k(t)$ by the induction:

$$p_1(t) = g_1(t - b_1)\chi_{[0;1]}(t),$$

$$p_k(t) = (p_{k-1}(t)(1 - \chi_{\Delta_k}(t - b_k)) + g_k(t - b_k)) \cdot \chi_{[0;1]}(t), \quad k > 1.$$

It is clear that $h(t)$ is decreasing and, therefore, $p_k(t) \leq p_{k+1}(t)$, for all $t \in [0;1]$ and all $k \in \mathbb{N}$. Also, for all $k \in \mathbb{N}$, we have

$$\int_0^1 p_k(t)dt \leq \int_0^1 h(t)dt \leq \int_0^1 \ln(e/t)dt = 2. \tag{2.5}$$

Define now the function $q(\cdot)$ by

$$q(t) = \lim_{k \to +\infty} p_k(t), \quad t \in [0;1].$$

By (2.5), we find that the function $q(\cdot)$ is almost everywhere finite. By the construction, it is clear that $q^*(t) \leq h(t) \leq p^*(t)$. Now, by the well-known result (see [3, Theorem 7.5]), there exists the measure-preserving transformation $\omega : [0;1] \to [0;1]$ such that $q(t) = q^*(\omega(t))$. Define now $\hat{p}(\cdot)$ by $\hat{p}(t) := p^*(\omega(t))$, $t \in [0;1]$. Since $q^*(t) \leq p^*(t)$, it is obvious that $q^*(\omega(t)) \leq p^*(\omega(t))$, then for all $t \in (0;1)$, we get the following inequality:

$$q(t) \leq \hat{p}(t). \tag{2.6}$$

Now, we construct an exponential function $\bar{p} : \Omega \to [1, \infty)$, for which the space of continuous functions will be a closed subspace inside the corresponding variable exponent Lebesgue space. To do this, we define a measure-preserving mapping $\rho : \Omega \to [0;1]$, by the following rule: Suppose that $x = (x_1, \ldots, x_n) \in \Omega$ and for every $i \in \{1, \ldots, n\}$ index, the representation of the corresponding coordinates follows as: $x_i = 0.a_{i1}a_{i2}a_{i3}\ldots$, then

$$\rho(x) = 0.a_{11}a_{21}\ldots a_{n1}a_{12}a_{22}\ldots a_{n2}\ldots .$$

This mapping, mentioned above, is well-known from the literature. Thus, we can define the function $\bar{p}(x) = \hat{p}(\rho(x))$. To complete the proof, we have to verify that the space of continuous functions is a closed subspace in the corresponding space $L^{\bar{p}(\cdot)}(\Omega)$. For this purpose, we show that there exists a positive number K such that for every rectangle $I \in \Omega$, we have $||\chi_I||_{\bar{p}} \geq K$. Consider any number $c > 1$, since, in view of the fact that the set of binary rational numbers is dense everywhere in the set of all real numbers, for this reason we can find an n-dimensional binary rectangle I^d for the given n-dimensional rectangle such that $I^d \subset I$. Then, by the properties of the function ρ, $c > 1$, and by (2.6), we get

$$\int_I c^{\bar{p}(x)}dx \geq \int_{I^d} c^{\bar{p}(x)}dx = \int_{(I^d)'} c^{\hat{p}(t)}dt \geq \int_{(I^d)'} c^{q(t)}dt,$$

where $(I^d)'$ denotes one-dimensional binary interval taken from $[0;1]$, for which $\rho(I^d) = (I^d)'$. By the construction of $q(\cdot)$, there exists a number k_0 such that $A_{k_0} \subset (I^d)'$. Then we get

$$\int_{(I^d)'} c^{q(t)}dt \geq \int_{A_{k_0}} c^{q(t)}dt \geq \int_{A_{k_0}} c^{p_{k_0}(t)}dt$$

$$= \int_{A_{k_0}} c^{g_{k_0}(t-d_{k_0})}dt = \int_{r_{k_0}}^{r_{k_0}+t_{k_0}-t_{k_0+1}} c^{h(t-d_{k_0})\cdot\chi_{\Delta_{k_0}}(t-d_{k_0})}dt$$

$$= \int_{t_{k_0+1}}^{t_{k_0}} c^{h(t)}dt \geq 1.$$

Now, by the definition of the norm in the variable Lebesgue space and by the above estimations, we get that for all n-dimensional rectangles $I \subset \Omega$, we have $||\chi_I||_{\bar{p}} > 1/c$. By Theorem 2.3, we get the proof of the sufficiency of Theorem 2.4.

3. Proof of Theorem 1.4

Before we proceed to constructing the functions whose Fourier series diverge over the sets of positive measure, we establish foundational principles. These principles will serve as the basis for the further construction of the corresponding functions. It is important to note that we will not explicitly prove the divergence of the Fourier series at this point. Instead, our focus will be on constructing these functions, followed by outlining a method for applying Bochkarev's theorem to establish their divergence.

Let us start with constructing the $p(\cdot)$. For all $t \in (0;1)$, we define the function $h(t) := \min\{(p')^*(t), \ln(e/t)\}$. By (1.3), it is obvious that

$$\limsup_{t\to 0+} \frac{h(t)}{\ln(e/t)} > 0.$$

Then there exists a sequence $t_k \downarrow 0$ such that

$$\frac{h(t_k)}{\ln(e/t_k)} \geq a, \quad k \in \mathbb{N},$$

for some positive number a.

It is obvious that we can choose a subsequence t_{k_n} such that $2t_{k_{n+1}} < t_{k_n}$. Let us choose a positive number c such that $c > e^{1/a}$, then we get

$$\int_0^1 c^{h(t)} dt > \int_{t_{k_{n+1}}}^{t_{k_n}} c^{a\cdot\ln(e/t_{k_n})} dt$$

$$= (t_{k_n} - t_{k_{n+1}}) \cdot e^{a\cdot\ln c\cdot\ln(e/t_{k_n})} > \frac{t_{k_n}}{2} \cdot \left(\frac{e}{t_{k_n}}\right)^{a\cdot\ln c} \to +\infty, \quad n \to +\infty. \tag{3.1}$$

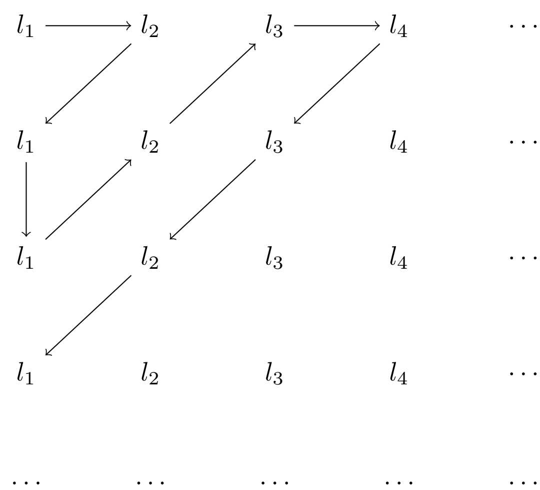
According to (3.1) and the fact that $t_k \downarrow 0$, we can choose the subsequence $(t_{k_{n_m}})$ from (t_{k_n}) such that

$$\int_{t_{k_{n_{m+1}}}}^{t_{k_{n_m}}} c^{h(t)} dt \geq 1, \quad m \in \mathbb{N}.$$

So, without loss of generality, we can assume that the sequence (t_k) is already such that

$$1 < a\ln(e/t_1), \quad 2t_{k+1} < t_k, \quad \int_{t_{k+1}}^{t_k} c^{h(t)} dt \geq 1, \quad k \in \mathbb{N}. \tag{3.2}$$

Let $\{l_k : k \in \mathbb{N}\}$ be a fixed dense set on $(0;1)$ (below we will choose l_k by using biorthonormal system Φ). Let r_k, $k \in \mathbb{N}$ be the following numeration of the table:



It is clear that for each l_k, there exists the sequence (r_{k_m}) such that $l_k = r_{k_m}$, $m \in \mathbb{N}$. Now, let $\Delta_k := [t_{k+1}; t_k]$, where t_k are the points possessing property (3.2). Define $d_k := -t_{k+1} + r_k$ and $E_k := \Delta_k + d_k = [r_k; r_k + t_k - t_{k+1}]$. Let $g_k(t) := h(t) \cdot \chi_{\Delta_k}(t)$, $k \in \mathbb{N}$. We introduce the functions $q_k(t)$ by the induction:

$$q_1(t) := g_1(t - b_1)\chi_{[0;1]}(t),$$
$$q_k(t) := [q_{k-1}(t)(1 - \chi_{\Delta_k}(t - d_k)) + g_k(t - d_k)] \cdot \chi_{[0;1]}(t), \quad k > 1.$$

It is clear that $h(t)$ is decreasing and therefore $q_k(t) \le q_{k+1}(t)$, for all $t \in [0;1]$ and all $k \in \mathbb{N}$. Also, for all $k \in \mathbb{N}$, we have

$$\int_0^1 q_k(t)dt \le \int_0^1 h(t)dt \le \int_0^1 \ln(e/t)dt = 2. \tag{3.3}$$

Define now a function

$$\hat{q}(t) = \lim_{k\to+\infty} q_k(t), \quad t \in [0;1].$$

It is clear that

$$\hat{q}(t) \ge q_k(t) \ge a \ln(e/t_k), \quad t \in E_k,\ k \in \mathbb{N}. \tag{3.4}$$

By (3.3), we get that the function $\hat{q}(\cdot)$ is a.e. finite. According to the construction, it is clear that $\hat{q}^*(t) \le h(t) \le (p')^*(t)$. It follows from the well-known result (see [3, Theorem 7.5]) that there exists a measure-preserving transformation $\zeta : [0;1] \to [0;1]$ such that $\hat{q}(t) = \hat{q}^*(\zeta(t))$. Now, define $\tilde{q}(\cdot)$ by $\tilde{q}(t) = (p')^*(\zeta(t))$. Since $\hat{q}^*(t) \le (p')^*(t)$, it is obvious that $\hat{q}^*(\zeta(t)) \le (p')^*(\zeta(t))$, then for all $t \in (0;1)$, we get the following inequality:

$$\hat{q}(t) \le \tilde{q}(t). \tag{3.5}$$

Now, as in the proof of (2.4), we can construct an exponential function $\bar{q} : \Omega \to [1, \infty)$, for which the space of continuous functions will be a closed subspace inside its corresponding variable exponent Lebesgue space. Then,we get

$$\int_I c^{\bar{q}(x)}dx \ge \int_{I^b} c^{\bar{q}(x)}dx = \int_{(I^b)'} c^{\tilde{q}(t)}dt \ge \int_{(I^b)'} c^{\hat{q}(t)}dt,$$

where $(I^b)'$ denotes one-dimensional dyadic interval taken from $[0;1]$, for which $\rho(I^b) = (I^b)'$. By the construction of $\hat{q}(\cdot)$, there exists the number k_0 such that $E_{k_0} \subset (I^b)'$. Then we get

$$\int\limits_{(I^d)'} c^{\hat{q}(t)}dt \geq \int\limits_{E_{k_0}} c^{\hat{q}(t)}dt \geq \int\limits_{E_{k_0}} c^{q_{k_0}(t)}dt$$

$$= \int\limits_{E_{k_0}} c^{g_{k_0}(t-d_{k_0})}dt = \int\limits_{r_{k_0}}^{r_{k_0}+t_{k_0}-t_{k_0+1}} c^{h(t-d_{k_0})\cdot\chi_{\Delta_{k_0}}(t-d_{k_0})}dt$$

$$= \int\limits_{t_{k_0+1}}^{t_{k_0}} c^{h(t)}dt \geq 1.$$

Now, by the definition of the norm in a variable Lebesgue space and by the above estimations, we get that for all n-dimensional rectangles $I \subset \Omega$, we have $||\chi_I||_{\bar{q}(\cdot)} > 1/c$. By the [8, Theorem 4], we find that the space of continuous functions is a closed subspace in $L^{\bar{q}}(\Omega)$.

Consider the function $\bar{p}(\cdot)$ which is the Hölder conjugate of $\bar{q}(\cdot)$. It is clear that $\bar{p}(\cdot)$ is equmeasurable to $p(\cdot)$. Since Ω is a finite nonatomic measure space and $\bar{p}$ is measurable, [3, Theorem 7.5] guarantees that there exists a measure-preserving transformation $\omega_1 : \Omega \to [0;1]$ such that $\bar{p}(x) = \bar{p}^*(\omega_1(x))$ for almost every $x \in \Omega$. Similarly, applying the theorem to p, there exists a measure-preserving transformation $\omega_2 : \Omega \to [0;1]$ such that $p(x) = p^*(\omega_2(x))$ for almost every $x \in \Omega$. Since any measure-preserving transformation between non-atomic finite measure spaces is a measure space isomorphism, ω_2 is also a bijective(up to null sets), therefore its inverse $\overset{-1}{\omega_2}$ is also measure-preserving. Now, if we define $\omega = \overset{-1}{\omega_2} \circ \omega_1$, we get that $\bar{p}(x) = p(\omega(x))$ for almost every $x \in \Omega$. Hence, there exists a measure-preserving transformation $\omega : \Omega \to \Omega$ such that $\bar{p}(x) = p(\omega(x))$.

Let $M_k := \rho^{-1}(E_k)$, $k \in \mathbb{N}$ and $C > \ln(e/t_1) \cdot (a \ln(e/t_1) - 1)^{-1}$, then by (3.4) and (3.5), it is obvious that

$$1 < \bar{p}(x) \leq 1 + \frac{C}{\ln(e/t_k)}, \quad x \in M_k.$$

Since ρ is measure-preserving $\rho(M_k) = E_k$ and $|E_k| \asymp t_k$, then by the last estimation, we obtain

$$||\chi_{M_k}||_{\bar{p}(\cdot)} \asymp |E_k| = t_k - t_{k+1} \asymp t_k. \tag{3.6}$$

Indeed, using [7, Theorem 2.45], we obtain

$$\frac{1}{2}||\chi_{M_k}||_1 \leq ||\chi_{M_k}||_{\bar{p}(\cdot)} \leq 2||\chi_{M_k}||_{1+\frac{C}{\ln(e/t_k)}} \asymp t_k.$$

Finally, by (3.6), we have

$$\left\| \sum_{k=1}^{\infty} a_k \chi_{M_k} \right\|_{\bar{p}(\cdot)} \asymp \left\| \sum_{k=1}^{\infty} a_k \chi_{M_k} \right\|_1. \tag{3.7}$$

Recall that for each k, there exists a sequence of natural numbers (k_m), $m \in \mathbb{N}$ such that $l_k = r_{k_m}$, $m \in \mathbb{N}$. Thus, we can rewrite (3.7) in the following form:

$$\left\| \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} a_{k_m} \chi_{M_{k_m}} \right\|_{\bar{p}(\cdot)} \asymp \left\| \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} a_{k_m} \chi_{M_{k_m}} \right\|_1. \tag{3.8}$$

Next, we should select $\{l_k \in \mathbb{N}\}$ set by using the system Φ in such a way that after definition corresponding functions F_1 and F_2 and using (3.8) we will obtain the proof of a theorem. Our construction of the functions F_1 and F_2 is analogous to the one from the work of Bochkarev [6].

For all $\theta \in \Omega$, consider a sequence of the binary cubes $Q_m(\theta) \subset \Omega$ such that $\theta \in Q_m(\theta)$, $\forall m \in \mathbb{N}$ and diameter tends to zero. By the Lebesgue differentiation theorem, for any $f \in L^1(\Omega)$, we have

$$\lim_{m\to\infty} \frac{1}{|Q(\theta)|} \int\limits_{Q(\theta)} f(x)dx = f(\theta),$$

for almost every $\theta \in \Omega$.

Let G^{2N} denote the set of points $\theta^{(2N)} = (\theta_1^{(2N)}, \dots, \theta_{2N}^{(2N)}) \in \Omega^{2N}$ for which

$$\lim_{m\to\infty} \frac{1}{|Q_m(\theta_{2i-1}^{(2N)})|} \int\limits_{Q_m(\theta_{2i-1}^{(2N)})} g_n(x)dx = g_n(\theta_{2i-1}^{(2N)})$$

and

$$\lim_{m\to\infty} \frac{1}{|Q_m(\theta_{2i}^{(2N)})|} \int\limits_{Q_m(\theta_{2i}^{(2N)})} f_n(x)dx = f_n(\theta_{2i}^{(2N)}),$$

for all $n \in \mathbb{N}$ and $i \in \{1, \dots, N\}$. It is clear that $|G^{2N}| = 1$. Consider the following set $\Theta' := \{\theta_i^{(2N)} : i \in \{1, \dots, 2N\}, N \in \mathbb{N}\}$. If this set is not dense in Ω, we examine a countable set Θ such that $\Theta' \subset \Theta$ and Θ is dense in Ω. Let l_k, $k \in \mathbb{N}$ be some numeration of Θ.

For all fixed $N \in \mathbb{N}$ and fixed $\theta_i^{(2N)}$, $i \in \{1, \dots, 2N\}$, there exists the sequence $r_{i_k}^{(2N)}$, $k \in \mathbb{N}$ such that $\theta_i^{(2N)} = r_{i_k}^{(2N)}$, $k \in \mathbb{N}$. Then by $E_i^{(2N)}$ we define the binary interval such that $\rho(\theta_i^{(2N)}) \in E_i^{(2N)}$ and $|E_i^{(2N)}| \le t_{i_1} - t_{i_1+1}$ and let $M_i^{(2N)} := \rho^{-1}(E_i^{(2N)})$.

For a sequence $\{N_n\}$ of positive integers and a decreasing sequence $\{\varepsilon_n\}$ of positive numbers (which we will specify below), we set

$$F_1(x) = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{N_n} \sum_{i=1}^{N_n} \frac{\chi_{M_{2i-1}^{(2N)}}(x)}{|M_{2i-1}^{(2N)}|}, \quad F_2(x) = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{N_n} \sum_{i=1}^{N_n} \frac{\chi_{M_{2i}^{(2N)}}(x)}{|M_{2i}^{(2N)}|}.$$

Now, for the proof of the existence of a set $E \subset \Omega$ such that $|E| > 0$ and for all $x \in E$, we have

$$\overline{\lim_{N\to\infty}} \sum_{n=1}^{N} ((F_1, g_n) f_n(x) + (F_2, f_n) g_n(x)) = \infty.$$

We simply need to replicate Bochkarev's proof step by step without making any changes. Consequently, the detailed proof will not be provided in this context.

Finally, by the definitions of the functions F_1, F_2, and (3.8), we have $F_1, F_2 \in L^{\bar{p}(\cdot)}(\Omega)$. As mentioned earlier, $\bar{p}$ is equimeasurable to p, implying the existence of a measure-preserving transformation $\omega : \Omega \to \Omega$ such that $\bar{p}(x) = p(\omega(x))$. Therefore, we conclude that $F_1, F_2 \in L^{p(\omega(\cdot))}(\Omega)$. This completes the proof.

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