Transactions of A. Razmadze
Mathematical Institute
Vol. **179** (2025), issue 3, 351–354

THE EXACT UPPER BOUNDS OF LENGTHS OF SUMS OF VECTORS OF CERTAIN FINITE SYSTEMS OF VECTORS IN THE EUCLIDEAN $\mathbb{R}^n$ SPACES

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Abstract. In the paper some theorems representing the exact upper bounds of lengths of sums of vectors of certain finite systems of vectors in the Euclidean $\mathbb{R}^n$ spaces are proved and several of their applications are indicated.

Geometric inequalities constitute an important topic in contemporary mathematics. They have numerous applications in various fields of mathematics (see [1–6]).

In this note, we prove inequalities that can be used to solve various purely theoretical problems, as well as the tasks of practical importance and closely related to estimation of certain values. Let us consider some of them.

Example 1. Let a point object located at point A be shifted successively by finitely many vectors $X_1, X_2, \dots, X_k$ to point B. What is the maximal distance between A and B if A is the common origin of $X_1, X_2, \dots, X_k$ and belongs to the convex hull of $\{X_1, X_2, \dots, X_k\}$, and the length of each vector of the set $\{X_1, X_2, \dots, X_k\}$ does not exceed d, where d is a fixed positive real number.

Example 2. Consider the following class of problems with parameters $V, m_1, m_2, \dots, m_k$, where k is any fixed natural number such that $k \geq 4$. Let V be a convex polyhedron with vertices $v_1, v_2, \dots, v_k$, and $B_1, B_2, \dots, B_k$ be point bodies with masses $m_1, m_2, \dots, m_k$ respectively, located at $v_1, v_2, \dots, v_k$ respectively. Consider the magnitude of the total gravitational force acting from the bodies $B_1, B_2, \dots, B_k$ on a point body C with unit mass located at some point A of V. Find the exact upper bound of the magnitude of the total gravitational force for this class, if A is at a distance of at least r from every vertex of V and $\max\{m_1, m_2, \dots, m_k\} = M$, where r and M are fixed positive real numbers.

Example 3. Consider the following class of problems with parameters $V, q_1, q_2, \dots, q_k$, where k is any fixed natural number such that $k \geq 4$. Let V be a convex polyhedron with vertices $v_1, v_2, \dots, v_k$, inscribed in a unit sphere S such that the center of S belongs to V. Let $B_1, B_2, \dots, B_k$ be the point bodies with positive charges $q_1, q_2, \dots, q_k$ respectively, located at $v_1, v_2, \dots, v_k$ respectively. Consider the magnitude of the total Coulomb force acting from the bodies $B_1, B_2, \dots, B_k$ on a point body C with unit charge located at the center of S. Find the exact upper bound of the magnitude of the total Coulomb force for this class, if $\max\{q_1, q_2, \dots, q_k\} = Q$, where Q is a fixed positive real number.

Let us formulate and prove some inequalities closely related to the above mentioned examples. Note that, everywhere below, $\mathbb{Z}$ stands for the set of all integer numbers and $\omega_{i,j}$ stands for the angle between radius vectors x_i and x_j, and $\operatorname{conv}\{x_1, \dots, x_k\}$ stands for the convex hull of the set $\{x_1, \dots, x_k\}$.

Theorem 1. *Let $x_1, x_2, \dots, x_k$ be finitely many radius vectors in the Euclidean $\mathbb{R}^n$ space such that $O \in \operatorname{conv}\{x_1, \dots, x_k\}$, where O is the origin in $\mathbb{R}^n$, and $k \in (0, +\infty) \cap \mathbb{Z}$ and $n \in (0, +\infty) \cap \mathbb{Z}$.*

If $|x_1| \leq 1$, $|x_2| \leq 1$, $\dots$, $|x_k| \leq 1$, then $|x_1 + x_2 + \dots + x_k| \leq k - 1$.

Proof. The relation $O \in \operatorname{conv}\{x_1, \dots, x_k\}$ implies that there exist non-negative real numbers $\alpha_1, \alpha_2, \dots, \alpha_k$ such that $\alpha_1 + \alpha_2 + \dots + \alpha_k = 1$ and

$$\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_k x_k = 0. \tag{1}$$

2020 *Mathematics Subject Classification.* 51N20, 52A40.
Key words and phrases. Convex hull; R^n space; Finite vector system; Exact upper bound.

Without loss of generality, suppose that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k$. Taking into account that $\alpha_1, \alpha_2, \ldots, \alpha_k$ are non-negative real numbers and $\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$, we have $\alpha_k > 0$. According to (1), we also have:

$$\left(-\frac{\alpha_1}{\alpha_k}\right) x_1 + \left(-\frac{\alpha_2}{\alpha_k}\right) x_2 + \cdots + \left(-\frac{\alpha_{k-1}}{\alpha_k}\right) x_{k-1} = x_k. \tag{2}$$

According to (2), we obtain

$$\begin{aligned} |x_1 + x_2 + \cdots + x_k| &= \left| \left(1 - \frac{\alpha_1}{\alpha_k}\right) x_1 + \left(1 - \frac{\alpha_2}{\alpha_k}\right) x_2 + \cdots + \left(1 - \frac{\alpha_{k-1}}{\alpha_k}\right) x_{k-1} \right| \\ &\leq \left|1 - \frac{\alpha_1}{\alpha_k}\right| |x_1| + \left|1 - \frac{\alpha_2}{\alpha_k}\right| |x_2| + \cdots + \left|1 - \frac{\alpha_{k-1}}{\alpha_k}\right| |x_{k-1}|. \end{aligned} \tag{3}$$

Taking into account that $0 \leq 1 - \frac{\alpha_i}{\alpha_k} \leq 1$ and $|x_i| \leq 1$ for each $i \in [1, k] \cap \mathbb{Z}$, by virtue of (3), we have:

$$|x_1 + x_2 + \cdots + x_k| \leq k - 1.$$

Theorem 1 is thus proved. □

Remark 1. It directly follows from the presented proof of Theorem 1 that $|x_1 + x_2 + \cdots + x_k| = k - 1$ if and only if $\alpha_1 = \alpha_2 = \cdots = \alpha_{k-1} = 0$, $\alpha_k = 1$, $x_1 = x_2 = \cdots = x_{k-1}$, $|x_1| = |x_2| = \cdots = |x_{k-1}| = 1$ and $|x_k| = 0$.

Remark 2. Note that Theorem 1 can also be proved using the principle of mathematical induction with respect to the dimension of the space $\mathbb{R}^n$. Namely, one can prove Theorem 1 for the case $n = 2$ and then use the orthogonal projection onto a specially chosen hyperplane of the space.

Theorem 2. *Let $x_1, x_2, \ldots, x_k$ be finitely many radius vectors in the Euclidean $\mathbb{R}^n$ space such that $O \in \operatorname{conv}\{x_1, \ldots, x_k\}$, where O is the origin in $\mathbb{R}^n$, and $k \in (1, +\infty) \cap \mathbb{Z}$ and $n \in (0, +\infty) \cap \mathbb{Z}$.*

If $|x_1| = |x_2| = \cdots = |x_k| = 1$, then $|x_1 + x_2 + \cdots + x_k| \leq k - 2$.

Proof. It is clear from the conditions of Theorem 2 that $k \geq 2$. The relation $O \in \operatorname{conv}\{x_1, \ldots, x_k\}$ implies that there exist non-negative real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$ and

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k = 0. \tag{4}$$

Without loss of generality, we may suppose that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k$. Taking into account that $\alpha_1, \alpha_2, \ldots, \alpha_k$ are non-negative real numbers and $\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$, we have that $\alpha_k > 0$. According to (4) we get:

$$\left(-\frac{\alpha_1}{\alpha_k}\right) x_1 + \left(-\frac{\alpha_2}{\alpha_k}\right) x_2 + \cdots + \left(-\frac{\alpha_{k-1}}{\alpha_k}\right) x_{k-1} = x_k. \tag{5}$$

According to (5), we have:

$$\begin{aligned} 1 &= \left(\frac{\alpha_1}{\alpha_k}\right)^2 + \left(\frac{\alpha_2}{\alpha_k}\right)^2 + \cdots + \left(\frac{\alpha_{k-1}}{\alpha_k}\right)^2 \\ &\quad + \cdots + 2\sum_{i<j} \left(\frac{\alpha_i}{\alpha_k}\right)\left(\frac{\alpha_j}{\alpha_k}\right) \cos w_{i,j} \\ &\leq \sum_{p=1}^{k-1} \left(\frac{\alpha_p}{\alpha_k}\right)^2 + 2\sum_{i<j} \left(\frac{\alpha_i}{\alpha_k}\right)\left(\frac{\alpha_j}{\alpha_k}\right) \\ &= \left(\frac{\alpha_1}{\alpha_k} + \frac{\alpha_2}{\alpha_k} + \cdots + \frac{\alpha_{k-1}}{\alpha_k}\right)^2 = \left(\frac{\alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}}{\alpha_k}\right)^2. \end{aligned} \tag{6}$$

It is clear that $\alpha_k > 0$ and $0 \leq \frac{\alpha_i}{\alpha_k} \leq 1$ for each $i \in [1, k] \cap \mathbb{Z}$. Therefore

$$0 \leq \frac{\alpha_1}{\alpha_k} + \frac{\alpha_2}{\alpha_k} + \cdots + \frac{\alpha_{k-1}}{\alpha_k} \leq k - 1. \tag{7}$$

According to (6) and (7), we have:

$$1 \leq \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}}{\alpha_k} \leq k - 1. \tag{8}$$

$$\begin{aligned} |x_1 + x_2 + \cdots + x_{k-1} + x_k| &= \bigg| x_1 + x_2 + \cdots + x_{k-1} + \left(-\frac{\alpha_1}{\alpha_k}\right) x_1 \\ &\quad + \left(-\frac{\alpha_2}{\alpha_k}\right) x_2 + \cdots + \left(-\frac{\alpha_{k-1}}{\alpha_k}\right) x_{k-1} \bigg| \\ &= \left| \left(1 - \frac{\alpha_1}{\alpha_k}\right) x_1 + \left(1 - \frac{\alpha_2}{\alpha_k}\right) x_2 + \cdots + \left(1 - \frac{\alpha_{k-1}}{\alpha_k}\right) x_{k-1} \right|. \end{aligned} \tag{9}$$

According to (9), we infer that:

$$\begin{aligned} |x_1 + x_2 + \cdots + x_{k-1} + x_k|^2 &= \left(1 - \frac{\alpha_1}{\alpha_k}\right)^2 \\ &\quad + \left(1 - \frac{\alpha_2}{\alpha_k}\right)^2 + \cdots + \left(1 - \frac{\alpha_{k-1}}{\alpha_k}\right)^2 \\ +2\sum_{i<j} \left(1 - \frac{\alpha_i}{\alpha_k}\right)\left(1 - \frac{\alpha_j}{\alpha_k}\right) \cos w_{i,j} &\leq \sum_{p=1}^{k-1} \left(1 - \frac{\alpha_p}{\alpha_k}\right)^2 + 2\sum_{i<j} \left(1 - \frac{\alpha_i}{\alpha_k}\right)\left(1 - \frac{\alpha_j}{\alpha_k}\right) \\ &= \left(\left(1 - \frac{\alpha_1}{\alpha_k}\right) + \left(1 - \frac{\alpha_2}{\alpha_k}\right) + \cdots + \left(1 - \frac{\alpha_{k-1}}{\alpha_k}\right)\right)^2 \\ &= \left((k-1) - \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}}{\alpha_k}\right)^2. \end{aligned} \tag{10}$$

Using (8) in (10), it is clear that:

$$|x_1 + x_2 + \cdots + x_k|^2 \leq (k-2)^2. \tag{11}$$

According to (11), we have:

$$|x_1 + x_2 + \cdots + x_k| \leq k - 2.$$

This completes the proof of Theorem 2. □

Remark 3. It can be derived from the presented proof of Theorem 2 that the equality $|x_1 + x_2 + \cdots + x_k| = k - 2$ in Theorem 2 takes place if and only if at least one of the following three conditions is satisfied:

1) $k = 2$;

2) $k = 3$, and there exist p and q such that $\{p, q\} \subset \{1, 2, 3\}$ and $x_p = -x_q$;

3) $k > 3$, and there exist p and q such that $\{p, q\} \subset [1, k] \cap \mathbb{Z}$ and $x_p = -x_q$ and $x_i = x_j$ for each $\{i, j\} \subset \big(([1, k] \setminus \{p, q\}) \cap \mathbb{Z}\big)$.

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(Received 19.09.2025)

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