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DYNAMIC CONTACT PROBLEM FOR VOLTERRA VISCOELASTIC MODEL

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Abstract. The dynaic boundary value contact problem for a half-space with an elastic inclusion is considered under the conditions of the Volterra viscoelastic model. Using the methods of contour integration and integral transformations, the contact problem is reduced to an integro-differential equation with respect to a tangential stress jump. Employing also the properties and method of orthogonal polynomials, the integro-differential equation is reduced to an infinite system of linear algebraic equations. The quasi-complete regularity of the obtained system is proven, and the reduction method for an approximate solution is developed.

1. Introduction

The problem under considered refers to a wide class of contact and mixed problems in viscoelasticity theory. There are many previously studied problems dealing with various domains reinforced with a thin elastic inclusion or a patch of variable stiffness, for which both the exact and approximate solutions have been obtained. In particular, effective solutions were obtained in [6, 16] for a contact problem of a piecewise-homogeneous orthotropic plate and two-dimensional integro-differential equations related to contact problems for a viscoelastic plate with finite (semi-infinite) inclusions of variable stiffness. Approximate solutions of static and dynamic contact problems for elastic and viscoelastic (Kelvin-Voigt material) half-space with rigid or elastic inclusion are considered in [14, 15, 18].

In [17], the dynamic boundary value problem for a half-space with a cut is considered under the condition of the Volterra viscoelastic model. The problems are reduced to a quasiregular infinite system of linear algebraic equations, which allow us to obtain an approximate solution with any accuracy.

The present paper considers the dynamical contact problem for a viscoelastic half-space under the Volterra model conditions. The half-space is reinforced by an elastic inclusion of variable rigidity. The Volterra integral model describes much more accurately the behavior of the material, making the problem under consideration more relevant from applied and mathematical points of view. The problem is reduced to the Carleman-type problem of the theory of analytic functions, and the effective solution of this boundary value problem is constructed using the method of factorization. Based on the contact condition along the contact surface, the integro-differential equation for the shear stress jump is derived. The quasi-regularity of the equivalent infinite system of linear algebraic equations is proved.

2. Statement of the Problem

The paper studies the dynamical contact problem for a viscoelastic body in the form of a half-space $(-\infty < x, z < \infty, \quad y > 0)$, reinforced by an elastic inclusion in the form of a strip $(0 \leq y \leq \ b, \quad -\infty < z < \infty)$ lying in the plane $x = 0$. The outer border of the inclusion is under the action of uniformly distributed shearing (acting along the Oz axis) load of intensity $\tau_0\delta(y)H(t-t_0)$, $H(t-t_0)$ is the Heaviside function, $\delta(y)$ is the Dirac Delta function, t is the time parameter, and t_0 is the ageing of the material at the beginning of the loading. Under the so-called anti-plane deformation conditions, only the displacement component $\omega = \omega(x, y, t)$ and the shear stress components τ_{xz}, τ_{yz} are other than zero.

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The basic equations of the linear theory of creep for the Volterra materials, expressing the relation between the components of strain and stress, have the form

$$
\begin{aligned}
\frac{\partial\omega(x,y,t)}{\partial x} &= \frac{\tau_{xz}(x,y,t)}{G(t)} - \int\limits_{t_0}^{t} \tau_{xz}(x,y,\tau)K(t,\tau)d\tau \\
\frac{\partial\omega(x,y,t)}{\partial y} &= \frac{\tau_{yz}(x,y,t)}{G(t)} - \int\limits_{t_0}^{t} \tau_{yz}(x,y,\tau)K(t,\tau)d\tau,
\end{aligned}
\tag{1}
$$

where $G(t)$ is the instantaneous shear modulus, $K(t,\tau) = \frac{\partial}{\partial\tau}\Big(\frac{1}{G(t)} + \mu(t,\tau)\Big)$ is the shear strain kernel, $\mu(t,\tau) = \varphi(\tau)\psi(t-\tau)$ is the creep size of shear strain, $\varphi(\tau)$ is the so-called ageing function and $\psi(t-\tau)$ characterizes the heredity properties of a material.

We approximate the function $\mu(t,\tau)$ by the relation: $\mu(t,\tau) = (A + Be^{-\lambda\tau})[1 - e^{-\gamma(t-\tau)}]$ and suppose that $G(t) = G = \text{const}$, A, B, λ, γ are the positive constants [2,3,8–10].

Considering the equilibrium equations of dynamical viscoelasticity, we obtain from (1) the following boundary value problem:

$$
\begin{gathered}
\Delta\omega(x,y,t) = \rho\bigg(\frac{1}{G}\frac{\partial^2\omega}{\partial t^2} - \int\limits_0^t \frac{\partial}{\partial\tau}\Big[(A + Be^{-1\tau})(1 - e^{-\gamma(t-\tau)}\Big]\frac{\partial^2\omega}{\partial\tau^2}d\tau\bigg), \quad |x| < \infty, \quad y > 0, \\
\frac{\partial\omega(x,0,t)}{\partial y} = 0,
\end{gathered}
\tag{2}
$$

Δ is the two-dimensional Laplace operator, and ρ is the material density of the half-space. Along the contact surface, the shear stress is discontinuous and the displacement is continuous,

$$
\begin{gathered}
\langle\tau_{xx}(0,y,t)\rangle = \mu(y,t), \quad 0 < y < b; \quad \mu(y,t) = 0, \quad y \geq b, \\
\omega(-0,y,t) = \omega(+0,y,t) = \omega^{(1)}(y,t), \qquad \langle f(0,y,t)\rangle \equiv f(-0,y,t) - f(+0,y,t),
\end{gathered}
\tag{3}
$$

the displacement of the inclusion points $\omega^{(1)}(y,t)$ satisfies the condition

$$
\frac{\partial}{\partial y}h(y)\frac{\partial\omega^{(1)}(y,t)}{\partial y} - \frac{\rho_0 h(y)}{E_0}\frac{\partial^2\omega^{(1)}(y,t)}{\partial t^2} = -\frac{1}{E_0}\mu(y,t) - \frac{1}{E_0}\tau_0 H(t-t_0)\delta(y),
\tag{4}
$$

where $\mu(y,t)$ is an unknown contact stress at the point y at time moment t, acting onto the inclusion along the surface of its contact with a half-space, ρ_0 is a density and E_0 is the elasticity modulus of the inclusion material, $h(y)$ is its thickness. It is required to find the stresses and displacements fields in the half-space.

3. Solution of the Problems

The contour integration, the integral Fourier transform with respect to the variables x, y and the Laplace integral transform with respect to the variables t from (2), (3) yield [19]

$$
\begin{gathered}
-\overline{\omega}_{\alpha,\beta}(p) + g(p)\overline{\omega}_{\alpha,\beta}(p+\lambda) = \frac{-\overline{\mu}_\beta(p)}{G\,[\beta^2 + \alpha^2 + k(p)]}, \quad p = i\zeta + \varepsilon, \\
\overline{\omega}'_\alpha(0,p) = 0,
\end{gathered}
\tag{5}
$$

where $\overline{\omega}_{\alpha,\beta}(p)$ is the Fourier integral transform (with respect to the variables x, y) and the Laplace integral transform (with respect to the variables t) of the function $\omega(x,y,t)$, $\overline{\omega}'_\alpha(y,p) = \frac{\overline{\omega}_\alpha(y,p)}{\partial y}$ is the Fourier transform (with respect to the variables x) and the Laplace integral transform (with respect to the variables t) of the function $\frac{\omega(x,y,t)}{\partial y}$, $\overline{\mu}_\beta(p)$ is the integral Fourier transform (with respect to the variable y) and the Laplace integral transform (with respect to the variables t) of the function $\mu(y,t)$,

$$
g(p) = \frac{q(p)}{\beta^2 + \alpha^2 + k(p)}, \quad q(p) = \frac{B\rho\gamma(p+\lambda)^3}{p(p+\gamma)}, \quad k(p) \equiv \frac{\rho}{G}p^2 + \frac{A\rho\gamma p^2}{p+\gamma}, \quad \varepsilon \text{ is a positive number.}
$$

Introducing the notation

$$\bar{\omega}_{\alpha,\beta}(i\zeta+\varepsilon) \equiv \psi_{\alpha,\beta}(\zeta), \qquad \bar{\omega}_{\alpha,\beta}(i\zeta+\varepsilon+\lambda) = \psi_{\alpha,\beta}(\zeta-i\lambda), \tag{6}$$

we obtain the following condition of the Carlemman-type boundary value problem for a strip:

$$\psi_{\alpha,\beta}(\zeta) + \mu\frac{g_0(\zeta)}{\varepsilon+\lambda-i\zeta}\psi_{\alpha,\beta}(\zeta-i\lambda) = \frac{\overline{\mu}_\beta(i\zeta+\varepsilon)}{G\left[\beta^2+\alpha^2+k(i\zeta+\varepsilon)\right]^2}, \quad |\zeta|<\infty, \tag{7}$$

where

$$g_0(\zeta) = \frac{-\rho(i\zeta+\varepsilon+\lambda)^2\left[\zeta^2+(\lambda+\varepsilon)^2\right]}{G(i\zeta+\varepsilon)(i\zeta+\gamma+\varepsilon)(\alpha^2+\beta^2+k(i\zeta+\varepsilon))}, \quad \mu = BG\gamma.$$

The boundary value problem is formulated as follows:
Find a function $\psi_{\alpha,\beta}(z)$ which is analytic in the strip $-\lambda < \operatorname{Im} z < 0$, continuously extendable to the strip boundary, bounded at infinity and satisfies condition (7) [4, 5].

It is easy to show that $g_0(\zeta) \to 1$, $\zeta \to \pm\infty$, $\operatorname{Ind} g_0(\zeta) = 0$ (index of the function on the real axis is equal to zero) and $\lg_0(\zeta) \in L_1(-\infty,\infty)$ (is integrable on the real axis), therefore, this function can be represented as follows:

$$g_0(\zeta) = \frac{\mathrm{X}_0(\zeta-i\lambda)}{\mathrm{X}_0(\zeta)}, \quad |\zeta|<\infty, \quad \mathrm{X}_0(z) = \exp\Big(-\frac{1}{2i\lambda}\int\limits_{-\infty}^{\infty}\ln g_0(\zeta)\operatorname{cth}\frac{\pi}{\lambda}(\zeta-z)d\zeta\Big).$$

The function $\varepsilon+\lambda-i\zeta$ has the form

$$\varepsilon+\lambda-i\zeta = \frac{X_2(\zeta)}{X_2(\zeta-\mathrm{i}\lambda)}, \quad X_2(z) = \lambda^{-\frac{iz}{\lambda}}\Gamma\Big(\frac{\varepsilon+2\lambda-iz}{\lambda}\Big).$$

Moreover,

$$\mu = \frac{X_1(\zeta-i\lambda)}{X_1(\zeta)}, \quad X_1(z) = \exp\Big(\frac{iz}{\lambda}\ln\mu\Big).$$

The functions $X_0(z)$ and $X_1(z)$ are holomorphic in the strip $-\lambda < \operatorname{Im} z < 0$, continuous at the border of this strip and bounded in a closed strip $-\lambda \le \operatorname{Im} z \le 0$.

Using Stirling's formula about the $\Gamma(z)$ function [1], for a sufficiently large $|z|$, the function $X_2(z)$ satisfies the estimate

$$X_2(z) = O\left(|x|^{\frac{3}{2}+\frac{y+5}{\lambda}}\,e^{-\frac{\pi}{2\lambda}|x|}\right), \quad z = x+iy, \quad |x|\to\infty, \quad -\lambda\le y\le 0. \tag{8}$$

Therefore, condition (7) can be rewritten as follows:

$$H(\zeta) + H(\zeta-i\lambda) = \frac{\bar{\mu}_\beta(i\zeta+\varepsilon)}{G\left[\beta^2+\alpha^2+k(i\zeta+\varepsilon)\right]}X(\zeta), \quad |\zeta|<\infty, \tag{9}$$

where

$$H(z) = \mathrm{X}(z)\psi_{\alpha,\beta}(z), \quad \mathrm{X}(z) = \mathrm{X}_0(z)\mathrm{X}_1(z)\mathrm{X}_2(z), \quad -\lambda < \mathrm{Im}z < 0.$$

The solution of the boundary value problem (9) can be represented in the following form:

$$\psi_{\alpha,\beta}(z) = \frac{1}{2i\lambda G\mathrm{X}(z)}\int\limits_{-\infty}^{\infty}\frac{\bar{\mu}_\beta(i\zeta+\varepsilon)\mathrm{X}(\zeta)d\zeta}{\left[\beta^2+\alpha^2+k(i\zeta+\varepsilon)\right]sh(\pi/\lambda)(\zeta-z)}, \quad -\lambda < \mathrm{Im}z < 0. \tag{10}$$

Since the function $X(z)$ exponentially vanishes at infinity, the last integral also has this property and the function $\psi_{\alpha,\beta}(z)$, determined by formula (10), is holomorphic in the strip $-\lambda < \operatorname{Im} z < 0$ and is bounded at infinity.

Taking into account the notation (6), the solution to problem (5) has the form

$$G\bar{\omega}_{\alpha,\beta}(p_0) = -\frac{\bar{\mu}_\beta(p_0)}{2\left[\beta^2+\alpha^2+k(p_0)\right]}$$
$$-\frac{1}{2\lambda\mathrm{X}(-ip_0+i\varepsilon)}\int\frac{\mu_p(i\zeta+\varepsilon)\mathrm{X}(\zeta)d\zeta}{\left[\beta^2+\alpha^2+k(i\zeta+\varepsilon)\right]sh(\pi/\lambda)(\zeta+ip_0-i\varepsilon)}, \quad p_0 = i\zeta_0+\varepsilon.$$

The inverse Fourier integral transform with respect to the variables β and α results in

$$G\bar{\omega}(x,y,p_0)=-\int\limits_{-b}^{b}\bar{\mu}(\eta,p_0)d\eta\frac{1}{2\pi}\int\limits_{0}^{\infty}\frac{e^{-\sqrt{\alpha^2+k(p_0)}|\eta-\eta|}}{\sqrt{\alpha^2+k(p_0)}}\cos\alpha x d\alpha$$

$$-\frac{1}{i\lambda}\int\limits_{-b}^{b}\bar{\mu}(\eta,p_0)d\eta\frac{1}{2\pi}\int\limits_{0}^{\infty}H_a(|y-\eta|,\zeta_0)\cos\alpha x d\alpha, \tag{11}$$

where

$$H_\alpha(|y-\eta|,\zeta_0)=\frac{1}{\mathrm{X}(\zeta_0)}\int\limits_{-\infty}^{\infty}\frac{\mathrm{e}^{-\sqrt{\alpha^2+k(i_5+\varepsilon)}|\nu-\eta|}}{\sqrt{\alpha^2+k(\mathrm{i}\zeta+\varepsilon)}}\frac{\mathrm{X}(\zeta)d\zeta}{sh(\pi/\lambda)(\zeta-\zeta_0)}.$$

Using the formula GR 3.951 (8) (see [11]), the right-hand side of formula (11) can be represented as a sum of integrals with principal and regular kernels:

$$G\bar{\omega}(x,y,p_0)=-\frac{1}{2\pi}A(\zeta_0)\int\limits_{-b}^{b}\ln\frac{1}{x^2+(y-\eta)^2}\overline{\mu}(\eta,p_0)d\eta$$

$$-\frac{1}{2\pi}\int\limits_{-b}^{b}R(x,|y-\eta|,p_0)\overline{\mu}(\eta,p_0)d\eta, \tag{12}$$

where

$$R(x,|y-\eta|,p_0)=\int\limits_{0}^{\infty}\left[\frac{e^{-\sqrt{\alpha^2+k(p_0)}|y-\eta|}}{\sqrt{\alpha^2+k(p_0)}}-\frac{e^{-\alpha|y-\eta|}}{\alpha}\right]\cos\alpha x d\alpha$$

$$+\frac{1}{i\lambda}\int\limits_{0}^{\infty}\left[H_\alpha(y-\eta,\zeta_0)-A(\zeta_0)\frac{e^{-\alpha|y-\eta|}}{\alpha}\right]\cos\alpha x d\alpha,$$

$$A(\zeta_0)=1+\frac{1}{\lambda i\mathrm{X}(\zeta_0)}\int\limits_{-\infty}^{\infty}\frac{\mathrm{X}(\zeta)d\zeta}{sh(\pi/\lambda)(\zeta-\zeta_0)},\quad p_0=i\zeta_0+\varepsilon.$$

Taking into account the condition of contact between the inclusion and the half-space $\overline{\omega}(0,y,p_0)=\overline{\omega}^{(1)}(y,p_0)$ and formula (12), performing then the Laplace integral transform ($L\{\cdot\}$) on the both parts of equation (4), we obtain the following integro-differential equation:

$$\left(\frac{d}{dy}h(y)\frac{d}{dy}+\frac{\rho_0p_0^2h(y)}{E_0}\right)\left(\frac{1}{2\pi}\int\limits_{-1}^{1}\ln\frac{1}{|y-\eta|}\overline{\mu}(\eta,p_0)d\eta+\int\limits_{-1}^{1}Q(|y-\eta|,p_0)\overline{\mu}(\eta,p_0)d\eta\right)$$

$$=\frac{G}{A(\zeta_0)E_0}\overline{\mu}(y,p_0)+\frac{G\tau_0e^{-p_0t_0}}{A(\zeta_0)E_0p_0}\delta(y) \tag{13}$$

under the condition that

$$\int\limits_{-1}^{1}\bar{\mu}(\eta,p_0)d\eta=2\tau_0\frac{e^{-t_0p_0}}{p_0}, \tag{14}$$

where $Q(|y-\eta|,p_0)=\frac{1}{2\pi}\frac{R(0,|y-\eta|,p_0)}{A(\zeta_0)}$, $\bar{\mu}(\eta,p_0)=L\{\mu(\eta,t)\}$.

Therefore, the problem is reduced to the integro-differential equation (13) with condition (14). A solution to problem (13), (14) is sought in the form

$$\overline{\mu}(\eta,p_0)=\frac{a_0(p_0)}{\sqrt{1-\eta^2}}+\frac{1}{\sqrt{1-\eta^2}}\sum_{m=1}^{\infty}a_m(p_0)T_m(\eta), \tag{15}$$

where $T_m(\eta)$ is the first kind Chebyshev orthogonal polynomial, $\{a_n(p_0)\}_{n\geq 1}$ an unknown functional sequence. By virtue of the inclusion equilibrium conditions (14), we obtain

$$a_0(p_0) = 2\tau_0 \frac{e^{-t_0 p_0}}{\pi p_0}\,. \tag{16}$$

Let us assume that the thickness of the inclusion varies according to the following law: $h(x) = h_0\sqrt{1-x^2}$, $|x|<1$, $h_0 = \text{const}$. Using the Rodrigues formula for the Jacoby orthogonal polynomials [20] and the following spectral relation

$$\frac{1}{\pi}\int_{-1}^{1} \ln\frac{1}{|x-y|}\frac{T_m(y)dy}{\sqrt{1-y^2}} = \mu_m T_m(x), \qquad \mu_m = \begin{cases} \ln 2, & m=0 \\ \frac{1}{m}, & m\neq 0 \end{cases}$$

and also the conditions of orthogonality of the first kind Chebyshev polynomials, from (13–15), we obtain the infinite system of linear algebraic equations ($n \geq 1$)

$$\delta_n(p_0)a_n(p_0) + \sum_{m-1}^{\infty} L_{mn}(p_0)a_m(p_0) = a_0(p_0)g_n(p_0), \quad n = 1,2,3,\ldots, \tag{17}$$

where

$$L_{mn}(p_0) = \frac{\rho_0 {p_0}^2 h_0}{2E_0} L_{mn}^{(1)} + L_{mn}^{(2)}(p_0), \quad L_{mn}^{(1)} = \frac{1}{m}\int_{-1}^{1}\sqrt{1-y^2}T_m(y)T_m(y)dy,$$

$$L_{mn}^{(2)}(p_0) = \int_{-1}^{1} T_n(y)\left(\int_{-1}^{1} K(|y-\eta|, p_0)\frac{T_m(\eta)d\eta}{\sqrt{1-\eta^2}}\right)dy,$$

$$g_n(p_0) = -\frac{G\gamma(p_0)}{2E_0}\int_{-1}^{1} T_n(y)\delta(y)dy$$

$$-\frac{\rho_0 p_0^2 h_0}{\pi E_0}\ln 2\int_{-1}^{1}\sqrt{1-y^2}T_n(y)dy - \frac{2}{\pi}\int_{-1}^{1} T_n(y)\left(\int_{-1}^{1}\frac{K(|y-\eta|\,, p_0)d\eta}{\sqrt{1-\eta^2}}\right)dy,$$

$$\delta_n(p_0) = \frac{\pi h_0}{4}n + \frac{\pi\gamma(p_0)G}{2E_0},$$

$$K(|y-\eta|\,, p_0) = \frac{\partial}{\partial y}\sqrt{1-y^2}\frac{\partial Q(|y-\eta|\,, p_0)}{\partial y}$$

$$+\frac{\rho_0 p_0^2}{E_0}\sqrt{1-y^2}Q(|y-\eta|\,, p_0), \quad \gamma(p_0) = \frac{1}{A(-ip_0 + i\varepsilon)}\,.$$

Using the properties of the first kind Chebyshev orthogonal polynomials and Gamma function, we get [1, 20]

$$L_{mn}^{(1)} = \frac{1}{m}\begin{cases} \pi/4, & m = n \neq 1, \\ \pi/8, & m = n = 1, \\ -\pi/8, & m = n \pm 2, \\ 0, & m \neq n, \quad m \neq n \pm 2, \end{cases}$$

$$L_{mn}^{(2)}(p_0) = \frac{\sqrt{\pi}\Gamma(m+1)}{8\Gamma(m+0.5)m(m-1)}\int_{-1}^{1} T_n(y) \times \left(\int_{-1}^{1}(1-\eta^2)^{3/2}P_{m-2}^{(3/2,3/2)}(\eta)\frac{\partial^2 K(|y-\eta|\,, p_6)}{\partial\eta^2}d\eta\right)dy,$$

$$g_n(p_0) = -\frac{G\gamma(p_0)}{2E_0}\cos\frac{\pi n}{2} - \frac{2}{\pi}\int_{-1}^{1} T_n(y)\left(\int_{-1}^{1}\frac{K(|y-\eta|\,, p_0)d\eta}{\sqrt{1-\eta^2}}\right)dy, \quad n \neq 2,$$

$$g_2(p_0) = \frac{G\gamma(p_0)}{E_0} + \frac{\rho_0 {p_0}^2 h_0}{4E_0} \ln 2 - \frac{2}{\pi} \int_{-1}^{1} T_2(y) \Bigg(\int_{-1}^{1} \frac{K(|y-\eta|\,, p_0) d\eta}{\sqrt{1-\eta^2}} \Bigg) dy,$$

$$\delta_n(p_0) = O(n), \quad n \to \infty.$$

We rewrite system (17) in the following form:

$$a_n(p_0) + \sum_{m-1}^{\infty} \tilde{L}_m(p_0) a_m(p_0) = a_0(p_0) \tilde{g}_n(p_0), \quad n = 1, 2, 3, \dots, \tag{18}$$

where

$$\tilde{g}_n(p_0) = \frac{g_n(p_0)}{\delta_n(p_0)}, \quad \tilde{L}_{nn}(p_0) = \frac{L_{nn}(p_0)}{\delta_n(p_0)}.$$

Based on the previous representations, for system (18), we obtain the conditions

$$\sum_{n-1, m-1,}^{\infty} \left|\tilde{L}_{nm}(p_0)\right|^2 < \infty, \quad \sum_{n-1}^{\infty} \left|\tilde{g}_n(p_0)\right| < \infty. \tag{19}$$

The above conditions (19) prove that the infinite system (18) is quasi-completely regular in the space l_2, that is, their solutions satisfy the condition $\sum_{n-1}^{\infty} a_n^2(p_0) < \infty$.

The results of [12, p. 534] are applicable to the infinite system (18). Relying on this fact, the system

$$a_n^N(p_0) + \sum_{m=1}^{N} \tilde{L}_{mn}(p_0) a_m^N(p_0) = a_0(p_0) \tilde{g}_n(p_0), \quad n = 1, 2, \dots, N, \tag{20}$$

is solvable for sufficiently large N, and the convergence of approximate solutions $\{a_n^N(p_0)\}_{n=1,\dots,N}$ to exact solution $\{a_n(p_0)\}_{n\geq 1}$ is valid in the sense of the norm of the space l_2.

The convergence rate is determined by the inequality

$$\left\|a(p_0) - \varphi_0^{-1} \overline{a}^N(p_0)\right\|_{l_2} \leq C_1(p_0) \Bigg[\sum_{n=N+1}^{\infty} \sum_{m=1}^{\infty} |\tilde{L}_{mn}(p_0)|^2 \Bigg]^{1/2} + C_2(p_0) \left(\frac{\sum\limits_{n=N+1}^{\infty} |\tilde{g}_n(p_0)|^2}{\sum\limits_{n=1}^{\infty} |\tilde{g}_n(p_0)|^2} \right)^{1/2},$$

where $a(p_0) = \{a_n(p_0)\}_{n\geq 1} = (a_1(p_0), a_2(p_0), \dots, a_n(p_0), \dots)$ is the solution of system (17), $\overline{a}^N(p_0) = (a_1^N(p_0), a_2^N(p_0), \dots, a_N^N(p_0))$ is the solution of system (20), $\varphi_0^{-1}\overline{a}^N(p_0) = (a_1^N(p_0), a_2^N(p_0), \dots, a_N^N(p_0), 0, 0, \dots)$.

Considering the expressions for $\widetilde{L}_{nm}(p_0)$ and $\widetilde{g}_n(p_0)$, we have

$$C_1(p_0) \left[\sum_{n=N+1}^{\infty} \sum_{m=1}^{\infty} |\tilde{L}_{nm}(p_0)|^2 \right]^{1/2} \leq C^*(p_0) \left[\sum_{n=1}^{\infty} \frac{1}{(n+N)^4} \right]^{1/2} = C^*(p_0) \left[\zeta(4, N)\right]^{1/2},$$

$$C_2(p_0) \left(\frac{\sum\limits_{n=N+1}^{\infty} |\tilde{g}_n(p_0)|^2}{\sum\limits_{n=1}^{\infty} |\tilde{g}_n(p_0)|^2} \right)^{1/2} = C^{**}(p_0) \left[\sum_{n=1}^{\infty} \frac{1}{(n+N)^2} \right]^{1/2} = C^{**}(p_0) \left[\zeta(2, N)\right]^{1/2},$$

where $\zeta(s, N)$ is the known generalized Zeta function.

Using the asymptotic formula of the generalized Zeta function [7], we obtain the following inequality:

$$\left\|a(p_0) - \varphi_0^{-1} \overline{a}^N(p_0)\right\|_{l_2} \leq C(p_0) N^{-1/2}. \tag{21}$$

Thus, the solutions of system (18) can be constructed by the reduction method with any accuracy [12, 13], and the rate of convergence is determined by inequality (21).

From the properties of the functions $K(|y-\eta|, p_0)$ and $g_n(p_0)$, with respect to the variable p_0, and from formula (16), it follows that the solution of system (18) (hence the function $\bar{\mu}(\eta, p_0)$) is an analytic function in the half-plane $\operatorname{Re} p_0$, tending to zero, so $\exp(-t_0 p_0)$, $|p_0| \to \infty$, $(t_0 > 0)$. Therefore, an inverse Laplace integral transform exists:

$$\mu(y,t) = \frac{1}{2\pi i} \int\limits_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \bar{\mu}(\eta, p_0) e^{p_0 t} dp_0, \quad \sigma_0 > 0.$$

The problem can be solved under the assumptions where the inclusion thickness varies according to a different law, including when it is constant.

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