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ON THE DEFORMATION OF CHIRAL POROUS CYLINDERS IN A STRAIN GRADIENT THERMOELASTIC THEORY

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Abstract. The paper presents the first form of the strain gradient theory of elasticity proposed by Mindlin and Eshel [21]. The theory was generalized by Papanicolopulos [25] to include non-centrosymmetric behaviours and make them suitable to investigate the problems related to size effects and nanotechnology. We study the thermoelastic deformation of a chiral porous cylinder subjected to a thermal field that is polynomial in the axial coordinate where, the coefficients of the polynomial are the functions of the two remaining coordinates. The problem is investigated by the method of induction and, as in the classical elasticity, is decomposed in terms of some generalized plane strain problems. Using the results established in [9], we obtain a closed-form solution of the starting problem $P^{(0)}$ of the inductive process, i.e., the deformation of a cylinder subjected to a thermal field independent of the axial coordinate. Then, we present a method for constructing the solution of the problem $P^{(n+1)}$ when the solution of the problem $P^{(n)}$ is known.

1. Introduction

This paper is aimed to extend the results established in an earlier study [9] on the deformation of chiral porous beams in the strain gradient thermoelasticity. In the previous paper, the cylinder was supposed to be loaded by forces acting on its bases and subjected to a thermal field, linear in the axial coordinate. The solution was obtained in closed form and, as in the classical thermoelasticity, was represented in terms of solutions of some plane strain problems. In the present paper we consider a cylinder free from mechanical loads and subjected to a temperature distribution that is polynomial in the axial coordinate, where the coefficients of the polynomial are the functions of the two remaining coordinates. The solution is obtained by the induction method. It is shown that, if we denote by $P^{(n)}$ the problem corresponding to the temperature $T_n = T_n(x_1, x_2)x_3^n$, then the solution of the problem $P^{(n+1)}$ depends on the solution of the problem $P^{(n)}$.

The elastic theory of chiral porous solids is a topic of growing theoretical and practical interest. The theory provides an efficient tool for modelling non-traditional media such as bones and other biological substances, crystalline solids and geomaterials. The theory has been also proved suitable for studying the behaviour of nanomaterials. For a long time nanoparticles have been pictured as spherical objects, but at atomic and even larger scale, they have complex features that display mirror asymmetry. Moreover nanotubes, nanospheres and other nanoparticles are porous due to the presence of their internal cavities. Chirality and porosity are intrinsic characteristic of nanomaterials and their effects on the behaviour of structural elements such as shells, plates and beams cannot be ignored. An accurate description of the unique physical and mechanical properties of nanomaterials is provided by Wu et al.in [35]. For a brief historical sketch on nanomaterials and technology see Guz and Dushchitskii [12]. For an overview of chirality in mechanics see Lakes [19, 20].

There are a number of theories which have been considered to introduce chirality in the mechanical behaviour of materials [18, 23, 24, 33]. In [25], Papanicolopulos generalized the Mindlin–Eshel strain gradient theory for centrosymmetric materials to the case of non-centrosymmetric solids. In his approach, the chiral behaviour is controlled by a single material parameter and the sign of parameter allows to distinguish between the right and left chirality. This theory has received widespread attention and many studies on theoretical developments and applications have been published [2, 11, 15, 17, 22].

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Following the model of materials with voids proposed by Cowin and Nunziato [3], the effects of porosity are described by means of a scalar function taken as an independent kinematic variable. Since his formulation, the theory of materials with voids attracted much interest and has been the object of intensive investigations (see e.g., [5, 7, 26, 27, 31, 34]). The basic results and a review on the theory of materials with voids can be found in the book by Iesan [14]. Recently, the theory has been generalized to the case of materials with multi-porosity structure (see e.g., [4, 8, 28, 30, 32]). For extensive references and an overview on the theories of multi-porosity materials, see Svanadze [29].

The strain gradient theory is constructed by adding the second-order partial derivative of the components of displacement and the first-order partial derivative of the microdilatation function in the classical set of independent constitutive variables [1, 16]. The paper is structured as follows. In Section 2, we present the basic equations of chiral porous elastic solids and formulate the problem of thermoelastic deformation of a right cylinder. In Section 3, we investigate the equilibrium problem of a cylinder free from mechanical loads and subjected to a temperature field independent of the axial coordinate. The solution is expressed in terms of plane strain problems. In Section 4, we establish the solution of the problem corresponding to a temperature distribution that is polynomial in the axial coordinate. We present a method of constructing the solution to the problem $P^{(n+1)}$ when the solution of problem $P^{(n)}$ is known. The solutions of chiral (non-porous) cylinders and porous (achiral) cylinders can be derived as special cases.

2. Basic Equations

In this section, we formulate the equilibrium problem of a porous chiral cylinder subjected to a temperature change polynomial of degree n in the axial coordinate. The cylinder is supposed to be homogeneous and isotropic. We denote by Π the lateral boundary, $\Sigma_\alpha(\alpha = 1, 2)$ is the terminal cross-sections, Σ is a generic cross-section, Γ_α is the boundary of Σ_α, Γ is the boundary of Σ,and h is the lenght of the cylinders. We choose a system of rectangular axes such that the $x_3 - axis$ is parallel to the cylinder generator and the $x_1Ox_2 - plane$ contains the basis Σ_1 at $x_3 = 0$. In what follows, we assume that the thermal field has the form

$$T = \sum_{k=1}^{n} T_k(x_1, x_2)x_3^k, \tag{2.1}$$

where the functions T_k are prescribed. The cylinder is supposed to be free of mechanical loads. Let u_i be the components of the displacement vector and φ the microdilatation function. The problem of equilibrium, of the cylinder consists in finding the functions u_i and φ satisfying the following systems of equations:

- *geometrical equations*

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \kappa_{ijk} = u_{k,ij}, \tag{2.2}$$

where e_{ij} is the strain tensor and κ_{ijk} is the strain gradient tensor.

- *Constitutive equations*

$$\begin{aligned} \tau_{ij} &= \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + d\varphi\delta_{ij} + f(\varepsilon_{ikm}\kappa_{jkm} + \varepsilon_{jkm}\kappa_{ikm}) - bT\delta_{ij}, \\ \mu_{ijk} &= \frac{1}{2}\alpha_1(\kappa_{rri}\delta_{jk} + 2\kappa_{krr}\delta_{ij} + \kappa_{rrj}\delta_{ik}) + \alpha_2(\kappa_{irr}\delta_{jk} + \kappa_{jrr}\delta_{ik}) \\ &\quad + 2\alpha_3\kappa_{rrk}\delta_{ij} + \beta_1\delta_{ij}\varphi_{,k} + \beta_2(\delta_{ik}\varphi_{,j} + \delta_{jk}\varphi_{,i}) \\ &\quad + 2\alpha_4\kappa_{ijk} + \alpha_5(\kappa_{kji} + \kappa_{kij}) + f(\varepsilon_{iks}e_{js} + \varepsilon_{jks}e_{is}), \\ \sigma_i &= \beta_1\kappa_{rri} + 2\beta_2\kappa_{irr} + a_0\varphi_{,i}, \quad g = de_{rr} + \xi\varphi - \beta T, \end{aligned} \tag{2.3}$$

where τ_{ij} is the stress tensor, μ_{ijk} is the dipolar stress tensor, σ_i is the equilibrated stress vector, g is the intrinsic body force, T is the temperature, δ_{ij} is the Kronecker delta, ε_{ijk} is the alternating symbol, λ, μ and b are the constitutive constants of the classical theory of elasticity; $\alpha_i(i = 1, 2, \ldots, 5)$ and $\beta_j(j = 1, 2)$ are the constitutive constants associated with the gradient terms; d, a_0, ξ and β are the constitutive constants linked to porosity, and f is a constant associated with the chiral behaviour.

- *Equilibrium equations*

$$\tau_{ji,j} - \mu_{kji,kj} = 0, \quad \sigma_{j,j} - g = 0. \tag{2.4}$$

In the equilibrium problems, the boundary conditions for a body B with boundary ∂B are given by [10, 21]

$$P_i = \widetilde{P}_i, \;\; R_i = \widetilde{R}_i, \;\; \sigma_i n_i = \widetilde{\sigma} \;\text{ on }\; \partial B \backslash C, Q_i = \widetilde{Q}_i \;\text{ on }\; C, \tag{2.5}$$

where $\widetilde{P}_i, \widetilde{R}_i, \widetilde{\sigma}$ and $\widetilde{Q}_i$ are the prescribed functions, C is the union of the edges, and n_j are the components of the outward unit normal of ∂B and

$$\begin{aligned} &P_i = (\tau_{ki} - \mu_{ski,s})n_k - D_j(n_r \mu_{rji}) + n_s n_p \mu_{spi}(D_k n_k), \\ &R_i = \mu_{rsi} n_r n_s, \;\; Q_i =< \mu_{pji} n_p n_q > \varepsilon_{jrq} s_r \;\text{ on }\; \partial B. \end{aligned} \tag{2.6}$$

Here, D_i are the components of the surface gradient, $D_i = (\delta_{ik} - n_i n_k)\partial/\partial x_k, s_i$ are the components of the unit vector, tangent to C, and $< g >$ denotes the difference of limits of g from both sides of C.

In the case of cylinder free of loading on its boundary, we have [9]

$$P_i = 0, \;\; R_i = 0, \;\; Q_i = 0, \;\; \sigma_\alpha n_\alpha = 0, \;\; \alpha = 1, 2 \;\text{ on }\; \Pi \tag{2.7}$$

and

$$\begin{aligned} &P_i = -\tau_{3i} + 2\mu_{\alpha 3i,\alpha} + \mu_{33i,3}, \quad R_i = \mu_{33i} \;\text{ on } \Sigma_1, \\ &Q_i = -2\mu_{\alpha 3i} n_\alpha, \;\text{ on }\; \Gamma_1. \end{aligned} \tag{2.8}$$

Further, we have to satisfy on the plane end Σ_1 the following conditions:

$$\int_{\Sigma_1} P_\alpha da + \int_{\Gamma_1} Q_\alpha ds = 0, \tag{2.9}$$

$$\int_{\Sigma_1} P_3 da + \int_{\Gamma_1} Q_3 ds = 0, \tag{2.10}$$

$$\int_{\Sigma_1} (x_\alpha P_3 + R_\alpha) da + \int_{\Gamma_1} x_\alpha Q_3 ds = 0, \tag{2.11}$$

$$\int_{\Sigma_1} \varepsilon_{\alpha\beta 3} x_\alpha P_\beta da + \int_{\Gamma_1} \varepsilon_{a\beta 3} x_\alpha Q_\beta ds = 0. \tag{2.12}$$

In the next sections, we seek a solution of the problem defined by equations (2.2), (2.3), (2.4) and the boundary conditions (2.7), (2.9)–(2.12) when temperature T is assigned.

3. Problem $P^{(0)}$

In what follows, we use the method of induction to solve the problem formulated in Section 2. Let us denote by $P^{(k)}$ the thermoelastic problem corresponding to the case where the temperature has the form

$$T = T_k(x_1, x_2) x_3^k,$$

where T_k is a given function and k is a positive integer. We have to find a solution of problem $P^{(k+1)}$ once a solution of the problem $P^{(k)}$ is known. The solution of problem $P^{(0)}$ corresponding to the thermal field

$$T = T_0(x_1, x_2), \tag{3.1}$$

is the starting point of the inductive process. Problem $P^{(0)}$ with different boundary conditions has been investigated in [9]. We look for a solution of the problem in the form

$$u_\alpha = -\frac{1}{2} a_\alpha x_3^2 + \varepsilon_{3\beta\alpha} a_4 x_\beta x_3 + \sum_{k=1}^{4} a_k u_\alpha^{(k)} + w_\alpha(x_1, x_2),$$

$$u_3 = (a_1 x_1 + a_2 x_2 + a_3) x_3 + \sum_{k=1}^{4} a_k u_3^{(k)} + w_3(x_1, x_2),$$

$$\varphi=\sum_{k=1}^{4}a_k\varphi^{(k)}+\psi,$$

where $u_j^{(k)},\varphi^{(k)},w_j$ and ψ are unknown functions which are independent of x_3, and a_k are unknown constants. We denote by $\tau_{\alpha j}^{(k)}$, $\mu_{\alpha\beta i}^{(k)}$, $\sigma_\alpha^{(k)}$, and $g^{(k)}$, $(k=1,2,3,4)$ the stress tensor, the dipolar stress tensor, the microstretch stress vector and the intrinsic body force associated with $u_i^{(k)}$ and $\varphi^{(k)}$, respectively. Moreover, we denote by $t_{\alpha j}$, $m_{\alpha\beta i}$, π_α and γ the stress tensor, the dipolar stress tensor, the microstretch stress vector and the intrinsic body force corresponding to w_i and ψ, respectively. The equilibrium equation (2.4) takes the form

$$\begin{gathered}\tau_{\alpha j,\alpha}^{(k)}-\mu_{\alpha\beta j,\alpha\beta}^{(k)}+\mathscr{F}_j^{(k)}=0,\ \ \sigma_{\alpha,\alpha}^{(k)}-g^{(\kappa)}+l^{(k)}=0\ \text{ on }\ \Sigma_1,\\ t_{\alpha i,\alpha}-m_{\alpha\beta i,\alpha}=0,\ \ \pi_{\alpha,\alpha}-\gamma=0.\end{gathered}\tag{3.2}$$

where

$$\mathscr{F}_j^{(\rho)}=\lambda\delta_{j\rho},\ \ \mathscr{F}_j^{(3)}=0,\ \ \mathscr{F}_j^{(4)}=0,\ \ l^{(\rho)}=-dx_\rho,\ \ l^{(3)}=-d,\ l^{(4)}=0.\tag{3.3}$$

The boundary conditions (2.5) become

$$P_i^{(k)}=\widetilde{P}_i^{(k)},\ \ R_i^{(k)}=\widetilde{R}_i^{(k)},\ \ \sigma_\alpha^{(k)}n_\alpha=\widetilde{\sigma}^{(k)}\ \text{ on }\ \Gamma_1,\tag{3.4}$$

where

$$\begin{aligned}P_i^{(k)}&=(\tau_{\beta i}^{(k)}-\mu_{\rho\beta i,\rho}^{(k)})n_\beta-D_\rho(n_\beta\mu_{\beta\rho i}^{(k)})+n_\beta n_\alpha\mu_{\beta\alpha i}^{(k)}(D_\rho n_\rho),\\ R_i^{(k)}&=\mu_{\rho\alpha i}^{(k)}n_\rho n_\alpha,\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}\widetilde{P}_1^{(1)}&=-\lambda x_1n_1+(\alpha_1-2\alpha_2)\varepsilon_{3\alpha\nu}(n_1n_2)_{,\nu}n_\alpha,\\ \widetilde{P}_2^{(1)}&=-\lambda x_1n_2+\frac{1}{2}(\alpha_1-2\alpha_2)\varepsilon_{3\alpha\rho}(n_1^2-n_2^2)_{,\alpha}n_\rho,\ \ \widetilde{P}_3^{(1)}=2fn_2,\\ \widetilde{R}_1^{(1)}&=2\alpha_3-\alpha_1+(\alpha_1-2\alpha_2)n_1^2,\ \ \widetilde{R}_2^{(1)}=(\alpha_1-2\alpha_2)n_1n_2,\\ \widetilde{R}_3^{(1)}&=0,\ \ \widetilde{\sigma}^{(1)}=(\beta_1-2\beta_2)n_1,\\ \widetilde{P}_1^{(2)}&=-\lambda x_2n_1+\frac{1}{2}(\alpha_1-2\alpha_2)\varepsilon_{3\alpha\nu}(n_1^2-n_2^2)_{,\alpha}n_\nu,\\ \widetilde{P}_2^{(2)}&=-\lambda x_2n_2+(\alpha_1-2\alpha_2)\varepsilon_{3\alpha\nu}(n_1n_2)_{,\nu}n_\alpha,\ \ \widetilde{P}_3^{(2)}=-2fn_1,\\ \widetilde{R}_1^{(2)}&=(\alpha_1-2\alpha_2)n_1n_2,\ \ \widetilde{R}_2^{(2)}=2\alpha_3-\alpha_1+(\alpha_1-2\alpha_2)n_2^2,\\ \widetilde{R}_3^{(2)}&=0,\ \ \widetilde{\sigma}^{(2)}=(\beta_1-2\beta_2)n_2,\\ \widetilde{P}_\alpha^{(3)}&=-\lambda n_\alpha,\ \ \widetilde{P}_3^{(3)}=0,\ \ \widetilde{R}_j^{(3)}=0,\ \ \widetilde{\sigma}^{(3)}=0,\\ \widetilde{P}_1^{(4)}&=\frac{1}{2}f[5n_1+D_1(x_2n_2)+D_2(x_2n_1-2x_1n_2)-2(x_2n_1n_2-x_1n_2^2)(D_\rho n_\rho)],\\ \widetilde{P}_2^{(4)}&=\frac{1}{2}f[5n_2+D_1(x_1n_2-2x_2n_1)+D_2(x_1n_1)-2(x_1n_1n_2-x_2n_1^2)(D_\rho n_\rho)],\\ \widetilde{P}_3^{(4)}&=\mu\varepsilon_{3\beta\nu}x_\nu n_\beta,\ \ \widetilde{R}_1^{(4)}=f(x_1n_2^2-x_2n_1n_2),\\ \widetilde{R}_2^{(4)}&=f(x_2n_1^2-x_1n_1n_2),\ \ \widetilde{R}_3^{(4)}=0,\ \ \widetilde{\sigma}^{(4)}=0.\end{aligned}\tag{3.6}$$

The functions $t_{\alpha j}$, $m_{\alpha\beta i}$ and π_i must satisfy the boundary conditions

$$P_i^*=0,\ \ R_i^*=0,\ \ \pi_\alpha n_\alpha=0\ \text{ on }\ \Gamma_1.$$

where

$$\begin{aligned}P_i^*&=(t_{\beta i}-m_{\rho\beta i,\rho})n_\beta-D_\rho(n_\beta m_{\beta\rho i})+n_\beta n_\alpha m_{\beta\alpha i}(D_\rho n_\rho),\\ R_i^*&=m_{\rho\alpha i}n_\rho n_\alpha,\ \text{ on }\ \Gamma_1.\end{aligned}$$

The constants a_k are determined by the boundary conditions (2.9)–(2.12) on the end Σ_1. Condition (2.9) is identically satisfied and conditions (2.10)–(2.12) reduce to the following system:

$$\sum_{k=1}^{4} D_{3k}a_k = -F_3^*,$$
$$\sum_{k=1}^{4} D_{\alpha k}a_k = \varepsilon_{3\alpha\beta}M_\beta^*,$$
$$\sum_{k=1}^{4} D_{4k}a_k = -M_3^*,$$

where

$$\begin{aligned}
&D_{\alpha k} = \int_{\Sigma_1}(x_\alpha S_{33}^{(k)} + 2N_{\alpha 33}^{(k)} - N_{33\alpha}^{(k)})da,\\
&D_{3k} = \int_{\Sigma_1} S_{33}^{(k)}da, D_{4k} = \int_{\Sigma_1}\varepsilon_{3\alpha\beta}(x_\alpha S_{3\beta}^{(k)} + 2N_{\alpha 3\beta}^{(k)})da,\\
&F_3^* = \int_{\Sigma_1} t_{33}da, M_\alpha^* = \varepsilon_{3\alpha\beta}\int_{\Sigma_1}(x_\beta t_{33} + 2m_{\beta 33} - m_{33\beta})da,\\
&M_3^* = \int_{\Sigma_1}\varepsilon_{3\alpha\beta}(x_\alpha t_{3\beta} + 2m_{\alpha 3\beta})da.
\end{aligned} \tag{3.7}$$

In (3.7), we have used the notations

$$\begin{aligned}
&S_{33}^{(\rho)} = (\lambda+2\mu)x_\rho + \tau_{33}^{(\rho)},\ \ S_{33}^{(3)} = \lambda + 2\mu + \tau_{33}^{(3)},\\
&S_{33}^{(4)} = 4f + \tau_{33}^{(4)},\ \ S_{3\alpha}^{(\rho)} = 2f\varepsilon_{\alpha\rho 3} + \tau_{\alpha 3}^{(\rho)},\\
&S_{3\alpha}^{(3)} = \tau_{\alpha 3}^{(3)},\ \ S_{3\alpha}^{(4)} = \mu\varepsilon_{3\beta\alpha}x_\beta + \tau_{\alpha 3}^{(4)},\\
&N_{\alpha 33}^{(i)} = \frac{1}{2}(2\alpha_2 - \alpha_1 + 4\alpha_4)\delta_{i\alpha} + \mu_{\alpha 33}^{(i)},\ \ (i=1,2,3),\\
&N_{\alpha 33}^{(4)} = -\frac{1}{2}fx_\alpha + \mu_{\alpha 33}^{(4)},\ \ N_{33\alpha}^{(i)} = (\alpha_1 - 2\alpha_3 - 2\alpha_4 + \alpha_5)\delta_{i\alpha} + \mu_{33\alpha}^{(i)},\\
&N_{33\alpha}^{(4)} = fx_\alpha + \mu_{33\alpha}^{(4)},\ \ N_{\alpha 3\beta}^{(\rho)} = \varepsilon_{3\alpha\beta}fx_\rho + \mu_{\alpha 3\beta}^{(\rho)},\\
&N_{\alpha 3\beta}^{(3)} = \varepsilon_{3\alpha\beta}f + \mu_{\alpha 3\beta}^{(3)},\ \ N_{\alpha 3\beta}^{(4)} = \varepsilon_{3\alpha\beta}(2\alpha_4 - \alpha_5) + \mu_{\alpha 3\beta}^{(4)}
\end{aligned}$$

and

$$\begin{aligned}
&\tau_{33}^{(k)} = \lambda e_{\rho\rho}^{(k)} + d\varphi^{(k)},\ \ \mu_{3\alpha\beta}^{(k)} = \frac{1}{2}\alpha_1\kappa_{\rho\rho 3}^{(k)}\delta_{\alpha\beta} + \alpha_5\kappa_{\beta\alpha 3}^{(k)} + f\varepsilon_{\beta\rho 3}e_{\alpha\rho}^{(k)},\\
&\mu_{3\alpha 3}^{(k)} = \frac{1}{2}\alpha_1\kappa_{\rho\rho\alpha}^{(k)} + \alpha_2\kappa_{\alpha\rho\rho}^{(k)} + f\varepsilon_{\rho\alpha 3}e_{3\rho}^{(k)} + \beta_2\varphi_{,\alpha}^{(k)},\\
&\mu_{33\alpha}^{(k)} = \alpha_1\kappa_{\alpha\rho\rho}^{(k)} + 2\alpha_3\kappa_{\rho\rho\alpha}^{(k)} + \beta_1\varphi_{,\alpha}^{(k)} + 2f\varepsilon_{3\alpha\rho}e_{3\rho}^{(k)},\\
&\mu_{333}^{(k)} = (\alpha_1 + 2\alpha_3)\kappa_{\rho\rho 3}^{(k)},\ \sigma_3^{(k)} = \beta_1\kappa_{\rho\rho 3}^{(k)},\\
&t_{33} = \lambda\eta_{\rho\rho} + d\psi - bT_0,\\
&m_{3\alpha\beta} = \frac{1}{2}\alpha_1\xi_{\rho\rho 3}\delta_{\alpha\beta} + \alpha_5\xi_{\beta\alpha 3} + f\xi_{\beta\rho 3}\eta_{\alpha\rho},\\
&m_{3\alpha 3} = \frac{1}{2}\alpha_1\xi_{\rho\rho\alpha} + \alpha_2\xi_{\alpha\rho\rho} + f\xi_{\rho\alpha 3}\eta_{3\rho} + \beta_2\psi_{,\alpha},\\
&m_{33\alpha} = \alpha_1\xi_{\alpha\rho\rho} + 2\alpha_3\xi_{\rho\rho\alpha} + \beta_1\psi_{,\alpha} + 2f\varepsilon_{3\alpha\rho}\eta_{3\rho},\\
&m_{333} = (\alpha_1 + 2\alpha_3)\xi_{\rho\rho 3}\xi_{\rho\rho 3}, \pi_3 = \beta_1\xi_{\rho\rho 3}.
\end{aligned} \tag{3.8}$$

By the solution of problem $P^{(0)}$ we conclude that the thermal field (3.1) produces axial extension, bending and torsion. The results of this Section were applied to study the deformation of a chiral porous circular cylinder [6].

4. Deformation Produced by Thermal field that is a Polynomial in the Axial Coordinate

In this section, we study the problem when the thermal field is given by (2.1). The solution of problem $P^{(n)}$ is assumed to be known for any function T_n so that we know the solution of the problem when $T = T_{n+1}(x_1, x_2)x_3^n$. Thus, the problem reduces to finding the functions u_i, φ, e_{ij}, κ_{ijk}, τ_{ij}, μ_{ijk}, σ_i, and g that satisfy equations (2.2)–(2.4) on B and the boundary conditions (2.5)–(2.9), when $T = T_{n+1}(x_1, x_2)x_3^{n+1}$, assuming that we know the functions u_i^*, φ^*, e_{ij}^*, κ_{ijk}^*, τ_{ij}^*, μ_{ijk}^*, and g^* that satisfy equations (2.2)–(2.4) on B and conditions (2.7), (2.9)–(2.12) with null mechanical data and temperature $T = T_{n+1}(x_1, x_2)x_3^n$. Following the method used in the theory of loaded cylinders [14], we seek a solution in the form

$$
\begin{aligned}
u_\alpha &= (n+1)\Big[\int_0^{x_3} u_\alpha^* dx_3 - \frac{1}{2}d_\alpha x_3^2 + \epsilon_{3\beta\alpha} d_4 x_\beta x_3 + \sum_{k=1}^4 d_k u_\alpha^{(k)} + V_\alpha(x_1, x_2)\Big],\\
u_3 &= (n+1)\Big[\int_0^{x_3} u_3^* dx_3 + (d_1 x_1 + d_2 x_2 + d_3)x_3 + \sum_{k=1}^4 d_k u_3^{(k)} + V_3(x_1, x_2)\Big],\\
\varphi &= (n+1)\Big[\int_0^{x_3} \varphi^* dx_3 + \sum_{k=1}^4 d_k \varphi^{(k)} + \Psi(x_1, x_2)\Big],
\end{aligned}
\tag{4.1}
$$

where $(u_j^{(k)}, \varphi^{(k)})$ are the solutions of problems $A^{(k)}$, $(k = 1, 2, 3, 4)$, V_j and Ψ are unknown functions, and d_j $(j = 1, 2, 3, 4)$ are unknown constants. From (2.2) and (4.1), we find that

$$
\begin{aligned}
e_{\alpha\beta} &= (n+1)\Big[\int_0^{x_3} e_{\alpha\beta}^* dx_3 + \sum_{k=1}^4 d_k e_{\alpha\beta}^{(k)} + g_{\alpha\beta}\Big],\\
e_{33} &= (n+1)\Big[\int_0^{x_3} e_{33}^* dx_3 + d_1 x_1 + d_2 x_2 + d_3 + u_3^*(x_1, x_2, 0)\Big],\\
e_{\alpha 3} &= (n+1)\Big[\int_0^{x_3} e_{\alpha 3}^* dx_3 + \frac{1}{2}\epsilon_{3\beta\alpha} d_4 x_\beta + \sum_{k=1}^4 d_k e_{\alpha 3}^{(k)} + g_{\alpha 3} + \frac{1}{2}u_\alpha^*(x_1, x_2, 0)\Big],\\
\varphi_{,\alpha} &= (n+1)\Big[\int_0^{x_3} \varphi_{,\alpha}^* dx_3 + \sum_{k=1}^4 d_k \varphi_{,\alpha}^{(k)} + \Psi_{,\alpha}\Big],\\
\varphi_{,3} &= (n+1)\Big[\int_0^{x_3} \varphi_{,3}^* dx_3 + \varphi^*(x_1, x_2, 0)\Big],
\end{aligned}
$$

$$
\begin{aligned}
\kappa_{\alpha\beta j} &= (n+1)\Big[\int_0^{x_3} \kappa_{\alpha\beta j}^* dx_3 + \sum_{k=1}^4 d_k \kappa_{\alpha\beta j}^{(k)} + y_{\alpha\beta j}\Big],\\
\kappa_{\alpha 33} &= (n+1)\Big[\int_0^{x_3} \kappa_{\alpha 33}^* dx_3 + u_{3,\alpha}^*(x_1, x_2, 0) + d_\alpha\Big],
\end{aligned}
$$

$$\kappa_{3\alpha\beta} = (n+1)\left[\int_0^{x_3} \kappa^*_{3\alpha\beta}dx_3 + u^*_{\beta,\alpha}(x_1,x_2,0) + \varepsilon_{3\alpha\beta}d_4\right], \tag{4.2}$$
$$\kappa_{33\alpha} = (n+1)\left[\int_0^{x_3} \kappa^*_{33\alpha}dx_3 + u^*_{\alpha,3}(x_1,x_2,0) - d_\alpha\right],$$
$$\kappa_{333} = (n+1)\left[\int_0^{x_3} \kappa^*_{333}dx_3 + u^*_{3,3}(x_1,x_2,0)\right],$$

where

$$g_{\alpha\beta} = \frac{1}{2}(V_{\alpha,\beta} + V_{\beta,\alpha}), \quad 2g_{\alpha 3} = V_{3,\alpha}, \quad y_{\alpha\beta j} = V_{j,\alpha\beta}. \tag{4.3}$$

We consider the isothermal plane problem corresponding to the displacements V_j and microdilatation function Ψ. In this problem, we denote the stress tensor, the dipolar stress tensor, the microstretch stress vector and the intrinsic body force by p_{ij}, q_{ijk}, ν_j and ρ, respectively. Then from the constitutive equations and (4.1)–(4.3), we get

$$\begin{aligned}
\tau_{\alpha\beta} &= (n+1)\left\{\int_0^{x_3} \tau^*_{\alpha\beta}dx_3 + \sum_{k=1}^{4} d_k\tau^{(k)}_{\alpha\beta} + p_{\alpha\beta} + [\lambda(d_1x_1 + d_2x_2 + d_3) - 2fd_4]\,\delta_{\alpha\beta} + \mathcal{K}_{\alpha\beta}\right\},\\
\tau_{\alpha 3} &= (n+1)\left[\int_0^{x_3} \tau^*_{\alpha 3}dx_3 + \sum_{k=1}^{4} d_k\tau^{(k)}_{\alpha 3} + p_{\alpha 3} + \varepsilon_{3\beta\alpha}(\mu d_4 x_\beta - 2fd_\beta) + \mathcal{K}_{\alpha 3}\right],\\
\tau_{33} &= (n+1)\left[\int_0^{x_3} \tau^*_{33}dx_3 + \sum_{k=1}^{4} d_k\tau^{(k)}_{33} + p_{33} + (\lambda+2\mu)(d_1x_1 + d_2x_2 + d_3) + 4fd_4 + \mathcal{K}_{33}\right],\\
\sigma_\alpha &= (n+1)\left[\int_0^{x_3} \sigma^*_\alpha dx_3 + \sum_{k=1}^{4} d_k\sigma^{(k)}_\alpha + \nu_\alpha + (\beta_2 - \beta_1)d_\alpha + \mathcal{I}_\alpha\right],\\
\sigma_3 &= (n+1)\left[\int_0^{x_3} \sigma^*_3 dx_3 + \sum_{k=1}^{4} d_k\sigma^{(k)}_3 + \nu_3 + \mathcal{I}_3\right],\\
g &= (n+1)\left[\int_0^{x_3} g^*_3 dx_3 + \sum_{k=1}^{4} d_k g^{(k)} + \rho + (d_1x_1 + d_2x_2 + d_3)d + \mathcal{G}\right].
\end{aligned} \tag{4.4}$$

The dipolar stress tensor is given by

$$\mu_{111} = (n+1)\left[\int_0^{x_3} \mu^*_{111}dx_3 + \sum_{k=1}^{4} d_k\mu^{(k)}_{111} + q_{111} + 2d_1(\alpha_2 - \alpha_3) + \mathcal{H}_{111}\right],$$
$$\mu_{222} = (n+1)\left[\int_0^{x_3} \mu^*_{222}dx_3 + \sum_{k=1}^{4} d_k\mu^{(k)}_{222} + q_{222} + 2d_2(\alpha_2 - \alpha_3) + \mathcal{H}_{222}\right],$$
$$\mu_{221} = (n+1)\left[\int_0^{x_3} \mu^*_{221}dx_3 + \sum_{k=1}^{4} d_k\mu^{(k)}_{221} + q_{221} + (\alpha_1 - 2\alpha_3)d_1 - fd_4x_1 + \mathcal{H}_{221}\right],$$
$$\mu_{112} = (n+1)\left[\int_0^{x_3} \mu^*_{112}dx_3 + \sum_{k=1}^{4} d_k\mu^{(k)}_{112} + q_{112} + (\alpha_1 - 2\alpha_3)d_2 - fd_4x_2 + \mathcal{H}_{112}\right],$$

$$\mu_{121} = (n+1)\Big[\int_0^{x_3} \mu^*_{121} dx_3 + \sum_{k=1}^4 d_k \mu^{(k)}_{121} + q_{121} + (\alpha_2 - \frac{1}{2}\alpha_1)d_2 + \frac{1}{2} f d_4 x_2 + \mathcal{H}_{121}\Big],$$

$$\mu_{122} = (n+1)\Big[\int_0^{x_3} \mu^*_{122} dx_3 + \sum_{k=1}^4 d_k \mu^{(k)}_{112} + q_{122} + (\alpha_2 - \frac{1}{2}\alpha_1)d_1 + \frac{1}{2} f d_4 x_1 + \mathcal{H}_{122}\Big],$$

$$\mu_{3\alpha\beta} = (n+1)\Big\{\int_0^{x_3} \mu^*_{3\alpha\beta} dx_3 + \sum_{k=1}^4 d_k \mu^{(k)}_{3\alpha\beta} + q_{3\alpha\beta} + \varepsilon_{3\alpha\beta}[(2\alpha_4 - \alpha_3)d_4 + f(d_1 x_1 + d_2 x_2 + d_3)] + \mathcal{H}_{3\alpha\beta}\Big\}, \tag{4.5}$$

$$\mu_{\alpha\beta 3} = (n+1)\Big\{\int_0^{x_3} \mu^*_{\alpha\beta 3} dx_3 + \sum_{k=1}^4 d_k \mu^{(k)}_{\alpha\beta 3} + q_{\alpha\beta 3} + \mathcal{H}_{\alpha\beta 3}\Big\},$$

$$\mu_{\alpha 33} = (n+1)\Big[\int_0^{x_3} \mu^*_{\alpha 33} dx_3 + \sum_{k=1}^4 d_k \mu^{(k)}_{\alpha 33} + q_{\alpha 33} + (\alpha_2 - \frac{1}{2}\alpha_1 + 2\alpha_4)d_\alpha - \frac{1}{2} f d_4 x_\alpha + \mathcal{H}_{\alpha 33}\Big],$$

$$\mu_{333} = (n+1)\Big[\int_0^{x_3} \mu^*_{333} dx_3 + \sum_{k=1}^4 d_k \kappa^{(k)}_{\rho\rho 3} + q_{333} + \mathcal{H}_{333}\Big],$$

$$\mu_{33\alpha} = (n+1)\Big[\int_0^{x_3} \mu^*_{33\alpha} dx_3 + \sum_{k=1}^4 d_k \mu^{(k)}_{33\alpha} + q_{33\alpha} + (\alpha_1 - 2\alpha_3 - 2\alpha_4 + 2\alpha_5)d_\alpha + f d_4 x_\alpha + \mathcal{H}_{33\alpha}\Big].$$

In (4.4) and (4.5), we have used the notations

$$\begin{aligned}
\mathcal{K}_{\alpha\beta} &= [\lambda\delta_{\alpha\beta} u^*_3 + f(\varepsilon_{3\rho\alpha} u^*_{\rho,\beta} + \varepsilon_{3\rho\beta} u^*_{\rho,\alpha})](x_1, x_2, 0),\\
\mathcal{K}_{\alpha 3} &= [\mu u^*_3 + f\varepsilon_{3\alpha\rho}(u^*_{3,\rho} - u^*_{\rho,3})](x_1, x_2, 0),\\
\mathcal{K}_{33} &= [(\lambda + 2\mu)u^*_3 + 2f\varepsilon_{3\beta\alpha} u^*_{\alpha,\beta}](x_1, x_2, 0),\\
\mathcal{I}_\alpha &= [\beta_1 u^*_{\alpha,3} + \beta_2 u^*_{3,\alpha}](x_1, x_2, 0),\\
\mathcal{I}_3 &= [\beta_1 u^*_{3,3} + \beta_2 u^*_{r,r}](x_1, x_2, 0),\\
\mathcal{G} &= d u^*_3(x_1, x_2, 0)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{H}_{111} &= [(\alpha_1 + 2\alpha_3)u^*_{1,3} + (\alpha_1 + 2\alpha_2)u^*_{3,1}](x_1, x_2, 0),\\
\mathcal{H}_{222} &= [(\alpha_1 + 2\alpha_3)u^*_{2,3} + (\alpha_1 + 2\alpha_2)u^*_{3,2}](x_1, x_2, 0),\\
\mathcal{H}_{112} &= [\alpha_1 u^*_{3,2} + 2\alpha_3 u^*_{2,3} + f u^*_1](x_1, x_2, 0),\\
\mathcal{H}_{121} &= [\frac{1}{2}\alpha_1 u^*_{2,3} + \alpha_2 u^*_{3,2} - \frac{1}{2} f u^*_1](x_1, x_2, 0),\\
\mathcal{H}_{122} &= [\frac{1}{2}\alpha_1 u^*_{1,3} + \alpha_2 u^*_{3,1} + \frac{1}{2} f u^*_2](x_1, x_2, 0),\\
\mathcal{H}_{221} &= [\alpha_1 u^*_{3,1} + 2\alpha_3 u^*_{1,3} - f u^*_2](x_1, x_2, 0),\\
\mathcal{H}_{3\alpha\beta} &= [(\frac{1}{2}\alpha_1 u^*_{3,3} + \alpha_2 u^*_{j,j} + \beta_2\varphi^*)\delta_{\alpha\beta} + 2\alpha_4 u^*_{\beta,\alpha} + \alpha_5 u^*_{\alpha,\beta} + f\varepsilon_{\alpha\beta 3} u^*_3](x_1, x_2, 0),\\
\mathcal{H}_{\alpha 33} &= [(\alpha_2 + 2\alpha_4 + \alpha_5)u^*_{3,\alpha} + \frac{1}{2}(\alpha_1 + 2\alpha_5)u^*_{a,3} + \frac{1}{2} f\varepsilon_{3\rho\alpha} u^*_\rho](x_1, x_2, 0),\\
\mathcal{H}_{\alpha\beta 3} &= [(\alpha_1 u^*_{j,j} + 2\alpha_3 u^*_{3,3} + \beta_1\varphi^*)\delta_{\alpha\beta} + \alpha_5(u^*_{\alpha,\beta} + u^*_{\beta,\alpha})](x_1, x_2, 0),\\
\mathcal{H}_{333} &= [(\alpha_1 + 2\alpha_2)u^*_{j,j} + (\alpha_1 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5)u^*_{3,3} + (\beta_1 + 2\beta_2)\varphi^*](x_1, x_2, 0),\\
\mathcal{H}_{33\alpha} &= [(\alpha_1 + 2\alpha_5)u^*_{3,\alpha} + 2(\alpha_3 + \alpha_4)u^*_{\alpha,3} + f\varepsilon^*_{3\alpha\rho} u^*_\rho](x_1, x_2, 0).
\end{aligned} \tag{4.6}$$

With help of (3.2), (3.3), (4.4) and (4.5), we find that equations (2.4) become

$$p_{\alpha j,\alpha} - q_{\alpha\beta j,\alpha\beta} + \mathcal{F}_j = 0, \quad \nu_{\alpha,\alpha} - \rho + \mathcal{L} = 0, \tag{4.7}$$

on Σ_1, where

$$\begin{aligned} &\mathcal{F}_j = \mathcal{K}_{\rho j,\rho} - \mathcal{K}_{\alpha\beta j,\alpha\beta} + [\tau^*_{3j} - \mu^*_{33j,3} - 2\mu^*_{3\alpha j,\alpha}](x_1, x_2, 0),\\ &\mathcal{L} = \sigma^*_3(x_1, x_2, 0) + \mathcal{I}_{\alpha,\alpha} - \mathcal{G}. \end{aligned} \tag{4.8}$$

In view of (3.4), (3.5), (3.6), (4.4) and (4.5), conditions (2.7) reduce to

$$\mathcal{P}_i = \mathcal{P}^0_i, \quad \mathcal{R}_i = \mathcal{R}^0_i, \quad \nu_\alpha n_\alpha = \nu^0 \text{ on } \Gamma, \tag{4.9}$$

where

$$\begin{aligned} &\mathcal{P}_i = (p_{\alpha i} - q_{j\alpha i,j})n_\alpha - \mathcal{D}_j(n_\rho q_{\rho j i}) + (\mathcal{D}_\rho n_\rho) n_\beta n_\alpha \mu_{\beta\alpha i},\\ &\mathcal{R}_i = q_{\rho\alpha i} n_\rho n_\alpha, \quad \mathcal{P}^0_i = (\mathcal{H}_{\rho\beta i,\rho} - \mathcal{K}_{\beta i}) n_\beta + \mathcal{D}_r(n_\rho \mathcal{H}_{\rho\gamma i})\\ &\quad - (\mathcal{D}_\rho n_\rho)\mathcal{H}_{\alpha\beta i} n_a n_\beta + 2n_\beta \mu^*_{3\beta i}(x_1, x_2, 0),\\ &\mathcal{R}^0_i = -\mathcal{H}_{\alpha\beta i} n_\alpha n_\beta, \quad \nu^0 = -\mathcal{I}_a n_a. \end{aligned} \tag{4.10}$$

Thus, the functions V_j and Ψ are components of the displacement vector and and microstretch function in a generalized plane strain problem defined by the equilibrium equations (4.7) and the boundary conditions (4.9). Using (4.8), (4.10) and the method from Section 3, we obtain

$$\begin{aligned} &\int_{\Sigma_1} \mathcal{F}_j da + \int_{\Gamma_1} \mathcal{P}^0_j ds = -\int_{\Sigma_1} \mathcal{P}^*_j da - \int_{\Gamma_1} \mathcal{Q}^*_j ds,\\ &\int_{\Sigma_1} \varepsilon_{3\alpha\beta} x_\alpha \mathcal{F}_\beta da + \int_{\Gamma_1} \varepsilon_{3\alpha\beta}(x_\alpha \mathcal{P}^0_\beta + n_\alpha \mathcal{R}^0_\beta) ds = -\int_{\Sigma_1} \varepsilon_{\alpha\beta 3} x_\alpha \mathcal{P}^*_\beta da - \int_{\Gamma_1} \varepsilon_{\alpha\beta 3} x_\alpha \mathcal{Q}^*_\beta ds. \end{aligned} \tag{4.11}$$

In problem $P^{(n)}$, the functions $\mathcal{P}^*_j$ and $\mathcal{Q}^*_j$ are equal to zero, so the necessary and sufficient conditions for the existence of the functions V_j and Ψ are satisfied. As in Section 3, we can show that conditions (2.9) are identically satisfied. Conditions (2.10)–(2.12) reduce for the unknown constants d_k ($k = 1, 2, 3, 4$) to the following system:

$$\sum_{j=1}^{4} D_{ij} d_j = -\theta_i, \tag{4.12}$$

where

$$\begin{aligned} \theta_\alpha &= \int_{\Sigma_1} \{2(q_{\alpha 33} + \mathcal{H}_{\alpha 33}) - q_{33\alpha} - \mathcal{H}_{33\alpha} + x_\alpha[(\lambda + 2\mu)u^*_3\\ &\quad + 2f\varepsilon_{3\beta\rho} u^*_{\rho,\beta}](x_1, x_2, 0) + x_\alpha(p_{33} - \mu^*_{333})\} da,\\ \theta_3 &= \int_{\Sigma_1} \{p_{33} - \mu^*_{333} + [(\lambda + 2\mu)u^*_3 + 2f\varepsilon_{3\beta\rho} u^*_{\rho,\beta}](x_1, x_2, 0)\} da,\\ \theta_4 &= \int_{\Sigma_1} \varepsilon_{\alpha\beta 3}\{x_\alpha(p_{3\beta} + \mathcal{K}_{3\beta} - \mu^*_{33\beta}) + 2q_{\alpha 3\beta} + 2\mathcal{H}_{\alpha 3\beta}\} da. \end{aligned}$$

As in the classical elasticity, the positive definiteness of the potential energy implies that [13]

$$\det(D_{mn}) \neq 0. \tag{4.13}$$

Relation (4.13) shows that system (4.12) determines the constants d_j ($j = 1, 2, 3, 4$).

5. Conclusions

The results presented in this paper can be summarized as follows:

- The basic equations of the strain gradient theory of chiral porous thermoelastic solids are presented and the equilibrium problem of a homogeneous and isotropic cylinder is formulated. The cylinder is subjected to a temperature field that is polynomial in the axial coordinate.
- The porosity and chirality are introduced in the constitutive equations by means of a scalar function and a material constant, respectively.
- The strain gradient theory of chiral porous materials is constructed by adding the second-order partial derivative of the displacement components and the first-order partial derivative of the microdilatation function to the set of independent constitutive variable.
- First, the problem of a cylinder subjected to a temperature field independent of the axial coordinate is solved and then the case of a cylinder deformed by a temperature field that is polynomial in the axial coordinate is considered.
- The analytical solution is obtained by the method of induction and is expressed in terms of solutions of the same plane strain problems.

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