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RESEARCH OF A NONLINEAR DYNAMIC SYSTEM DESCRIBING THE PROCESS OF INTERACTION BETWEEN COLCHIAN, GEORGIAN AND SVAN POPULATIONS

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Abstract. This paper investigates a three-dimensional nonlinear dynamic system describing the interaction of three populations speaking Colchian, Georgian and Svan languages. The period when the Colchian population prevailed over two others is considered. In the absence of quadratic terms characterizing the self-restriction of population growth and some relationships between the constant coefficients of a nonlinear three-dimensional dynamic system, the first integral in the form of a hyperbolic paraboloid is found. Using the first integral, the three-dimensional nonlinear dynamic system is reduced to a two-dimensional system. For some relations between the constant coefficients of a two-dimensional dynamical system, the theorems on the existence of a closed trajectory in a simply connected domain of the first quarter of the phase plane of solutions are proved using the Berdixson principle.

In the case of quadratic terms characterizing the self-restriction of population growth and certain relationships between the constant coefficients of a nonlinear three-dimensional dynamic system, the first integrals are found. In one case, the integral is a cone, and in the other, a hyperbolic paraboloid. In both cases, using the first integral, the general three-dimensional nonlinear dynamic system is reduced to a two-dimensional system. For some relations between the constant coefficients of a two-dimensional dynamical system, the theorems on the existence of a closed trajectory in certain simply connected domain of the first quarter of the phase plane of solutions are proved using the Bendixson principle.

Thus, in all these cases, it is proved that all three populations living in the same region and speaking three different languages coexist and do not completely assimilate them.

Indroduction

Synergetics has given a powerful impetus to the use of mathematical models in the social sciences. Mathematical modeling of social processes compared to modeling in natural science is more original due to the complexity of model justifications [2–4, 9–14].

From a historical point of view, we consider mathematical modeling as an innovative approach to describing the distribution domain of the Proto–Kartvelian speaking population and the process of further language transformation, determining the number of the population speaking the corresponding language in each time period.

To describe the process of transformation of the Proto–Kartvelian population into four populations speaking four different languages, Georgian, Mengrelian, Laz and Svan, we divided mathematical modeling into four separate stages: the first stage (L–XXV BC), at the end of which there is a division of the Proto–Kartvelian population into three parts: - the first part emigrated to Europe and gradually completely or partially assimilated, the second part speaking the Svan language, the third part speaking the Colchian and-Georgian languages; the second stage (XXV–X centuries BC), at the end of which the Colchian and Georgian populations split into two parts, the Colchian and Georgian populations; the third stage (X–I centuries BC. e.), at the end of which the Colchian population splits into the Mengrelian and Laz populations. Moreover, in the X–III centuries BC, the Colchian population predominated over others, and in the III–I centuries BC. e. it was the Georgian population that predominated over others; the fourth stage (from the Ist century BC – to the present days), when

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four multilingual populations Georgian, Mengrelian, Laz and Svan live peacefully in a small territory in the Caucasus and Western Asia.

Mathematical modeling of the first stage is considered in [5].

Mathematical and computer modeling of the second stage is considered in [6–8].

1. General Mathematical Model. System of Equations

Let us consider a general mathematical model describing the interactions between three populations speaking different lnguages Colchian, Georgian and Svan, taking into account the existence of self-limiting conditions for population growth terms.

The mathematical model is described by a three-dimensional nonlinear system of differential equations with variable coefficients:

$$\begin{cases} \dfrac{dw(t)}{dt} = \alpha_5(t)w(t) - \gamma_1(t)w^2(t) + \beta_6(t)u(t)w(t) + \beta_7(t)w(t)v(t) - p_3(t)w(t), \\ \dfrac{du(t)}{dt} = \alpha_4(t)u(t) - \gamma_2(t)u^2(t) + \beta_4(t)u(t)w(t) + \beta_5(t)u(t)v(t) - p_2(t)u(t), \\ \dfrac{dv(t)}{dt} = \alpha_6(t)v(t) - \gamma_3(t)v^2(t) - \beta_8(t)u(t)v(t) - \beta_9(t)w(t)v(t), \end{cases} \tag{1.1}$$

with the initial conditions:

$$w(t_2) = w_2,\ u(t_2) = u_2,\ v(t_2) = v_2, \tag{1.2}$$

$w(t), u(t), v(t) \in C^1[t_2, t_4]$, $t \in (t_2, t_4)$,
$\beta_i(t) \in C[t_2, t_4]$, $i \in \{4, 5, 6, 7, 8, 9\}$,
$\gamma_1(t), \gamma_2(t), \gamma_3(t), p_2(t), p_3(t) \in C[t_2, t_4]$,
$\alpha_4(t), \alpha_5(t), \alpha_6(t) \in C[t_2, t_4]$,
$w(t)$ is the number of Colchian speaking population at time t;
$u(t)$ is the number of Georgian speaking population at time t;
$v(t)$ is the number of the population speaking the Svan language at the t time;
t_2 is the Xth century BC, t_3 is the IIIrd Century BC, t_4 is the Ist Century BC;
$\alpha_4(t), \alpha_5(t), \alpha_6(t)$ are natural demographic factors, of the Georgian, Colchian and Svan populations, respectively;
$\gamma_1(t), \gamma_2(t), \gamma_3(t)$ are co-factors of self-limiting growth, respectively, of the Colchian, Georgian and Svan populations, non-negative functions, moreover, they are identically equal to zero, in case of non-accounting for self-limiting population growth;
$\beta_5(t), \beta_8(t), \beta_7(t), \beta_9(t)$ are co-factors of assimilation of the Svan population by the Georgian, Colchian population;
$\beta_4(t), \beta_6(t)$ are sign-variable functions describing assimilation of the population speaking Colchian by the population speaking Georgian, or vice versa;
$\beta_4(t)$ is negative to the segment $[t_2, t_3]$ and positive to the segment $[t_3, t_4]$;
$\beta_6(t)$ is positive to the segment $[t_2, t_3]$ and negative to the segment $[t_3, t_4]$;
$p_2(t) > 0$ are co-factors of unnatural reduction of the Georgian population due to forced hostilities with neighboring peoples;
$p_3(t) > 0$ are co-factors of unnatural reduction of the Colchian population due to forced hostilities with neighboring peoples.

An analytical study of a three-dimensional dynamic system of differential equations (1.1) with variable coefficients is impossible.

Therefore, in order to qualitatively describe the process of interaction and mutual influence of populations speaking three different languages (Colchian, Georgian, Svan), we consider some particular cases that are amenable to analytical research.

2. A Special Case of a Mathematical Model without Terms of Self-limiting Population Growth

As a first mathematical model, we consider the case without nonlinear terms of self-limitating population growth during the period of Colchian language dominance.

We also assume that all coefficients of the system of equations are constant:

$$\begin{gathered}\gamma_1(t)\equiv\gamma_2(t)\equiv\gamma_3(t)=0,\ \alpha_4(t)=\alpha_4,\ \alpha_5(t)=\alpha_5,\ \alpha_6(t)=\alpha_6,\\ \beta_4(t)=-\beta_4,\ \beta_5(t)=\beta_5,\ \beta_6(t)=\beta_6,\ \beta_7(t)=\beta_7,\ \beta_8(t)=\beta_8,\ \beta_9(t)=\beta_9,\\ p_2(t)=p_2,\ p_3(t)=p_3=\text{const}.\end{gathered}\tag{2.1}$$

Then, taking into account (2.1) and considering the Cauchy problem on the interval $t\in(t_2;t_3)$, where the coefficients $\beta_4>0$, $\beta_6>0$ are of a certain sign and the Colchian population spreads its language among two other populations, the system of equations (1.1) can be rewritten in the following form:

$$\begin{cases}\dfrac{dw(t)}{dt}=\alpha_5w(t)+\beta_6u(t)w(t)+\beta_7w(t)v(t)-p_3w(t),\\ \dfrac{du(t)}{dt}=\alpha_4u(t)-\beta_4u(t)w(t)+\beta_5u(t)v(t)-p_2u(t),\\ \dfrac{dv(t)}{dt}=\alpha_6v(t)-\beta_8u(t)v(t)-\beta_9w(t)v(t),\end{cases}\tag{2.2}$$

with the initial conditions

$$w(t_2)=w_2,\quad u(t_2)=u_2,\quad v(t_2)=v_2,\quad t\in(t_2;t_3).\tag{2.3}$$

The system of equations (2.2) is considered in the interval $t\in(t_2;t_3)$ and assumes that the Colchian language is dominant, i.e., the population speaking this language assimilates the population speaking Georgian and Svan languages.

A qualitative analysis of the system of equations (2.2), taking into account the adequacy and non-triviality of the mathematical model, leads to a system of restrictions on the variable coefficients of the dynamic system

$$\begin{cases}\alpha_6>0,\ \beta_6>0,\\ \beta_7>0,\ \beta_4>0,\\ \beta_5>0,\ \beta_8>0, & t\in(t_2;t_3),\\ \beta_9>0,\ p_3>0,\\ p_2>0,\ \alpha_5-p_3<0,\end{cases}\tag{2.4}$$

α_4, α_5 are the coefficients that can take different values and be alternating or equal to zero.

In the system of equations (2.2), we make some transformations:

$$\begin{cases}\dfrac{dw(t)}{wdt}=(\alpha_5-p_3)+\beta_6u(t)+\beta_7v(t),\\ \dfrac{du(t)}{udt}=(\alpha_4-p_2)-\beta_4w(t)+\beta_5v(t),\\ \dfrac{dv(t)}{vdt}=\alpha_6-\beta_8u(t)-\beta_9w(t).\end{cases}\tag{2.5}$$

Now, let us assume that the constant coefficients of the system of equations (2.2) satisfy additional conditions that do not contradict (2.4)

$$\begin{cases}\beta_6=\beta_8,\ \beta_9=\beta_4,\\ \beta_5=\beta_7,\ \alpha_6=(\alpha_4-p_2)-(\alpha_5-p_3).\end{cases}\tag{2.6}$$

Let us add the first and third equations of system (2.5) and subtract the second equation of this system. Then, taking into account (2.6), we obtain the relation

$$\frac{dw(t)}{wdt}+\frac{dv(t)}{vdt}-\frac{du(t)}{udt}=0.\tag{2.7}$$

From (2.7), we obtain the first integral of the system of equations (2.2), (2.3) under the assumption (2.6)

$$\frac{wv}{u} = \frac{w_2 v_2}{u_2}\,. \tag{2.8}$$

The first integral (2.8) in the solution space $(O, w(t), u(t), v(t))$ of the system of equations (2.2), (2.3) is a hyperbolic paraboloid.

Introduce the notation

$$p = \frac{u_2}{w_2 v_2}\,. \tag{2.9}$$

Then,taking into account (2.8), (2.9), the three-dimensional dynamic system (2.2) can be reduced to a two-dimensional dynamic system

$$\begin{cases} \dfrac{dw(t)}{dt} = (\alpha_5 - p_3)w + \beta_6 p w^2 v + \beta_7 wv, \\ \dfrac{dv(t)}{dt} = \alpha_6 v - \beta_9 wv - \beta_8 p v^2 w, \end{cases} \tag{2.10}$$

with the initial conditions

$$w(t_2) = w_2,\ v(t_2) = v_2.$$

A qualitative analysis of the system of equations (2.2), taking into account the adequacy and nontriviality of the mathematical model, leads to a system of restrictions on the coefficients of the dynamic system

$$\alpha_6 = (\alpha_4 - p_2) - (\alpha_5 - p_3) > 0. \tag{2.11}$$

Let us introduce the notation

$$\begin{cases} F_1(w,v) = (\alpha_5 - p_3)w + \beta_6 p w^2 v + \beta_7 wv, \\ F_2(w,v) = \alpha_6 v - \beta_9 wv - \beta_8 p v^2 w. \end{cases} \tag{2.12}$$

Taking into account (2.12), the nonlinear dynamic system (2.10) can be rewritten in a vector form

$$\frac{d\overrightarrow{V}}{dt} = \overrightarrow{F}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad \overrightarrow{V}(t) = \begin{pmatrix} w(t) \\ v(t) \end{pmatrix}, \quad \overrightarrow{V}(t_2) = \begin{pmatrix} w_2 \\ v_2 \end{pmatrix}. \tag{2.13}$$

We calculate the divergence of a vector field $\overrightarrow{F}(F_1, F_2)$,

$$\operatorname{div} \overrightarrow{F} = \frac{\partial F_1}{\partial w} + \frac{\partial F_2}{\partial v} = (\alpha_5 - p_3) + 2\beta_6 pwv + \beta_7 v + \alpha_6 - \beta_9 w - 2\beta_8 pvw. \tag{2.14}$$

Using (2.6) and (2.11), equation (2.14) can be rewritten as follows:

$$\operatorname{div} \overrightarrow{F} = \frac{\partial F_1}{\partial w} + \frac{\partial F_2}{\partial v} = (\alpha_4 - p_2) + \beta_7 v - \beta_9 w. \tag{2.15}$$

Introduce the notation

$$\operatorname{div} \overrightarrow{F} \equiv G(v,w) = \alpha_4 - p_2 + \beta_7 v - \beta_9 w. \tag{2.16}$$

In the phase plane $(O, v(t), w(t))$ of solutions of the two-dimensional dynamic system (2.12), (2.13), we consider the line on which the divergence of the vector field $\overrightarrow{F}(F_1, F_2)$ vanishes,

$$w = \frac{\alpha_4 - p_2}{\beta_9} + \frac{\beta_7}{\beta_9} v. \tag{2.17}$$

Consider a few cases.

Case 1:

$$\alpha_4 = p_2. \tag{2.18}$$

In the case of (2.18), (2.17) has the form

$$w = \frac{\beta_7}{\beta_9} v. \tag{2.19}$$

Theorem 2.1. *Problem* (2.12), (2.13), (2.18) *in some simply connected domain* $D \in (O, v(t), w(t))$ *of the first quarter of the phase plane* $(O, v(t), w(t))$ *has the solution in the form of the closed trajectory that lies entirely in this domain.*

Proof. Consider a line on the phase plane (O, v, w) on which the divergence of vector field vanishes. According to (2.17), (2.18), this curve will be a line (2.19).

Thus, the divergence of the vector field in the physically meaningful first quarter of the phase plane of solutions turns to zero on a half-line (2.19) with the left endpoint $O(0, 0)$. Suppose that the starting point $M(v_2, w_2)$ also belongs to the half-line (2.19), when $v > 0$.

It is clear that the $G(v(t), w(t))$ divergence (2.16) of the vector field $\overrightarrow{F}(F_1, F_2)$ in some domain $D \in (O, v(t), w(t))$, containing the point $M(v_2, w_2)$, changes its sign (Figure 1).

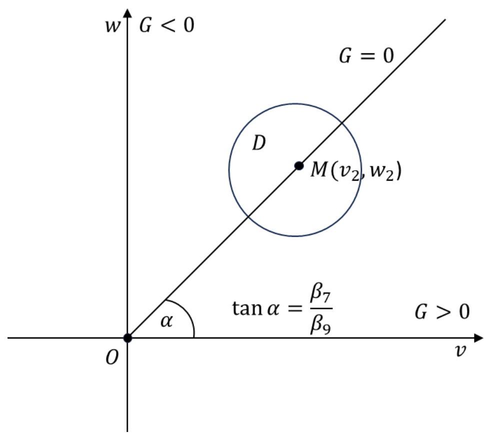

FIGURE 1

According to the Bendixson criterion, there is a closed integral trajectory of the dynamic system (2.12), (2.13), (2.18), that lies entirely in this domain [1]. □

Case 2:

$$\alpha_4 > p_2. \tag{2.20}$$

Theorem 2.2. *Problem* (2.12), (2.13), (2.20) *in some simply connected domain* $D \in (O, v(t), w(t))$ *of the first quarter of the phase plane* $(O, v(t), w(t))$ *has the solution in the form of the closed trajectory which completely lies in this domain.*

Proof. Consider a line on the phase plane (O, v, w) on which the divergence of the vector field becomes equal to zero. According to (2.17), (2.20), this curve will be a line (2.17).

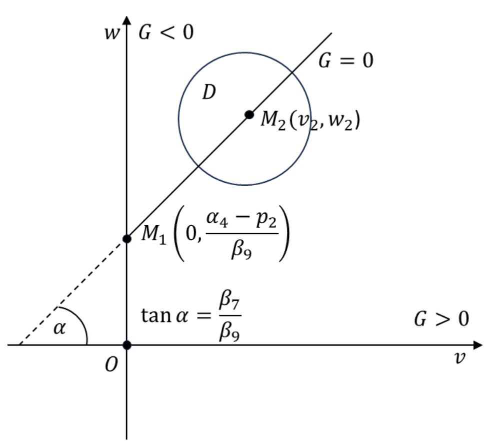

FIGURE 2

The divergence of the vector field in the physically meaningful first quarter of the phase plane of solutions turns to zero on a half-line with the left endpoint $M_1\big(0, \frac{\alpha_4 - p_2}{\beta_9}\big)$. Suppose that the starting point $M_2(v_2, w_2)$ also belongs to the half-line (2.19), when $v > 0$.

It is clear that the $G(v(t), w(t))$ divergence (2.16) of the vector field $\overrightarrow{F}(F_1, F_2)$, in some domain $D \in (O, v(t), w(t))$, containing the point $M_2(v_2, w_2)$, changes its sign (Figure 2).

According to the Bendixson criterion, there is a closed integral trajectory of the dynamic system (2.12), (2.13), (2.20), that lies entirely in this domain [1]. □

Case 3:

$$\alpha_4 < p_2. \tag{2.21}$$

Theorem 2.3. *Problem* (2.12), (2.13), (2.21) *in some simply connected domain* $D \in (O, v(t), w(t))$ *of the first quarter of the phase plane* $(O, v(t), w(t))$, *has the solution in the form of the closed trajectory lying completely in this domain.*

Proof. Consider a line on the phase plane (O, v, w) where the vector field divergence becomes equal to zero. According to (2.17), (2.21), this curve will be a line (2.17).

The divergence of the vector field in the physically meaningful first quarter of the phase plane of solutions turns to zero on a half-line with the left endpoint $M_3\big(\frac{p_2 - \alpha_4}{\beta_7}, 0\big)$. Suppose that the starting point $M_4(v_2, w_2)$ also belongs to the half-line (2.17), when $v > \frac{p_2 - \alpha_4}{\beta_7}$.

It is clear that the $G(v(t), w(t))$ divergence (2.16) of the vector field $\overrightarrow{F}(F_1, F_2)$, in some domain $D \in (O, v(t), w(t))$, containing the point $M_4(v_2, w_2)$, changes its sign (Figure 3).

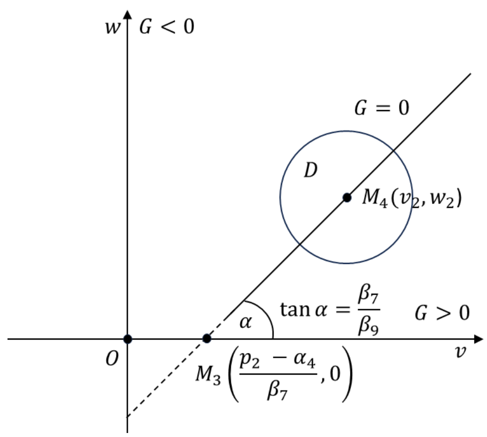

Figure 3

According to the Bendixson criterion, there is a closed integral trajectory of the dynamic system (2.12), (2.13), (2.21), that lies entirely in this domain [1]. □

3. Mathematical Model with Terms Self-limiting Population Growth

Let us consider a system of nonlinear differential equations (1.1) in the case of constant coefficients

$$\begin{cases} \dfrac{dw(t)}{dt} = \alpha_5 w(t) - \gamma_1 w^2(t) + \beta_6 u(t) w(t) + \beta_7 w(t) v(t) - p_3 w(t), \\ \dfrac{du(t)}{dt} = \alpha_4 u(t) - \gamma_2 u^2(t) - \beta_4 u(t) w(t) + \beta_5 u(t) v(t) - p_2 u(t), \\ \dfrac{dv(t)}{dt} = \alpha_6 v(t) - \gamma_3 v^2(t) - \beta_8 u(t) v(t) - \beta_9 w(t) v(t), \end{cases} \tag{3.1}$$

with the initial conditions

$$w(t_2) = w_2, \quad u(t_2) = u_2, \quad v(t_2) = v_2, \quad t \in (t_2; t_3). \tag{3.2}$$

This suggests the case

$$\gamma_1 \neq 0, \quad \gamma_2 \neq 0, \quad \gamma_3 \neq 0. \tag{3.3}$$

After some mathematical transformations of the system of equations (3.1), we obtain the following system:

$$\begin{cases} \dfrac{dw(t)}{wdt} = (\alpha_5 - p_3) - \gamma_1 w(t) + \beta_6 u(t) + \beta_7 v(t), \\ 2\dfrac{du(t)}{udt} = 2(\alpha_4 - p_2) - 2\gamma_2 u(t) - 2\beta_4 w(t) + 2\beta_5 v(t), \\ \dfrac{dv(t)}{vdt} = \alpha_6 - \gamma_3 v(t) - \beta_8 u(t) - \beta_9 w(t). \end{cases} \tag{3.4}$$

Adding the first and third equations of system (3.4) and subtracting the second equation of this system, we obtain the relation

$$-2\frac{du(t)}{udt} + \frac{dv(t)}{vdt} + \frac{dw(t)}{wdt} = (\ln \frac{wv}{u^2})' = (\alpha_5 - p_3) + \alpha_6 - 2(\alpha_4 - p_2)$$
$$+2(\beta_4 - \gamma_1 - \beta_9)w(t) + (\beta_7 - \gamma_3 - 2\beta_5)v(t) + (\beta_6 - \beta_8 + 2\gamma_2)u(t). \tag{3.5}$$

Consider a special case

$$\begin{cases} 2\beta_4 - \gamma_1 - \beta_9 = 0, \\ \beta_7 - \gamma_3 - 2\beta_5 = 0, \\ \beta_6 - \beta_8 + 2\gamma_2 = 0, \\ (\alpha_5 - p_3) + \alpha_6 - 2(\alpha_4 - p_2) = 0. \end{cases} \tag{3.6}$$

Then, taking into account (3.6), we rewrite (3.5) in the following form:

$$-2\frac{du(t)}{udt} + \frac{dv(t)}{vdt} + \frac{dw(t)}{wdt} = (\ln \frac{wv}{u^2})' = 0, \tag{3.7}$$

whence we obtain the first integral of the system of nonlinear differential equations (3.1),

$$\frac{wv}{u^2} = \frac{w_2 v_2}{u_2^2} = \text{const} = q. \tag{3.8}$$

Now, (3.8) in the solutions space $(O, w(t), u(t), v(t))$ of the system of differential equations (3.1) is a cone

$$wv = qu^2. \tag{3.9}$$

We express the function $v(t)$ from (3.9) and substitute it into (3.1), which in this case reduces to a system of two differential equations

$$\begin{cases} \dfrac{dw(t)}{dt} = (\alpha_5 - p_3)w(t) - \gamma_1 w^2(t) + \beta_6 u(t)w(t) + \beta_7 q u^2(t), \\ \dfrac{du(t)}{dt} = (\alpha_4 - p_2)u(t) - \gamma_2 u^2(t) - \beta_4 u(t)w(t) + \beta_5 q \frac{u^3(t)}{w(t)}, \end{cases} \tag{3.10}$$

with the initial conditions

$$w(t_2) = w_2, \; u(t_2) = u_2, \; t \in (t_2; t_3). \tag{3.11}$$

Introduce the notation

$$\begin{cases} F_1(w,u) \equiv (\alpha_5 - p_3)w(t) - \gamma_1 w^2(t) + \beta_6 u(t)w(t) + \beta_7 q u^2(t), \\ F_1(w,u) \equiv (\alpha_4 - p_2)u(t) - \gamma_2 u^2(t) - \beta_4 u(t)w(t) + \beta_5 q \frac{u^3(t)}{w(t)}. \end{cases} \tag{3.12}$$

Taking into account (3.12), the nonlinear dynamic system (3.10), (3.11) can be rewritten in a vector form

$$\frac{d\overrightarrow{V}}{dt} = \overrightarrow{F}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad \overrightarrow{V}(t) = \begin{pmatrix} w(t) \\ v(t) \end{pmatrix}, \quad \overrightarrow{V}(t_2) = \begin{pmatrix} w_2 \\ v_2 \end{pmatrix}. \tag{3.13}$$

We calculate the divergence of a vector field $\overrightarrow{F}(F_1, F_2)$,

$$\begin{aligned}\operatorname{div}\overrightarrow{F} &= \frac{\partial F_1}{\partial w} + \frac{\partial F_2}{\partial u} = (\alpha_5 - p_3) - 2\gamma_1 w + \beta_6 u + (\alpha_4 - p_2) - 2\gamma_2 u - \beta_4 w + 3\beta_5 q\frac{u^2}{w}, \\ \operatorname{div}\overrightarrow{F} &= (\alpha_5 - p_3) + (\alpha_4 - p_2) + (\beta_6 - 2\gamma_2)u - (2\gamma_1 + \beta_4)w + 3\beta_5 q\frac{u^2}{w}\end{aligned} \tag{3.14}$$

and introduce the notation

$$G_1(u, w) \equiv (\alpha_5 - p_3) + (\alpha_4 - p_2) + (\beta_6 - 2\gamma_2)u - (2\gamma_1 + \beta_4)w + 3\beta_5 q\frac{u^2}{w}\,. \tag{3.15}$$

On the phase plane of solutions of a two-dimensional nonlinear dynamic system (3.10), we study the curves on which the divergence of the vector field vanishes,

$$(\alpha_5 - p_3) + (\alpha_4 - p_2) + (\beta_6 - 2\gamma_2)u - (2\gamma_1 + \beta_4)w + 3\beta_5 q\frac{u^2}{w} = 0. \tag{3.16}$$

Assume that the constant coefficients of the system of equations (3.13), except for (3.6), satisfy additional conditions that do not contradict the adequacy of the mathematical model

$$\begin{cases}(\alpha_5 - p_3) + (\alpha_4 - p_2) = 0, \\ \beta_6 = 2\gamma_2.\end{cases} \tag{3.17}$$

Then, taking into account (3.17), in the first quarter of the phase plane of solutions, which has a physical meaning, (3.16) is a half-line passing through the origin of coordinates (the left boundary of the half-line is not included in the consideration).

Thus, on the half-line the divergence of the vector field becomes zero, in the left angular sector it is negative, and in the right angular sector it is positive,

$$\begin{cases}w = \sqrt{\frac{3\beta_5 q}{2\gamma_1 + \beta_4}}u, \\ u > 0.\end{cases} \tag{3.18}$$

Theorem 3.1. *Problem* (3.12), (3.13), (3.6), (3.17) *in some simply connected domain* $D \in (O, u(t), w(t))$ *of the first quarter of the phase plane* $(O, u(t), w(t))$ *has the solution in the form of the closed trajectory lying completely in this domain.*

Proof. Consider a line on the phase plane (O, u, w) where the vector field divergence becomes equal to zero. According to (3.14), (3.17) this curve will be a half-line (3.18). The divergence of the vector field in the physically meaningful first quarter of the phase plane of solutions turns to zero on a half-line (3.18). Suppose that the starting point $M_5(u_2, w_2)$ also belongs to half-line (3.18), when $u > 0$.

It is clear that the $G_1(u(t), w(t))$ divergence (3.14) of the vector field $\overrightarrow{F}(F_1, F_2)$, in some domain $D \in (O, u(t), w(t))$, containing the point $M_5(u_2, w_2)$, changes its sign (Figure 4).

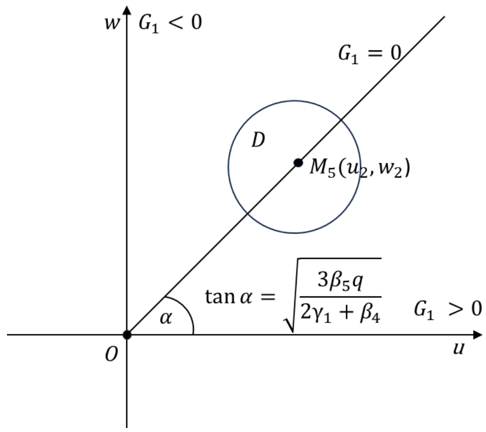

FIGURE 4

According to the Bendixson criterion, there is a closed integral trajectory of the dynamic system (3.12), (3.13), (3.6), (3.17) that lies entirely in this domain [1]. □

Now, let us consider the case when in (3.15) only the equality

$$(\alpha_5 - p_3) + (\alpha_4 - p_2) = 0 \tag{3.19}$$

holds. Then, taking into account (3.19), after some mathematical transformations in (3.15), the divergence of the vector field in this case vanishes on the curve

$$(2\gamma_1 + \beta_4)w^2 - (\beta_6 - 2\gamma_2)uw - 3\beta_5 q u^2 = 0. \tag{3.20}$$

Now, (3.20) in the physically meaningful phase plane of solutions (3.13), is a half-line passing through the origin of coordinates (without it)

$$\begin{cases} w = au, \\ a = \frac{(\beta_6 - 2\gamma_2) + \sqrt{12q\beta_5(2\gamma_1+\beta_4) + (\beta_6 - 2\gamma_2)^2}}{2(2\gamma_1+\beta_4)} > 0, \\ u > 0. \end{cases} \tag{3.21}$$

Theorem 3.2. *Problem* (3.12), (3.13), (3.6), (3.19) *in some simply connected domain* $D \in (O, u(t), w(t))$ *of the first quarter of the phase plane* $(O, u(t), w(t))$ *has the solution in the form of the closed trajectory lying completely in this domain.*

Proof. Consider a line on the phase plane (O, u, w) where the vector field divergence becomes equal to zero. According to (3.14), (3.19), (3.20), this curve will be a line (3.21).

The divergence of the vector field in the physically meaningful first quarter of the phase plane of solutions turns to zero on a half-line (3.21). Suppose that the starting point $M_6(u_2, w_2)$ also belongs to half-line (3.21), when $u > 0$.

It is clear that the $G_1(u(t), w(t))$ divergence (3.14) of the vector field $\overrightarrow{F}(F_1, F_2)$, in some domain $D \in (O, u(t), w(t))$, containing the point $M_6(u_2, w_2)$, changes its sign (Figure 5).

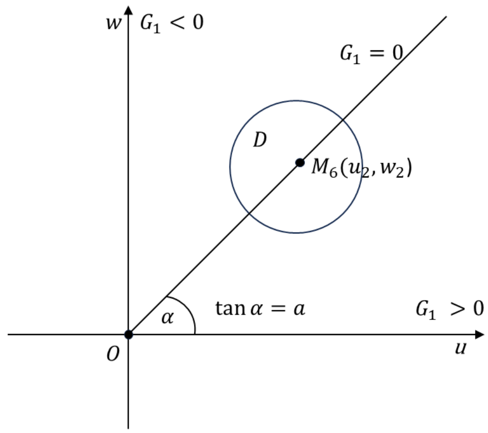

FIGURE 5

According to the Bendixson criterion, there is a closed integral trajectory of the dynamic system (3.12), (3.13), (3.6), (3.19) that lies entirely in this domain [1]. □

After some mathematical transformations of the system of equations (3.1), we can get the following system:

$$\begin{cases} \dfrac{dw(t)}{wdt} = (\alpha_5 - p_3) - \gamma_1 w(t) + \beta_6 u(t) + \beta_7 v(t), \\ \dfrac{du(t)}{udt} = (\alpha_4 - p_2) - \gamma_2 u(t) - \beta_4 w(t) + \beta_5 v(t), \\ \dfrac{dv(t)}{vdt} = \alpha_6 - \gamma_3 v(t) - \beta_8 u(t) - \beta_9 w(t). \end{cases} \tag{3.22}$$

Adding the first and third equations of system (3.22) and subtracting the second equation of this system, we obtain the relation

$$\begin{aligned} -\frac{du(t)}{udt} + \frac{dv(t)}{vdt} + \frac{dw(t)}{wdt} = (\ln \frac{wv}{u})' = (\alpha_5 - p_3) + \alpha_6 - (\alpha_4 - p_2 \\ +(\beta_4 - \gamma_1 - \beta_9)w(t) + (\beta_7 - \gamma_3 - \beta_5)v(t) + (\beta_6 - \beta_8 + \gamma_2)u(t). \end{aligned} \tag{3.23}$$

Now, let us assume that the constant coefficients of the system of equations (3.1) satisfy the additional conditions

$$\begin{cases} \beta_6 - \beta_8 + \gamma_2 = 0, \\ \beta_7 - \gamma_3 - \beta_5 = 0, \\ \beta_4 - \gamma_1 - \beta_9 = 0, \\ \alpha_6 - (\alpha_4 - p_2) + (\alpha_5 - p_3) = 0. \end{cases} \tag{3.24}$$

From (3.23), under the assumption (3.24), we obtain the first integral of the system of equations (3.1), (3.2),

$$\frac{wv}{u} = \frac{w_2 v_2}{u_2}\,. \tag{3.25}$$

The first integral (3.25) in the solution space $(O, w(t), u(t), v(t))$ of the system of equations (3.1), (3.2) is a hyperbolic paraboloid.

Introduce the notation

$$p_1 = \frac{w_2 v_2}{u_2}\,. \tag{3.26}$$

Then, taking into account (3.25), (3.26), the three-dimensional dynamic system (3.1) can be reduced to a two-dimensional dynamic system

$$\begin{cases} \dfrac{dw(t)}{dt} = (\alpha_5 - p_3)w(t) - \gamma_1 w^2(t) + \beta_6 u(t)w(t) + \beta_7 p_1 u(t), \\ \dfrac{du(t)}{dt} = (\alpha_4 - p_2)u(t) - \gamma_2 u^2(t) - \beta_4 u(t)w(t) + \beta_5 p_1 \frac{u^2(t)}{w(t)}, \end{cases} \tag{3.27}$$

with the initial conditions

$$w(t_2) = w_2,\ u(t_2) = u_2.$$

Introduce the notation

$$\begin{cases} F_1(w,u) \equiv (\alpha_5 - p_3)w(t) - \gamma_1 w^2(t) + \beta_6 u(t)w(t) + \beta_7 p_1 u(t), \\ F_2(w,u) \equiv (\alpha_4 - p_2)u(t) - \gamma_2 u^2(t) - \beta_4 u(t)w(t) + \beta_5 p_1 \frac{u^2(t)}{w(t)}\,. \end{cases} \tag{3.28}$$

Taking into account (3.24), the nonlinear dynamic system (3.27), (3.28) can be rewritten in a vector form

$$\frac{d\overrightarrow{V}}{dt} = \overrightarrow{F}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad \overrightarrow{V}(t) = \begin{pmatrix} w(t) \\ v(t) \end{pmatrix}, \quad \overrightarrow{V}(t_2) = \begin{pmatrix} w_2 \\ v_2 \end{pmatrix}\,. \tag{3.29}$$

We calculate the divergence of a vector field $\overrightarrow{F}(F_1, F_2)$,

$$\begin{gathered} \operatorname{div}\overrightarrow{F} = \frac{\partial F_1}{\partial w} + \frac{\partial F_2}{\partial u} = (\alpha_5 - p_3) - 2\gamma_1 w + \beta_6 u + (\alpha_4 - p_2) - 2\gamma_2 u - \beta_4 w + 2\beta_5 p_1 \frac{u}{w}, \\ \operatorname{div}\overrightarrow{F} = (\alpha_5 - p_3) + (\alpha_4 - p_2) + (\beta_6 - 2\gamma_2)u - (2\gamma_1 + \beta_4)w + 2\beta_5 p_1 \frac{u}{w} \end{gathered} \tag{3.30}$$

and introduce the notation

$$G_2(u,w) \equiv (\alpha_5 - p_3) + (\alpha_4 - p_2) + (\beta_6 - 2\gamma_2)u - (2\gamma_1 + \beta_4)w + 2\beta_5 p_1 \frac{u}{w}\,. \tag{3.31}$$

On the phase plane of solutions of a two-dimensional nonlinear dynamic system (3.27), we study the curves on which the divergence of the vector field vanishes,

$$(\alpha_5 - p_3) + (\alpha_4 - p_2) + (\beta_6 - 2\gamma_2)u - (2\gamma_1 + \beta_4)w + 2\beta_5 p_1 \frac{u}{w} = 0. \tag{3.32}$$

Assume that the constant coefficients of the system of equations (3.1), except for (3.24), satisfy additional conditions (3.17) that do not contradict the adequacy of the mathematical model.

Constraints (3.17), (3.24) on the constant coefficients of the nonlinear dynamic system (3.1) lead to the following constraint system, which does not contradict the adequacy and nontriviality of the mathematical model,

$$\begin{cases} \beta_6 = 2\gamma_2, \\ \beta_8 = 3\gamma_2, \\ \beta_7 - \gamma_3 - \beta_5 = 0, \\ \beta_4 - \gamma_1 - \beta_9 = 0, \\ \alpha_6 = 2(\alpha_4 - p_2) > 0, \\ (\alpha_5 - p_3) = -(\alpha_4 - p_2) < 0. \end{cases} \tag{3.33}$$

Taking into account (3.33), equality (3.32) can be rewritten as follows:

$$u = \frac{2\gamma_1 + \beta_4}{2\beta_5 p_1} w^2. \tag{3.34}$$

Now, (3.34) in the physical value of the first quarter of the phase plane of solutions of the resulting system (3.27) is a parabola passing through the origin of coordinates.

Theorem 3.3. *Problem* (3.28), (3.29), (3.33) *in some simply connected domain* $D \in (O, u(t), w(t))$ *of the first quarter of the phase plane* $(O, u(t), w(t))$ *has the solution in the form of the closed trajectory which completely lies in this domain.*

Proof. Consider a curve on the phase plane (O, u, w) where the vector field divergence becomes equal to zero. According to (3.34) this curve will be a parabola.

The divergence of the vector field in the physically miningful first quarter of the phase plane of solutions turns to zero on a parabola. Suppose that the starting point $M_7(u_2, w_2)$ also belongs to parabola (3.34), when $u > 0$. It is clear that the $G_2(u(t), w(t))$ divergence (3.30), (3.33) of the vector

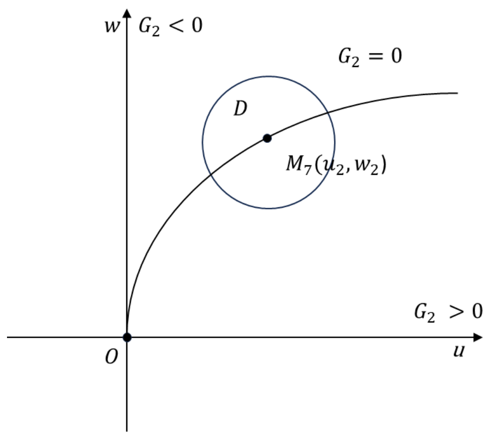

FIGURE 6

field $\overrightarrow{F}(F_1, F_2)$, in some domain $D \in (O, u(t), w(t))$, containing the point $M_7(u_2, w_2)$, changes its sign (Figure 6).

According to the Bendixson criterion, there is a closed integral trajectory of the dynamic system (3.28), (3.29), (3.33) that lies entirely in this domain [1]. □

Now, consider the case where the coefficients of the dynamic system (3.1) satisfy the system

$$\begin{cases} \beta_6 = 2\gamma_2, \\ \beta_8 = 3\gamma_2, \\ \beta_7 - \gamma_3 - \beta_5 = 0, \\ \beta_4 - \gamma_1 - \beta_9 = 0, \\ \alpha_6 = (\alpha_4 - p_2) - (\alpha_5 - p_3), \\ (\alpha_5 - p_3) + (\alpha_4 - p_2) \neq 0. \end{cases} \tag{3.35}$$

Taking into account (3.32), system (3.35) can be rewritten as follows:

$$u = \frac{2\gamma_1 + \beta_4}{2\beta_5 p_1} w^2 - \frac{(\alpha_5 - p_3) + (\alpha_4 - p_2)}{2\beta_5 p_1} w. \tag{3.36}$$

Now, (3.36) in the physical meaning of the first quarter of the phase plane of solutions of system (3.27) is a branch of the parabola in the case of $(\alpha_5 - p_3) + (\alpha_4 - p_2) > 0$ when passing through the point $M_8\big(0, \frac{(\alpha_5 - p_3) + (\alpha_4 - p_2)}{2\gamma_1 + \beta_4}\big)$ (Figure 7), and in the case of $(\alpha_5 - p_3) + (\alpha_4 - p_2) < 0$ when passing through the origin of coordinates (Figure 8).

Theorem 3.4. *Problem* (3.28), (3.29), (3.35) *in some simply connected domain* $D \in (O, u(t), w(t))$ *the first quarter of the phase plane* $(O, u(t), w(t))$ *has the solution in the form of the closed trajectory which completely lies in this domain.*

Proof. Consider a curve on the phase plane (O, u, w) where the vector field divergence becomes equal to zero. According to (3.32), (3.35) this curve will be a parabola (3.36).

The divergence of the vector field in the physically meaningful first quarter of the phase plane of solutions turns to zero on a parabola. Suppose that the starting point $M_9(u_2, w_2)$ (Figure 7) or $M_{10}(u_2, w_2)$ (Figure 8) also belongs to parabola (3.36), when $u > 0$, $w > 0$.

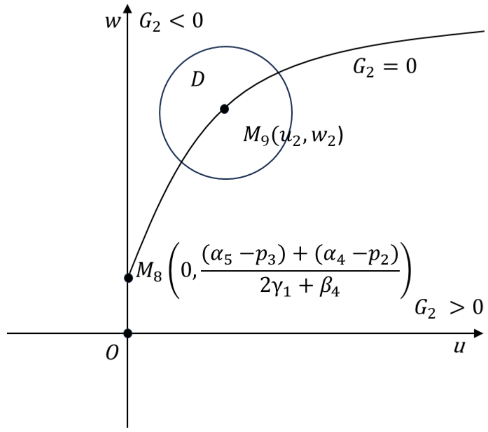

Figure 7

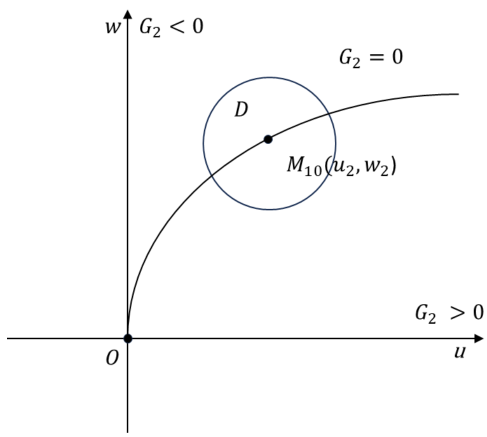

Figure 8

It is clear that the $G_2(u(t), w(t))$ divergence (3.30), (3.33) of the vector field $\overrightarrow{F}(F_1, F_2)$, in some domain $D \in (O, u(t), w(t))$, containing the point $M_9(u_2, w_2)$ (Figure 7), or the point $M_{10}(u_2, w_2)$ (Figure 8), changes its sign.

According to the Bendixson criterion, there is a closed integral trajectory of the dynamic system (3.28), (3.29), (3.35) that lies entirely in this domain [1]. □

Conclusion

Thus, the paper examines a three-dimensional nonlinear dynamic system describing the interaction of three populations speaking three different Colchian, Georgian and Svan languages. The period when the Colchian population predominated over the other two is considered. In the absence of quadratic terms characterizing the self-restriction of population growth and certain relationships between the constant coefficients of a nonlinear three-dimensional dynamic system, the first integral in the form of a hyperbolic paraboloid is found. Using the first integral, the three-dimensional nonlinear dynamic system is reduced to a two-dimensional system. For some relations between the constant coefficients of a two-dimensional dynamical system, the theorems on the existence of a closed trajectory in a simply connected domain of the first quarter of the phase plane of solutions are proved using the Bendixson principle.

In the case of quadratic terms characterizing the self-restriction of population growth and certain relationships between the constant coefficients of a nonlinear three-dimensional dynamic system, the first integrals are found. In one case, a system is a cone, and in the other, a hyperbolic paraboloid. In both cases, using the first integral, the general three-dimensional nonlinear dynamic system is reduced to a two-dimensional system. For some relations between the constant coefficients of a two-dimensional dynamical system, the theorems on the existence of a closed trajectory in a simply connected domain of the first quarter of the phase plane of solutions are proved using the Bendixson principle.

Thus, in all these cases, it is proved that all three populations living in the same region and speaking three different languages, coexist and do not completely assimilate them.

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