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A NUMERICAL ALGORITHM OF SOLVING A NONLINEAR INTEGRO–DIFFERENTIAL STRING EQUATION AND ITS ERROR

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Abstract. The paper considers an initial-boundary value problem for the Kirchhoff equation $w_{tt} = \varphi\big(\int_0^\pi w_x^2 dx\big) w_{xx}$ describing the oscillation of a string. It is assumed that $\varphi(z) \in C^p[0,\infty)$, $\varphi(z) \geq \alpha > 0$, where p is either 1 or 2, and the coefficients $a_i^{(l)}$ of expansions into a Fourier sine-series, $i = 1, 2, \dots$, of the initial functions $w^l(x)$, $l = 0, 1$, satisfy the inequality $|a_i^{(l)}| \leq \omega i^{l-(p+s+2.5)}$, where ω and s are the positive numbers. As S. Bernstein showed, this requirement guarantees the existence of a local solution to the problem posed. To find it, the numerical algorithm is constructed, consisting of three parts: the Galerkin method, a modified Crank-Nicolson difference scheme, and a Picard type iterative process. The algorithm error is estimated.

1. The Problem

1.1. Formulation of the problem and its background.

In 1876 [19], G. Kirchhoff, when refining D'Alembert's linear model, derived an equation for a string of the form

$$w_{tt}(x,t) - \left(\alpha_0 + \alpha_1 \int_0^\pi w_x^2(x,t)\,dx\right) w_{xx}(x,t) = 0, \qquad (1.1)$$

$$0 < x < \pi, \quad 0 < t \leq T,$$

where $\alpha_l = \text{const} > 0$, $l = 0, 1$.

In the present paper, we consider the following initial boundary value problem for the generalized Kirchhoff equation

$$w_{tt}(x,t) = \varphi\left(\int_0^\pi w_x^2(x,t)\,dx\right) w_{xx}(x,t), \quad 0 < x < \pi, \quad 0 < t \leq T, \qquad (1.2)$$

$$w(x,0) = w^0(x), \quad w_t(x,0) = w^1(x), \quad w(0,t) = w(\pi,t) = 0, \qquad 0 \leq x \leq \pi, \quad 0 \leq t \leq T, \qquad (1.3)$$

with the aim to construct for it an approximate method and estimate the method accuracy. In (1.2), (1.3), the functions $\varphi(z)$ and $w^l(x)$ are known, $l = 0, 1$, and

$$\varphi(z) \geq \alpha, \quad 0 \leq z < \infty, \quad \alpha = \text{const} > 0. \qquad (1.4)$$

The mechanical meaning of equation (1.2) [2] is to describe a dynamic string under conditions of nonlinear stress-strain dependence, while equation (1.1), being a particular form of equation (1.2) with $\varphi(z) = \alpha_0 + \alpha_1 z$, is applicable in the case of the linear Hooke's law.

Many experts have studied equations (1.1), (1.2) and their various versions and generalizations, including the equation $w_{tt}(t) + a(\|A^{\frac{1}{2}} w(t)\|^2) A w(t) = f(t)$, $a(s) \geq a_0 > 0$, $A = A^* > 0$. We mention only some of the works [1–3,5–8,10,12–15,20–23,25,26,28–34,38,39] and especially accentuate paper [8] as a pioneering mathematical study of equation (1.2). Most frequently, the authors were interested in such questions as the solvability and uniqueness of a solution, its continuous dependence on the initial data, as well as control and stability. As for the problem of constructing and finding the properties of numerical methods for the above equations, it should be noted that there are relatively few published

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works in this direction. Various aspects of the algorithm construction for the Kirchhoff type equation (1.1) are considered in the works of Attigui [4], Bilbao [9], Chaudhary et al. [11], Kachakhidze et al. [18], Mbehou et al. [27], Liu and Rincon [24], Oplinger [35], Rogava and Vashakidze [40], Truong et al. [41]. In our papers, an approximate algorithm was constructed and its total error was estimated for equation (1.1) in [36] and for equation (1.2) in [37], which for these equations was done for the first time. Approximate methods for solution a certain class of parabolic integro-differential equations are investigated by Jangveladze et al. [17].

In this paper, we return to equation (1.2) considered in [37] under initial boundary conditions of the form (1.3). The difference between these two papers is that in [37] it is assumed that the initial functions $w^l(x)$, $l=0,1$, are analytic, whereas in the present paper the requirement on these functions is relaxed so much that it becomes sufficient for these functions to have derivatives up to a certain finite order. The latter assumption, expanding the class of possible functions $w^l(x)$, $l=0,1$, however,affects the properties of an exact solution of the problem, as well as the rate of error reduction of the numerical algorithm under consideration. As follows from [8], in this case the upper bound of the time interval on which the solvability of problem (1.2), (1.3) is guaranteed, may be less than T. Therefore the solution will already be local. It is proposed to find it by the same computational algorithm that was applied by us in [37]. The main objective of the paper is to estimate the total error of the result. This is achieved by including into the algorithm the parts related to the energy preserving property (Lemma 5.2, Lemma 6.1), inherent in the Kirchhoff equation (1.2). Owing to such an approach, we obtain a priori inequalities allowing us to estimate finally the total error of the algorithm. The upper bound of the total error can be calculated. The formulas needed for this and also for estimating T are written.

1.2. **Restrictions.** Let the function $\varphi(z)$ from equation (1.2) satisfy, in addition to (1.4), the requirement

$$\varphi(z) \in C^p[0,\infty), \tag{1.5}$$

where p can be equal both to 1 and to 2.

Let in conditions (1.3), the functions

$$w^0(x) \text{ and } w^1(x) \text{ be of the form } w^l(x) = \sum_{i=1}^{\infty} a_i^{(l)} \sin ix, \quad l=0,1, \tag{1.6}$$

and

$$|a_i^{(0)}| \le \frac{\omega}{i^{p+s+2,5}}\,, \quad |a_i^{(1)}| \le \frac{\omega}{i^{p+s+1,5}}\,, \quad i=1,2,\dots\,, \tag{1.7}$$

where ω and s are some positive constants.

2. The Algorithm

2.1. **Space discretization – Galerkin's method.** An approximate solution of problem (1.2), (1.3) is written in the form

$$w_n(x,t) = \sum_{i=1}^{n} w_{ni}(t) \sin ix, \quad 0 \le x \le \pi, \quad 0 \le t \le T, \tag{2.1}$$

where the coefficients $w_{ni}(t)$ are defined by Galerkin's method from the system of nonlinear differential equations and the conditions

$$w''_{ni}(t) + \varphi\biggl(\frac{\pi}{2}\sum_{j=1}^{n} j^2 w_{nj}^2(t)\biggr) i^2 w_{ni}(t) = 0, \quad i=1,2,\dots,n, \ \ 0<t\le T, \tag{2.2}$$

$$w_{ni}(0) = a_i^{(0)}, \quad w'_{ni}(0) = a_i^{(1)}, \quad i=1,2,\dots,n.$$

We introduce the functions

$$u_{ni}(t) = w'_{ni}(t), \quad v_{ni}(t) = i w_{ni}(t), \quad i=1,2,\dots,n, \tag{2.3}$$

and replace system (2.2) by an equivalent system

$$
\begin{gathered}
u'_{ni}(t)+\varphi\biggl(\frac{\pi}{2}\sum_{j=1}^{n}v_{nj}^2(t)\biggr)iv_{ni}(t)=0,\\
v'_{ni}(t)=iu_{ni}(t),\quad 0<t\le T,\quad i=1,2,\dots,n,\\
u_{ni}(0)=a_i^{(1)},\quad v_{ni}(0)=ia_i^{(0)},\quad i=1,2,\dots,n.
\end{gathered}
\tag{2.4}
$$

Let us write system (2.4) in a different form. For this we need some definitions. First of all, we define respectively the scalar product and the norm

$$
(u,v)_n=\frac{\pi}{2}\sum_{i=1}^{n}u_iv_i,\quad \|u\|_n=(u,u)_n^{\frac12}
\tag{2.5}
$$

for the vectors $u,v\in R^n$, $u=(u_i)_{i=1}^n$, $v=(v_i)_{i=1}^n$. Next, by functions (2.3), $0\le t\le T$, and the coefficients $a_i^{(j)}$, $i=1,2,\dots,n$, $j=0,1$, from (1.6), we form the vectors

$$
\boldsymbol{u}_n(t)=(u_{ni}(t))_{i=1}^n,\quad \boldsymbol{v}_n(t)=(v_{ni}(t))_{i=1}^n,\quad \boldsymbol{a}_n^j=(a_i^{(j)})_{i=1}^n,\ \ j=0,1.
\tag{2.6}
$$

We also define the matrix

$$
K_n=\operatorname{diag}(1,2,\dots,n).
\tag{2.7}
$$

Following these definitions, system (2.4) can be rewritten as

$$
\begin{gathered}
\boldsymbol{u}'_n(t)+\varphi\left(\|\boldsymbol{v}_n(t)\|_n^2\right)K_n\boldsymbol{v}_n(t)=0,\quad \boldsymbol{v}'_n(t)=K_n\boldsymbol{u}_n(t),\quad 0<t\le T,\\
\boldsymbol{u}_n(0)=\boldsymbol{a}_n^1,\quad \boldsymbol{v}_n(0)=K_n\boldsymbol{a}_n^0.
\end{gathered}
\tag{2.8}
$$

Using the vectors in (2.6), we define the block vectors

$$
\boldsymbol{s}_n(t)=(\boldsymbol{u}_n(t),\boldsymbol{v}_n(t)),\quad \boldsymbol{a}_n=(\boldsymbol{a}_n^1,K_n\boldsymbol{a}_n^0).
\tag{2.9}
$$

Here and in what follows, the transposition sign of the vectors is omitted. Definitions (2.9) allow us to write system (2.8) in the form

$$
\begin{gathered}
\boldsymbol{s}'_n(t)=\begin{pmatrix}0 & -\varphi(\|\boldsymbol{v}_n(t)\|_n^2)K_n\\ K_n & 0\end{pmatrix}\boldsymbol{s}_n(t),\\
\boldsymbol{s}_n(0)=\boldsymbol{a}_n.
\end{gathered}
\tag{2.10}
$$

2.2. **Time discretization – the difference scheme.** Using the function $\varphi(z)$ from equation (1.2), we introduce two new functions $\Phi(z)$ and $\Phi(z_1,z_2)$, $0\le z,\ z_1,z_2<\infty$. Let the first of them mean the primitive function of $\varphi(z)$, and the second one be the divided difference of the function $\Phi(z)$ for $z_1\ne z_2$ and be equal to $\varphi(z)$ for $z_1=z_2$. Therefore

$$
\begin{gathered}
\phi(z)=\int_0^z\varphi(\zeta)\,d\zeta,\\
\phi(z_1,z_2)=\frac{\phi(z_2)-\phi(z_1)}{z_2-z_1},\quad z_1\ne z_2,\qquad \Phi(z,z)=\varphi(z),\\
0\le z,\ z_1,z_2<\infty.
\end{gathered}
\tag{2.11}
$$

To a pair of block vectors s_1, s_2, $s_i=(u_i,v_i)$, $u_i,v_i\in R^n$, $i=1,2$, we put into correspondence the block vector $N_n(s_1,s_2)$ defined by the formula

$$
N_n(s_1,s_2)=\begin{pmatrix}0 & -\phi(\|v_1\|_n^2,\|v_2\|_n^2)\,K_n\\ K_n & 0\end{pmatrix}\frac{s_2+s_1}{2}\,.
\tag{2.12}
$$

Now, let us proceed to solving problem (2.10) by using the difference method. On the time interval $[0,T]$, we introduce the grid $\{t_m\mid 0=t_0<t_1<\cdots<t_M=T\}$ with a generally variable step $\tau_m=t_m-t_{m-1}>0$, $m=1,2,\dots,M$. An approximate value of $\boldsymbol{s}_n(t)$ on the mth time level, i.e., for $t=t_m$, $m=0,1,\dots,M$, denoted by the vector

$$
\boldsymbol{s}_n^m=(\boldsymbol{u}_n^m,\boldsymbol{v}_n^m),
\tag{2.13}
$$

$\boldsymbol{u}_n^m, \boldsymbol{v}_n^m \in R^n$, is defined by the implicit symmetric scheme

$$\frac{\boldsymbol{s}_n^m - \boldsymbol{s}_n^{m-1}}{\tau_m} = N_n(\boldsymbol{s}_n^{m-1}, \boldsymbol{s}_n^m), \quad m = 1, 2, \dots, M, \qquad \boldsymbol{s}_n^0 = \boldsymbol{a}_n. \tag{2.14}$$

2.3. **Solution of a discrete system – the iteration process.** The last part of the algorithm is aimed in solving the system of nonlinear equations (2.14).

Note that equation in (2.14) contains the vectors $\boldsymbol{s}_n^{m-l}$, $l = 0, 1$, from two time levels. It is assumed that the counting is performed levelwise and the iteration process is applied to each mth level, $m \geq 1$. In the equation from (2.14), the vector $\boldsymbol{s}_n^{m-1}$ is replaced by the vector $\boldsymbol{s}_n^{m-1,L} = (\boldsymbol{u}_n^{m-1,L}, \boldsymbol{v}_n^{m-1,L})$, $\boldsymbol{u}_n^{m-1,L}, \boldsymbol{v}_n^{m-1,L} \in R^n$, which is the last (L) approximation for $\boldsymbol{s}_n^{m-1}$ obtained on the $(m-1)$th level. Therefore the vector $\boldsymbol{s}_n^m$ cannot be found exactly. Instead of $\boldsymbol{s}_n^m$, it is the vector $\boldsymbol{s}_{n,R}^m = (\boldsymbol{u}_{n,R}^m, \boldsymbol{v}_{n,R}^m)$, $\boldsymbol{u}_{n,R}^m, \boldsymbol{v}_{n,R}^m \in R^n$, which is a real (R) solution of the resulting equation. Thus the equation $\frac{\boldsymbol{s}_{n,R}^m - \boldsymbol{s}_n^{m-1,L}}{\tau_m} = N_n(\boldsymbol{s}_n^{m-1,L}, \boldsymbol{s}_{n,R}^m)$ corresponds to the mth level, $m > 1$.

Since, starting from the second level, one and the same situation occurs on every level, it is natural to replace $\boldsymbol{s}_n^{m-1,L}$ in the latter equation by

$$\boldsymbol{s}_{n,R}^{m-1,L} = (\boldsymbol{u}_{n,R}^{m-1,L}, \boldsymbol{v}_{n,R}^{m-1,L}), \tag{2.15}$$

$\boldsymbol{u}_{n,R}^{m-1,L}, \boldsymbol{v}_{n,R}^{m-1,L} \in R^n$. As a result, for $\boldsymbol{s}_{n,R}^m$, we obtain the equation

$$\frac{\boldsymbol{s}_{n,R}^m - \boldsymbol{s}_{n,R}^{m-1,L}}{\tau_m} = N_n(\boldsymbol{s}_{n,R}^{m-1,L}, \boldsymbol{s}_{n,R}^m). \tag{2.16}$$

Not to introduce a special equation for the case $m = 1$, let us assume that (2.16) holds likewise for $m = 1$, provided that

$$\boldsymbol{s}_{n,R}^{0,L} = \boldsymbol{s}_{n,R}^0 = \boldsymbol{s}_n^0. \tag{2.17}$$

The nonlinear equation (2.16) is solved by a Picard type iteration process

$$\boldsymbol{s}_{n,R}^{m,k} = \boldsymbol{s}_{n,R}^{m-1,L} + \tau_m N_n(\boldsymbol{s}_{n,R}^{m-1,L}, \boldsymbol{s}_{n,R}^{m,k-1}), \quad k = 1, 2, \dots, \tag{2.18}$$

where

$$\boldsymbol{s}_{n,R}^{m,k-l} = (\boldsymbol{u}_{n,R}^{m,k-l}, \boldsymbol{v}_{n,R}^{m,k-l}), \tag{2.19}$$

$\boldsymbol{u}_{n,R}^{m,k-l}, \boldsymbol{v}_{n,R}^{m,k-l} \in R^n$, is an approximation of the vector $\boldsymbol{s}_{n,R}^m$ on the $(k-l)$th, $l = 0, 1$, iteration step. Thus the approximation to $\boldsymbol{s}_n^m$ is performed by using the vectors $\boldsymbol{s}_{n,R}^{m,k}$, $k = 0, 1, \dots$. For a slight simplification of our further reasoning it will be assumed that as the initial approximation on the mth level, $m = 1, 2, \dots, M$, we take the last iteration approximation on the preceding layer, i.e.,

$$\boldsymbol{s}_{n,R}^{m,0} = \boldsymbol{s}_{n,R}^{m-1,L}. \tag{2.20}$$

Let us write the iteration process (2.18) componentwise. Toward this end, we first represent the vectors from (2.15) and (2.19) as

$$\begin{gathered} \boldsymbol{u}_{n,R}^{m-1,L} = (u_{ni,R}^{m-1,L})_{i=1}^n, \quad \boldsymbol{v}_{n,R}^{m-1,L} = (v_{ni,R}^{m-1,L})_{i=1}^n, \\ \boldsymbol{u}_{n,R}^{m,k-l} = (u_{ni,R}^{m,k-l})_{i=1}^n, \quad \boldsymbol{v}_{n,R}^{m,k-l} = (v_{ni,R}^{m,k-l})_{i=1}^n. \end{gathered} \tag{2.21}$$

Using also (2.5) and (2.12), we eventually obtain two simple recurrent formulas

$$\begin{gathered} u_{ni,R}^{m,k} = u_{ni,R}^{m-1,L} - \tau_m i\, \phi\left(\frac{\pi}{2}\sum_{j=1}^n (v_{nj,R}^{m-1,L})^2, \frac{\pi}{2}\sum_{j=1}^n (v_{nj,R}^{m,k-1})^2\right) \\ \times \frac{v_{ni,R}^{m,k-1} + v_{ni,R}^{m-1,L}}{2}, \quad v_{ni,R}^{m,k} = v_{ni,R}^{m-1,L} + \frac{\tau_m i}{2}(u_{ni,R}^{m,k-1} + u_{ni,R}^{m-1,L}), \\ m = 1, 2, \dots, M, \quad k = 1, 2, \dots, \quad i = 1, 2, \dots, n. \end{gathered} \tag{2.22}$$

Thus the algorithm proposed for the solution of problem (1.2), (1.3) should be understood as performing the counting by formulas (2.22).

3. Preliminaries

3.1. **Some results of S. Bernstein.** In [8], the solvability of problem (1.2), (1.3) is studied under the conditions somewhat different from (1.4)–(1.7). The difference is that the particular case $p = 2$ is not considered in that work and, besides, it is assumed that the coefficients $a_i^{(0)}$ and $a_i^{(1)}$ of expansions in (1.6) are such that the series $\sum_{i=1}^{\infty} i^{5+\varepsilon} a_i^{(0)\,2}$ and $\sum_{i=1}^{\infty} i^{3+\varepsilon} a_i^{(1)\,2}$ converge, where ε is an arbitrary positive constant. As compared with the latter requirement, condition (1.7) for $p = 2$ is more restrictive. After making the corresponding minor correction in order to make the results of [8] applicable to our case, we present some statements therefrom on the solvability of problem (1.2), (1.3) and the fulfillment of certain inequalities. To do this, we introduce some values. We apply the functions $\varphi(z)$, $w^l(x)$, $l = 0, 1$, $\phi(z)$ from equalities (1.2), (1.3), (2.11), the number α from condition (1.4) and the coefficients $a_i^{(l)}$, $l = 1, 2$ from expansion in (1.6). Using also assumptions (1.5)–(1.7), we define

$$z_* = \frac{1}{\alpha}\left[\int_0^{\pi} (w^1(x))^2 dx + \phi\left(\int_0^{\pi} \left(w^{0\prime}(x)\right)^2 dx\right)\right], \qquad N = \frac{\pi}{2} \max_{0 \le z \le z_*} \left|\frac{\varphi'(z)}{\sqrt{\varphi(z)}}\right| \tag{3.1}$$

and

$$\rho_1(0) = \sum_{i=1}^{\infty} i\left[i^2 a_i^{(0)\,2} + \left(\varphi\left(\frac{\pi}{2}\sum_{j=1}^{\infty} j^2 a_j^{(0)\,2}\right)\right)^{-1} a_i^{(1)\,2}\right].$$

It is proved that problem (1.2), (1.3) is solvable on the time interval, which can be expressed by using the parameters given herein. Namely, the existence of a solution is guaranteed for

$$0 < T < \left(N\rho_1(0)\right)^{-1}. \tag{3.2}$$

This solution $w(x,t)$, which we call local, is representable in the form of a series

$$w(x,t) = \sum_{i=1}^{\infty} w_i(t)\sin ix, \quad 0 \le x \le \pi, \quad 0 \le t \le T, \tag{3.3}$$

whose coefficients satisfy the system of equations with the initial conditions

$$w_i''(t) + \varphi\left(\frac{\pi}{2}\sum_{j=1}^{\infty} j^2 w_j^2(t)\right) i^2\, w_i(t) = 0, \quad i = 1, 2, \dots, \quad 0 < t \le T, \qquad w_i(0) = a_i^{(0)}, \quad w_i'(0) = a_i^{(1)}, \quad i = 1, 2, \dots . \tag{3.4}$$

For $l \ge 0$, using the coefficients $w_{ni}(t)$ and $w_i(t)$ from expansions (2.1) and (3.3), we define the functions

$$\rho_{ln}(t) = \sum_{i=1}^{n} i^l\left[i^2 w_{ni}^2(t) + \left(\varphi\left(\frac{\pi}{2}\sum_{j=1}^{n} j^2 w_{nj}^2(t)\right)\right)^{-1} w_{ni}'^2(t)\right], \qquad \rho_l(t) = \sum_{i=1}^{\infty} i^l\left[i^2 w_i^2(t) + \left(\varphi\left(\frac{\pi}{2}\sum_{j=1}^{\infty} j^2 w_j^2(t)\right)\right)^{-1} w_i'^2(t)\right]. \tag{3.5}$$

It is proved that

$$\rho_{ln}(t),\ \rho_l(t) \le \rho_l(0)(1 - N\rho_1(0)t)^{-1}. \tag{3.6}$$

It will be further assumed that condition (3.2) is fulfilled. Otherwise, we may shorten the time interval and T will be understood as the value satisfying this condition.

3.2. **Auxiliary inequalities.** Like in the case of (2.3), using the coefficients of expansion (3.3) of an exact solution of problem (1.2), (1.3), we introduce the functions

$$u_i(t) = w_i'(t), \quad v_i(t) = i w_i(t), \quad i = 1, 2, \dots, \tag{3.7}$$

by which, analogously to (2.6), we construct the vectors

$$p_n \boldsymbol{u}(t) = (u_i(t))_{i=1}^n, \quad p_n \boldsymbol{v}(t) = (v_i(t))_{i=1}^n. \tag{3.8}$$

Lemma 3.1. *The following inequality:*

$$\|p_n \boldsymbol{v}(t)\|_n^2 \le z_*, \quad n = 1, 2, \dots, \quad 0 \le t \le T, \tag{3.9}$$

holds.

Proof. We multiply equation (1.2) by $2\, w_t(x,t)$ and integrate the resulting relation with respect to x from 0 to π. By (1.3), (1.6), (1.7) and (2.11), (3.3), we come first to $\big(\int_0^\pi w_t^2(x,t)dx + \phi\,\big(\int_0^\pi w_x^2(x,t)dx\big)\big)'$ $= 0$ and then to $\int_0^\pi w_t^2(x,t)dx + \phi\,\big(\int_0^\pi w_x^2(x,t)dx\big) = \int_0^\pi \big(w^1(x)\big)^2 dx + \phi\big(\int_0^\pi (w^{0\prime}(x))^2 dx\big)$. Let us use the latter equality and the relation

$$\|p_n \boldsymbol{v}(t)\|_n^2 = \frac{\pi}{2}\sum_{i=1}^n v_i^2(t) = \frac{\pi}{2}\sum_{i=1}^n i^2 w_i^2(t) \le \frac{\pi}{2}\sum_{i=1}^\infty i^2 w_i^2(t) = \int_0^\pi w_x^2(x,t)dx$$
$$\le \frac{1}{\alpha}\phi\bigg(\int_0^\pi w_x^2(x,t)dx\bigg), \quad n = 1, 2, \dots, \quad 0 < t \le T, \tag{3.10}$$

obtained from (2.5), (3.3), (3.7), (3.8) and from the inequality

$$\phi(z) \ge \alpha z, \tag{3.11}$$

which is a consequence of (1.4) and (2.11). This and (3.1) give (3.9) for $n = 1, 2, \dots,\ 0 < t \le T$. Using (1.3) and (3.1) in (3.10), we conclude that (3.9) is true for $t = 0$, as well. □

Applying the vectors from (2.6) and matrix (2.7), the number α and the function $\phi(z)$ from (1.4) and (2.11), we define the value

$$z_{n*} = \frac{1}{\alpha}\bigg(\|\boldsymbol{a}_n^1\|_n^2 + \phi(\|K_n \boldsymbol{a}_n^0\|_n^2)\bigg), \quad n = 1, 2, \dots. \tag{3.12}$$

Comparing (3.1) and (3.12), by virtue of (1.4), (1.6), (2.5)–(2.7) and (2.11), we find that

$$z_{n*} \le z_*, \quad n = 1, 2, \dots. \tag{3.13}$$

Lemma 3.2. *Problem* (2.10) *has a solution and the estimate*

$$\|\boldsymbol{v}_n(t)\|_n^2 \le z_{n*}, \quad n = 1, 2, \dots, \quad 0 \le t \le T, \tag{3.14}$$

is true.

Proof. Consider system (2.8) whose operator form is (2.10). Note that the validity of (3.14) for $t = 0$ is a consequence of (2.8) and (3.11), (3.12). Further, multiplying scalarly the first equation in (2.8) by $2\boldsymbol{u}_n(t)$ and taking into account the second equation and definition (2.7), we obtain $(\|\boldsymbol{u}_n(t)\|_n^2)' + \varphi(\|\boldsymbol{v}_n(t)\|_n^2)(\|\boldsymbol{v}_n(t)\|_n^2)' = 0$. From this and (2.11) follows the equality $(\|\boldsymbol{u}_n(t)\|_n^2 + \phi(\|\boldsymbol{v}_n(t)\|_n^2))' = 0$, which, together with (2.8), gives $\|\boldsymbol{u}_n(t)\|_n^2 + \phi(\|\boldsymbol{v}_n(t)\|_n^2) = \|\boldsymbol{a}_n^1\|_n^2 + \phi(\|K_n \boldsymbol{a}_n^0\|_n^2)$. The latter relation and (3.11), (3.12) imply inequality (3.14) for $\boldsymbol{v}_n(t)$, $0 < t \le T$, and an analogous estimate for $\boldsymbol{u}_n(t)$, namely, $\|\boldsymbol{u}_n(t)\|_n^2 \le \alpha z_{n*}$. From this and (3.14), by virtue of (1.6), (1.7), (3.1) and (3.13), we obtain the uniform boundedness of the norms $\boldsymbol{u}_n(t)$ and $\boldsymbol{v}_n(t)$ with respect to n and t, which guarantees the solvability of problem (2.10). □

Below, we will use the estimates

$$\frac{\pi}{2}\sum_{i=1}^{\infty} v_i^2(t) = \frac{\pi}{2}\sum_{i=1}^{\infty} i^2 w_i^2(t) \le z_*, \quad 0 \le t \le T, \tag{3.15}$$

$$\|\boldsymbol{v}_n(t)\|_n^2 = \frac{\pi}{2}\sum_{i=1}^{n} v_{ni}^2(t) = \frac{\pi}{2}\sum_{i=1}^{n} i^2 w_{ni}^2(t) \le z_*, \tag{3.16}$$
$$n = 1, 2, \dots, \quad 0 \le t \le T,$$

obtained respectively from (3.1), (3.7), (3.9), (3.10) and (2.3), (2.5), (2.6), (3.13), (3.14).

Further, we will need some inequalities for the coefficients in (2.1) and (3.3).

Lemma 3.3. *The estimates*

$$\left|\sum_{i=1}^{n} i^2 w_{ni}(t) w'_{ni}(t)\right| \le \mu_1(t), \quad \left|\sum_{i=1}^{n} i^2 w_i(t) w'_i(t)\right| \le \mu_1(t), \tag{3.17}$$

$$\sum_{i=1}^{n} i^4 w_{ni}^2(t) \le \mu_2(t), \ \sum_{i=1}^{n} i^4 w_i^2(t) \le \mu_2(t), \tag{3.18}$$

$$\sum_{i=1}^{n} i^{2j-2}\left(i^2 w_{ni}^2(t) + w'^2_{ni}(t)\right) \le \mu_j(t), \quad j = 3, \dots, p+2, \tag{3.19}$$
$$n = 1, 2, \dots, \quad 0 \le t \le T,$$

are valid, where the values $\mu_j(t)$, $j = 1, 2, \dots, p+2$, do not depend on n.

Proof. To simplify the notation, we introduce the values

$$\varphi_l = \max_{0 \le z \le z_*}\left|\frac{d^l \varphi}{dz^l}(z)\right|, \quad l = 0, 1, \dots, p, \qquad \varphi_0^1 = \max(1, \varphi_0). \tag{3.20}$$

In expression (3.6), let us represent $\rho_l(0)$ and $\rho_1(0)$ in terms of $a_i^{(0)}$ and $a_i^{(1)}$, $i = 1, 2, \dots$, for which we use the initial conditions in (3.4) and (3.5). We also take into account relations (1.4) and (1.7). As a result, if $l < 2(p+s+1)$, we obtain

$$\rho_{ln}(t),\ \rho_l(t) \le \zeta\left(2(p+s) + 3 - l\right) H(t), \quad 0 \le t \le T, \tag{3.21}$$

where $H(t)$ is a positive-valued function of the form

$$H(t) = \left(\frac{1}{2\omega^2}\ \min(1, \alpha) - \zeta\left(2(p+s+1)\right) N t\right)^{-1} \tag{3.22}$$

and $\zeta(z)$ is the Riemann zeta-function

$$\zeta(z) = \sum_{i=1}^{\infty} \frac{1}{i^z}, \quad z > 1, \tag{3.23}$$

whose numerical values are obtained, for example, in [16].

From (3.22) and (3.23), it follows that $H(t)$ is a monotonically increasing function

$$H(t_1) < H(t_2), \quad 0 \le t_1 < t_2 \le T. \tag{3.24}$$

Further, by virtue of (1.5), (3.5), (3.16) and (3.20), we have

$$\left|\sum_{i=1}^{n} i^2 w_{ni}(t) w'_{ni}(t)\right| \le \frac{1}{2}\varphi_0^1 \rho_{1n}(t)$$

and therefore, after applying (3.21), we conclude that the first inequality in (3.17) is fulfilled with

$$\mu_1(t) = \frac{1}{2}\,\varphi_0^1 \zeta\left(2(p+s+1)\right) H(t). \tag{3.25}$$

Analogously, after replacing (3.16) by (3.15), we prove the second inequality in (3.17).

The fulfillment of inequalities (3.18) follows from relations (3.5) and (3.21). Obviously,

$$\mu_2(t) = \zeta\,(2(p+s)+1)\,H(t). \tag{3.26}$$

Finally, by virtue of (1.5), (3.5), (3.16) and (3.20), the left-hand side of inequality (3.19) is estimated through $\varphi_0^1\rho_{2j-2,n}(t)$. For $\rho_{2j-2,n}(t)$, we have estimate (3.21). Therefore in (3.19) it can be assumed that for $j = 3, \dots, p+2$,

$$\mu_j(t) = \varphi_0^1\zeta\,(2(p+s-j)+5)\,H(t). \tag{3.27}$$

The lemma is proved. □

Lemma 3.4. *The inequality*

$$\left(\frac{\pi}{2}\sum_{i=n+1}^{\infty}\left(i^2w_i^2(t) + l{w'_i}^2(t)\right)\right)^{\frac{1}{2}} \le c_l(t)\frac{1}{n^{p+s+1}}\,, \tag{3.28}$$
$$l = 0,1, \quad n = 1,2,\dots, \quad 0 \le t \le T,$$

where

$$c_l(t) = \frac{1}{2}\left(\pi(2(p+s+1))^{-1}(\varphi_0^1)^lH(t)\right)^{\frac{1}{2}}, \quad l = 0,1, \tag{3.29}$$

is valid.

Proof. We introduce into our consideration the function

$$y_n(t) = \sum_{i=n+1}^{\infty}\left(i^2w_i^2(t) + \left(\varphi\left(\frac{\pi}{2}\sum_{j=1}^{\infty}j^2w_j^2(t)\right)\right)^{-1}w_i'^2(t)\right). \tag{3.30}$$

After multiplying the differential equation in (3.4) by $2w'_i(t)$ and performing summation over $i = n+1, n+2, \dots$, we write

$$y'_n(t) = -\pi\sum_{i=n+1}^{\infty}w_i'^2(t)\sum_{j=1}^{\infty}j^2w_j(t)w'_j(t)\prod_{r=0}^{1}\left(\frac{d^{1-r}\varphi}{dt^{1-r}}\left(\frac{\pi}{2}\sum_{j=1}^{\infty}j^2w_j^2(t)\right)\right)^{1-3r}.$$

This and (3.1), (3.5) and (3.30) imply $|y'_n(t)| \le N\rho_1(t)y_n(t)$. Now, recalling (3.6), we obtain $y_n(t) \le y_n(0)(1-N\rho_1(0)t)^{-1}$. From the latter relation and (3.15), (3.20), (3.30), we have the inequality $\sum_{i=n+1}^{\infty}\left(i^2w_i^2(t) + l{w'_i}^2(t)\right) \le (\varphi_0^1)^l y_n(0)(1-N\rho_1(0)t)^{-1}$. We apply to it (1.4), (1.7), (3.4), (3.21), (3.22) and (3.30). Besides, we use the estimate

$$\sum_{i=n+1}^{\infty}\frac{1}{i^{2(p+s)+3}} \le \frac{1}{2(p+s+1)n^{2(p+s+1)}}\,, \tag{3.31}$$

which follows from the integral test of the series convergence. Hence we conclude that inequality (3.28) for $0 < t \le T$ is fulfilled with $c_l(t)$ calculated by formula (3.29). Expression (3.29) for $c_l(t)$ is applicable in (3.28) in the case $t = 0$ as well, which can be verified by (1.7), (3.4), (3.20), (3.22) and (3.31). □

We will need the following form of formula (3.29) separately for the case $l = 1$. He is following

$$c_1(t) = \frac{1}{2}\left(\frac{\pi}{p+s+1}\varphi_0^1H(t)\right)^{\frac{1}{2}}. \tag{3.32}$$

4. The Error of Galerkin's Method

4.1. **Error relations.** Applying the vectors (3.8), let us construct the block vector

$$p_n\boldsymbol{s}(t)=(p_n\boldsymbol{u}(t),p_n\boldsymbol{v}(t)) \tag{4.1}$$

which will be an exact solution of problem (1.2), (1.3). The error of Galerkin's method is understood as a difference between the vectors from (4.1) and (2.9),

$$\Delta\boldsymbol{s}_n(t)=p_n\boldsymbol{s}(t)-\boldsymbol{s}_n(t). \tag{4.2}$$

The above formulas imply

$$\Delta\boldsymbol{s}_n(t)=\big(\Delta\boldsymbol{u}_n(t),\Delta\boldsymbol{v}_n(t)\big), \tag{4.3}$$

where

$$\Delta\boldsymbol{u}_n(t)=p_n\boldsymbol{u}(t)-\boldsymbol{u}_n(t),\quad \Delta\boldsymbol{v}_n(t)=p_n\boldsymbol{v}(t)-\boldsymbol{v}_n(t). \tag{4.4}$$

Let us derive the equations needed to estimate the norm of the error $\Delta\boldsymbol{s}_n(t)$. To this end, using functions (3.7), we take n first equalities in each relation of (3.4) and replace the resulting system by an equivalent system

$$u_i'(t)+\varphi\Big(\frac{\pi}{2}\sum_{j=1}^{\infty}v_j^2(t)\Big)iv_i(t)=0,\ \ v_i'(t)=iu_i(t),\ \ i=1,2,\dots,n,\ \ 0<t\le T,$$
$$u_i(0)=a_i^{(1)},\ v_i(0)=ia_i^{(0)},\quad i=1,2,\dots,n,$$

which, recalling notation (2.5)–(2.7) and (3.8), can be rewritten as

$$\begin{gathered}p_n\boldsymbol{u}'(t)+\varphi\big(\|p_n\boldsymbol{v}(t)\|_n^2\big)K_np_n\boldsymbol{v}(t)+\boldsymbol{\psi}_n(t)=0,\quad p_n\boldsymbol{v}'(t)=K_np_n\boldsymbol{u}(t),\\ p_n\boldsymbol{u}(0)=\boldsymbol{a}_n^1,\quad p_n\boldsymbol{v}(0)=K_n\boldsymbol{a}_n^0,\end{gathered} \tag{4.5}$$

where $\boldsymbol{\psi}_n(t)$ is the truncation error of the method equal to

$$\boldsymbol{\psi}_n(t)=\left(\varphi\left(\frac{\pi}{2}\sum_{j=1}^{\infty}v_j^2(t)\right)-\varphi\Big(\frac{\pi}{2}\sum_{j=1}^{n}v_j^2(t)\Big)\right)K_np_n\boldsymbol{v}(t). \tag{4.6}$$

Now, subtracting (2.8) from (4.5) and using (4.4), we obtain the desired equations and the conditions

$$\begin{gathered}\Delta\boldsymbol{u}_n'(t)+\varphi\big(\|p_n\boldsymbol{v}(t)\|_n^2\big)K_n\Delta\boldsymbol{v}_n(t)+\big(\varphi(\|p_n\boldsymbol{v}(t)\|_n^2)-\varphi(\|\boldsymbol{v}_n(t)\|_n^2)\big)\\ \times K_n\boldsymbol{v}_n(t)+\boldsymbol{\psi}_n(t)=0,\quad \Delta\boldsymbol{v}_n'(t)=K_n\Delta\boldsymbol{u}_n(t),\\ \Delta\boldsymbol{u}_n(0)=0,\quad \Delta\boldsymbol{v}_n(0)=0.\end{gathered} \tag{4.7}$$

4.2. **The estimate of the method error.** This subsection deals with error (4.2). To begin with, we extend the definition of the norm $\|\cdot\|_n$ given by (2.5) for the vectors from R^n to the block vectors. Let $s=(u,v)$, $u,v\in R^n$. Assume that

$$\|s\|_n=\big(\|u\|_n^2+\|v\|_n^2\big)^{\frac{1}{2}}. \tag{4.8}$$

Lemma 4.1. *The error of Galerkin's method is estimated by*

$$\|\Delta\boldsymbol{s}_n(t_m)\|_n\le c_2(t)\frac{1}{n^{2(p+s+1)}},\quad n=1,2,\dots,\quad 0<t\le T, \tag{4.9}$$

where

$$c_2(t) = \frac{\pi\omega}{4(p+s+1)} \max\Big(1, \frac{1}{\alpha}\Big)\Big(\frac{1}{2a}\,\varphi_1 T H(T)\Big)^{\frac{1}{2}}$$
$$\times \max_{i=0,1} e^{i\nu_0}\Big(\frac{1}{2\omega^2}\,\min(1,\alpha) H(t)\Big)^{\nu_{i+1}}, \tag{4.10}$$
$$\nu_0 = \frac{1}{2\alpha b}\,\varphi_1 z_* t, \quad \nu_1 = \frac{1}{N}\,\pi(a+b)\varphi_1 \prod_{j=1}^{2} \left(\zeta(2(p+s)+j)\right)^{3-2j},$$
$$\nu_2 = \frac{1}{4\alpha N}\,\pi\varphi_0^1\varphi_1$$

and a, b are arbitrary positive constants.

Proof. After multiplying scalarly the first equation in (4.7) by $2\Delta\boldsymbol{u}_n(t)$ and recalling the second equation and (1.5), (2.7), we obtain

$$F_n'(t) = \|\Delta\boldsymbol{v}_n(t)\|_n^2 \,\frac{d}{dt}\,\varphi(\|p_n\boldsymbol{v}(t)\|_n^2) + 2\left(\varphi(\|\boldsymbol{v}_n(t)\|_n^2) - \varphi(\|p_n\boldsymbol{v}(t)\|_n^2)\right)$$
$$\times \left(K_n\boldsymbol{v}_n(t), \Delta\boldsymbol{u}_n(t)\right)_n - 2(\boldsymbol{\psi}_n(t), \Delta\boldsymbol{u}_n(t))_n, \tag{4.11}$$

where we use the notation

$$F_n(t) = \|\Delta\boldsymbol{u}_n(t)\|_n^2 + \varphi(\|p_n\boldsymbol{v}(t)\|_n^2)\|\Delta\boldsymbol{v}_n(t)\|_n^2. \tag{4.12}$$

We are to estimate the terms on the right-hand side of (4.11) for $0 < t \le T$. Formulas (1.5), (2.5) and (3.20) will be used repeatedly.

Applying (3.9), (3.10) and (3.17), we get

$$\left|\frac{d}{dt}\,\varphi(\|p_n\boldsymbol{v}(t)\|_n^2)\right| \le \pi\left|\varphi'(\|p_n\boldsymbol{v}(t)\|_n^2)\right| \left|\sum_{i=1}^{n} i^2 w_i(t) w_i'(t)\right| \le \pi\varphi_1\mu_1(t). \tag{4.13}$$

By (3.9), (3.16) and (4.4), we conclude that

$$\left|\varphi(\|\boldsymbol{v}_n(t)\|_n^2) - \varphi(\|p_n\boldsymbol{v}(t)\|_n^2)\right| \le 2z_*^{\frac{1}{2}}\,\varphi_1\,\|\Delta\boldsymbol{v}_n(t)\|_n. \tag{4.14}$$

By (2.3), (2.6) and (2.7), we write $\|K_n\boldsymbol{v}_n(t)\|_n^2 = \frac{\pi}{2}\sum_{i=1}^{n} i^4 w_{ni}^2(t)$ and, taking additionally into account (3.18), we have

$$\|K_n\boldsymbol{v}_n(t)\|_n^2 \le \frac{\pi}{2}\,\mu_2(t). \tag{4.15}$$

Now, let us estimate the norm of the truncation error $\boldsymbol{\psi}_n(t)$ defined by equality (4.6). Using (2.7), (3.7), (3.8) and (3.18), we get $\|K_n p_n\boldsymbol{v}(t)\|_n^2 \le \frac{\pi}{2}\sum_{i=1}^{n} i^4 w_i^2(t) \le \frac{\pi}{2}\,\mu_2(t)$. Moreover, by (3.7) and (3.15), we infer

$$\|\boldsymbol{\psi}_n(t)\|_n \le \Big(\frac{\pi}{2}\Big)^{\frac{3}{2}}\,\varphi_1\mu_2^{\frac{1}{2}}(t) \sum_{i=n+1}^{\infty} i^2 w_i^2(t). \tag{4.16}$$

Further, from (4.7) and (4.12) follows

$$F_n(0) = 0. \tag{4.17}$$

By (1.4) and (4.11)–(4.17), using arbitrary positive constants a and b, we write the relation $F_n(t) \le \pi\varphi_1\big[\frac{1}{16a}\,\pi\int_0^T \big(\sum_{i=n+1}^{\infty} i^2 w_i^2(t)\big)^2 dt + \int_0^t \max\big(\frac{1}{\alpha}\big(\frac{z_*}{\pi b} + \mu_1(\tau)\big),\ 2(a+b)\times\mu_2(\tau)\big)\,F_n(\tau)\,d\tau\big]$. Let us apply to it the Gronwall inequality together with (1.4), (3.22), (3.25), (3.26), and relations (3.28), (3.29) for the case $l = 0$. Lastly, using (4.3), (4.8) and (4.12), we obtain estimate (4.9) and formula (4.10). □

5. The Difference Scheme Error

5.1. **The error equation.** The difference scheme (2.14) enables us to solve approximately problem (2.10). Let us define the error of scheme (2.14) on the mth time level, i.e., for $t = t_m$,

$$\Delta \boldsymbol{s}_n^m = (\Delta \boldsymbol{u}_n^m, \Delta \boldsymbol{v}_n^m),$$

as a difference of the vectors from (2.9) and (2.13)

$$\Delta \boldsymbol{s}_n^m = \boldsymbol{s}_n(t_m) - s_n^m, \quad m = 0, 1, \dots, M. \tag{5.1}$$

If now we write $\boldsymbol{s}_n^m = \boldsymbol{s}_n(t_m) - \Delta \boldsymbol{s}_n^m$ and use this equality together with (2.10) in scheme (2.14), then we obtain

$$\frac{\Delta \boldsymbol{s}_n^m - \Delta \boldsymbol{s}_n^{m-1}}{\tau_m} = \boldsymbol{f}_n^{m,m-1} + \boldsymbol{\psi}_n^{m,m-1}, \quad m = 1, 2, \dots, M, \qquad \Delta \boldsymbol{s}_n^0 = 0. \tag{5.2}$$

In (5.2), denote by $\boldsymbol{f}_n^{m,m-1}$ the vector defined by the formula

$$\boldsymbol{f}_n^{m,m-1} = N_n(\boldsymbol{s}_n(t_{m-1}), \boldsymbol{s}_n(t_m)) - N_n(\boldsymbol{s}_n^{m-1}, \boldsymbol{s}_n^m) \tag{5.3}$$

and by $\boldsymbol{\psi}_n^{m,m-1}$ the truncation error of scheme (2.14) equal to

$$\boldsymbol{\psi}_n^{m,m-1} = \frac{\boldsymbol{s}_n(t_m) - \boldsymbol{s}_n(t_{m-1})}{\tau_m} - N_n\left(\boldsymbol{s}_n(t_{m-1}), \boldsymbol{s}_n(t_m)\right). \tag{5.4}$$

5.2. **Auxiliary propositions.** In this subsection, several lemmas will be proved.

Lemma 5.1. *If the vectors s_1 and s_2, where $s_i = (u_i, v_i)$, $u_i, v_i \in R^n$, $i = 1, 2$, satisfy the equation*

$$\frac{s_2 - s_1}{\tau} = N_n(s_1, s_2), \tag{5.5}$$

$\tau > 0$, then

$$\sum_{i=1}^{2} (-1)^i \left(\|u_i\|_n^2 + \phi(\|v_i\|_n^2)\right) = 0. \tag{5.6}$$

Proof. The combination of (2.12) and (5.5) yields $\frac{u_2 - u_1}{\tau} + \phi(\|v_1\|_n^2, \|v_2\|_n^2) \times K_n \frac{v_2 + v_1}{2} = 0$, $\frac{v_2 - v_1}{\tau} = K_n \frac{u_2 + u_1}{2}$. Multiplying scalarly the first equation of this system by $u_2 + u_1$ and using the second equation, we come to (5.6). □

Let us estimate the term that forms the nonlinearity in the difference scheme (2.14).

Lemma 5.2. *System* (2.14) *has a solution and the estimate*

$$\|\boldsymbol{v}_n^m\|_n^2 \le z_{n*}, \qquad n = 1, 2, \dots, \quad m = 0, 1, \dots, M, \tag{5.7}$$

is valid.

Proof. Inequality (5.7) is fulfilled for $m = 0$, which can be verified by means of (2.9), (2.13), (2.14) and (3.11), (3.12). As to (5.7) for $m > 0$, note that the first equality in (2.14) is inscribed in scheme (5.5) for every $m = 1, 2, \dots, M$. Following (2.9), (2.13), (2.14) and (5.6), we find $\|\boldsymbol{u}_n^m\|_n^2 + \phi(\|\boldsymbol{v}_n^m\|_n^2) = \|\boldsymbol{a}_n^1\|_n^2 + \phi(\|K_n \boldsymbol{a}_n^0\|_n^2)$. This equality and (3.11), (3.12) give (5.7) for $m = 1, 2, \dots, M$. Moreover, we obtain $\|\boldsymbol{u}_n^m\|_n^2 \le \alpha z_{n*}$. (5.7) and the latter estimate together with (3.1) and (3.13) imply the uniform boundedness of the norms of $\boldsymbol{u}_n^m$ and $\boldsymbol{v}_n^m$, and therefore of $\boldsymbol{s}_n^m$ too, which guarantees the solvability of system (2.14). □

From (3.13) and (5.7) follows

$$\|\boldsymbol{v}_n^m\|_n^2 \le z_*, \quad n = 1, 2, \dots, \quad m = 0, 1, \dots, M. \tag{5.8}$$

We will need the following property of the block vector (2.12).

Lemma 5.3. *For the vectors* $s_i=(u_i,v_i)$, $u_i,v_i\in R^n$, $i=1,2,3,4$, *the inequality*

$$\|N_n(s_1,s_2)-N_n(s_3,s_4)\|_n\le\frac{1}{2}\,n\max(1,\eta)\left(\|s_1-s_3\|_n+\|s_2-s_4\|_n\right), \tag{5.9}$$

holds, where

$$\begin{gathered}\eta=\frac{1}{2}\sum_{i=1}^{2}\Gamma_0^{2i-1,2i}+\frac{1}{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{2}\right)\Gamma_1^{1,4}\sum_{i=1}^{4}\|v_i\|_n^2,\\ \Gamma_j^{l_1,l_2}=\max_z\left|\frac{d^j\varphi}{dz^j}(z)\right|,\quad 0\le z\le\max\left(\|v_{l_1}\|_n^2,\dots,\|\boldsymbol{v}_{l_2}\|_n^2\right),\\ j=0,1,\quad l_1=1,3,\quad l_2=2,4,\quad l_1<l_2.\end{gathered}\tag{5.10}$$

Proof. By (2.12), we have

$$N_n(s_1,s_2)-N_n(s_3,s_4)=(A,B), \tag{5.11}$$

where

$$\begin{aligned}A&=\frac{1}{2}\left[\phi\left(\|v_1\|_n^2,\|v_2\|_n^2\right)K_n(v_1+v_2)-\phi\left(\|v_3\|_n^2,\|v_4\|_n^2\right)K_n(v_3+v_4)\right],\\ B&=\frac{1}{2}K_n\left[(u_1-u_3)+(u_2-u_4)\right].\end{aligned}\tag{5.12}$$

To derive (5.9), (5.10), we use the equality

$$\|N_n(s_1,s_2)-N_n(s_3,s_4)\|_n=\left(\|A\|_n^2+\|B\|_n^2\right)^{\frac{1}{2}} \tag{5.13}$$

arising from definitions (4.8) and (5.11). Let us estimate $\|A\|_n$ and $\|B\|_n$. By (5.12),

$$\begin{aligned}A=\frac{1}{4}\Bigg\{&\sum_{i=1}^{2}\phi(\|v_{2\,i-1}\|_n^2,\|v_{2i}\|_n^2)K_n[(v_1-v_3)+(v_2-v_4)]\\ &+(\phi(\|v_1\|_n^2,\|v_2\|_n^2)-\phi(\|v_3\|_n^2,\|v_4\|_n^2))K_n\sum_{i=1}^{4}v_i\Bigg\}.\end{aligned}$$

We here transform the expressions $\phi(\|v_{2\,i-1}\|_n^2,\|v_{2i}\|_n^2)$, $i=1,2$, and $\phi(\|v_1\|_n^2,\|v_2\|_n^2)-\phi(\|v_3\|_n^2,\|v_4\|_n^2)$ by the Taylor formula and (1.5), (2.11). We also use the relations

$$\begin{gathered}\|K_n\|_n\le n,\\ \left|\|v_i\|_n^2-\|v_{i+2}\|_n^2\right|\le\left(\|v_i\|_n+\|v_{i+2}\|_n\right)\|v_i-v_{i+2}\|_n,\quad i=1,2,\end{gathered}\tag{5.14}$$

which are a consequence of (2.5) and (2.7). Thus, using notation (5.10), we get

$$\begin{aligned}\|A\|_n&\le\frac{1}{4}\,n\Big[(\Gamma_0^{1,2}+\Gamma_0^{3,4})\left(\|v_1-v_3\|_n+\|v_2-v_4\|_n\right)\\ &\quad+\frac{1}{2}\Gamma_1^{1,4}\left(\left|\|v_1\|_n^2-\|v_3\|_n^2\right|+\left|\|v_2\|_n^2-\|v_4\|_n^2\right|\right)\sum_{i=1}^{4}\|v_i\|_n\Big]\\ &\le\frac{1}{4}\,n\sum_{i=1}^{2}\Big[\Gamma_0^{1,2}+\Gamma_0^{3,4}+\frac{1}{2}\Gamma_1^{1,4}\left(\|v_i\|_n+\|v_{i+2}\|_n\right)\\ &\quad\times\sum_{j=1}^{4}\|v_j\|_n\Big]\|v_i-v_{i+2}\|_n.\end{aligned}\tag{5.15}$$

As to the estimate of $\|B\|_n$, it is obtained from (5.12) and (5.14) as

$$\|B\|_n\le\frac{1}{2}\,n\left(\|u_1-u_3\|_n+\|u_2-u_4\|_n\right). \tag{5.16}$$

To complete proving the lemma, it remains to use (5.15), (5.16) in (5.13) and to take into account the estimate

$$(\|v_i\|_n+\|v_{i+2}\|_n)\sum_{j=1}^{4}\|v_j\|_n\le(1+\sqrt{2})\sum_{j=1}^{4}\|v_j\|_n^2,\quad i=1,2,$$

as well as the equality

$$\|s_i - s_{i+2}\|_n = \left(\|u_i - u_{i+2}\|_n^2 + \|v_i - v_{i+2}\|_n^2\right)^{\frac{1}{2}}, \quad i = 1, 2,$$

arising from definition (4.8). □

Now, we will estimate the norm of the truncation error of the difference scheme $\boldsymbol{\psi}_n^{m,m-1}$ defined by (5.4).

Lemma 5.4. *The estimate*

$$\|\boldsymbol{\psi}_n^{m,m-1}\|_n \le \varkappa^{m,m-1}\tau_m^p, \qquad n = 1, 2, \dots, \quad m = 1, 2, \dots, M, \tag{5.17}$$

is valid, where $\varkappa^{m,m-1}$ is a positive value not depending on τ_m and n.

Proof. Let us write $\boldsymbol{\psi}_n^{m,m-1}$ as the block vector $\boldsymbol{\psi}_n^{m,m-1} = (\boldsymbol{\psi}_{n1}^{m,m-1}, \boldsymbol{\psi}_{n2}^{m,m-1})$, $\boldsymbol{\psi}_{nl}^{m,m-1} \in R^n$, $l = 1, 2$. By virtue of (5.4), we obtain the formulas for the ith components $\psi_{n1i}^{m,m-1}$ and $\psi_{n2i}^{m,m-1}$ of the vectors $\boldsymbol{\psi}_{n1}^{m,m-1}$ and $\boldsymbol{\psi}_{n2}^{m,m-1}$, $i = 1, 2, \dots, n$. Taking into account (2.6), (2.7), (2.9) and (2.12), we write

$$\begin{aligned}
\psi_{n1i}^{m,m-1} &= \frac{u_{ni}(t_m) - u_{ni}(t_{m-1})}{\tau_m} \\
&\quad + \phi(\|\boldsymbol{v}_n(t_{m-1})\|_n^2, \|\boldsymbol{v}_n(t_m)\|_n^2)\, i\, \frac{v_{ni}(t_m) + v_{ni}(t_{m-1})}{2}, \\
\psi_{n2i}^{m,m-1} &= \frac{v_{ni}(t_m) - v_{ni}(t_{m-1})}{\tau_m} - i\,\frac{u_{ni}(t_m) + u_{ni}(t_{m-1})}{2}.
\end{aligned}$$

Performing here the Taylor expansion at the point $t = t_{m-1}$, using condition (1.5), relations (2.3), (2.4), (2.11) and, in the case $p = 2$, also the relations obtained from (2.3), (2.4) by differentiation, we get for $\psi_{n1i}^{m,m-1}$ the following equality:

$$\begin{aligned}
\psi_{n1i}^{m,m-1} &= \left(1 - \frac{p}{2}\right)\tau_m\Bigg[w_{ni}''' + \varphi\left(\frac{\pi}{2}\sum_{j=1}^n j^2 w_{nj}^2\right) i^2 w_{ni}' \\
&\quad + \varphi'\left(\frac{\pi}{2}\sum_{j=1}^n j^2 w_{nj}^2\right)\left(\frac{\pi}{2}\sum_{j=1}^n j^2 w_{nj}^2\right)' i^2 w_{ni}\Bigg] \\
&\quad + (p-1)\tau_m^2\Bigg\{\frac{1}{6}\, w_{ni}'^{v} + \frac{1}{4}\, i^2\Bigg[\varphi\left(\frac{\pi}{2}\sum_{j=1}^n j^2 w_{nj}^2\right) w_{ni}'' \\
&\quad + \varphi'\left(\frac{\pi}{2}\sum_{j=1}^n j^2 w_{nj}^2\right)\left(\left(\frac{\pi}{2}\sum_{j=1}^n j^2 w_{nj}^2\right)' w_{ni}' + \left(\frac{\pi}{2}\sum_{j=1}^n j^2 w_{nj}^2\right)'' w_{ni}\right) \\
&\quad + \frac{2}{3}\varphi''\left(\frac{\pi}{2}\sum_{j=1}^n j^2 w_{nj}^2\right)\left(\left(\frac{\pi}{2}\sum_{j=1}^n j^2 w_{nj}^2\right)'\right)^2 w_{ni}\Bigg]\Bigg\},
\end{aligned}\tag{5.18}$$

where, for brevity, the values of the argument of the functions $\frac{d^l w_{ni}}{dt^l}(t)$, $l = 0, 1, \dots, 4$, are omitted. These values belong to the interval $[t_{m-1}, t_m]$. As to the formula for $\psi_{n2i}^{m,m-1}$, it can be written as

$$\begin{aligned}
\psi_{n2i}^{m,m-1} &= \left(1 - \frac{p}{2}\right)\tau_m i\left(w_{ni}''(\theta_{1i}) - w_{ni}''(\theta_{2i})\right) + (p-1)\tau_m^2 i \\
&\quad \times \left(\frac{1}{6}\, w_{ni}'''(\theta_{3i}) - \frac{1}{4}\, w_{ni}'''(\theta_{4i})\right), \quad t_{m-1} \le \theta_{li} \le t_m, \quad l = 1, 2, 3, 4.
\end{aligned}\tag{5.19}$$

Note that formulas (5.18) and (5.19) are valid provided the inclusion $w_{ni}(t) \in C^{p+2}[0, T]$, $i = 1, 2, \dots, n$, holds. Lemma 3.2 states that (2.10), i.e., system (2.2), is solvable. This result together with the form of the equation in (2.2) and condition (1.5) imply that this requirement is fulfilled.

Now, to obtain estimate (5.17), using (5.18) and (5.19), we need to verify that the series $\sum_{i=1}^{n} i^4 w_{ni}^2(t)$, $\sum_{i=1}^{n} i^4 w_{ni}'^2(t)$, $\sum_{i=1}^{n} i^{2p} w_{ni}''^2(t)$, $\sum_{i=1}^{n} i^{2(p-1)} w_{ni}'''^2(t)$, $(p-1)\sum_{i=1}^{n} i^4 w_{ni}'^{v\,2}(t)$ are uniformly bounded with respect to $n=1,2,\dots$ and $0<t\le T$. Toward this end, it suffices for (3.19) to be fulfilled. Indeed, by virtue of (2.2), we write

$$\sum_{i=1}^{n} i^{2(p-l)}\Big(\frac{d^{l+2}w_{ni}}{dt^{l+2}}\,(t)\Big)^2=\sum_{i=1}^{n} i^{2(p-l+2)}\bigg\{\frac{d^l}{dt^l}\,\bigg[\varphi\bigg(\frac{\pi}{2}\sum_{j=1}^{n} j^2 w_{nj}^2(t)\bigg)w_{ni}(t)\bigg]\bigg\}^2$$

and then, keeping in mind (1.5), consider successively the cases $l=0,1,\dots,p$. □

Further, it will be useful to define concretely the value $\varkappa^{m,m-1}$ from estimate (5.17). For this, note that if (3.19) holds, then the first inequalities in (3.17) and (3.18) are fulfilled. Combining (1.5), (2.2), (3.16)–(3.20), (4.8) and (5.17)–(5.19), we obtain

$$\begin{aligned}
&\text{for } p=1,\quad \varkappa^{m,m-1}=\sqrt{\pi}\max_t\Big(\pi\varphi_1\mu_1(t)\mu_2^{\frac12}(t)+\sqrt{\frac32}\varphi_0\varphi_0^1\mu_3^{\frac12}(t)\Big),\\
&\qquad\qquad t_{m-1}\le t\le t_m,\\
&\text{for } p=2,\quad \varkappa^{m,m-1}=\frac{\pi}{3\sqrt2}\max_t\bigg[\frac{5}{\pi}\,\mu_4(t)\sum_{l=1}^{2}(\varphi_0)^l\\
&\qquad+\varphi_0^1\mu_1(t)\mu_2^{\frac12}(t)\sum_{l=1}^{2}(2l+1)\varphi_l+9\varphi_1\mu_1(t)\mu_3^{\frac12}(t)\\
&\qquad+\mu_2(t)\sum_{i=1}^{2}(i+1)\mu_{i+1}^{\frac12}(t)\prod_{l=0}^{1}\varphi_l\bigg],\ t_{m-1}\le t\le t_m.
\end{aligned}\tag{5.20}$$

5.3. **The estimate of the difference scheme error.** For error (5.1), we have

Lemma 5.5. *If for every mth level, $m=1,2,\dots,m_0$, $1\le m_0\le M$, the grid step satisfies the inequality*

$$\tau_m\le\frac{2(1-\sigma)}{n}\bigg[\max\bigg(1,\sum_{l=0}^{1}((1+\sqrt2)\,z_*)^l\varphi_l\bigg)\bigg]^{-1},\tag{5.21}$$

where

$$\sigma\ \text{ is an arbitrary parameter such that }\ 0<\sigma<1,\tag{5.22}$$

then for the m_0th level, i.e., for $t=t_{m_0}$, the error of scheme (2.14) *is estimated by*

$$\|\Delta\boldsymbol{s}_n^{m_0}\|_n\le c_3(t_{m_0})\frac1n\bigg[\bigg(1+\frac{1}{m_0}\,\lambda n\bigg)^{m_0}-1\bigg]\max_{1\le m\le m_0}\tau_m^p.\tag{5.23}$$

Here

$$c_3(t_{m_0})=\max_{1\le m\le m_0}\varkappa^{m,m-1}\bigg[\max\bigg(1,\sum_{l=0}^{1}((1+\sqrt2)\,z_*)^l\varphi_l\bigg)\bigg]^{-1},\tag{5.24}$$

$$\lambda=\frac{t_{m_0}}{\sigma}\max_{1\le i,j\le m_0}\frac{\tau_i}{\tau_j}\max\bigg(1,\sum_{l=0}^{1}((1+\sqrt2)\,z_*)^l\varphi_l\bigg).\tag{5.25}$$

Proof. Formulas in (5.2) and (5.3) give the inequality

$$\|\Delta\boldsymbol{s}_n^m\|_n\le\|\Delta\boldsymbol{s}_n^{m-1}\|_n+\tau_m\left(\|N_n(\boldsymbol{s}_n(t_{m-1}),\boldsymbol{s}_n(t_m))-N_n(\boldsymbol{s}_n^{m-1},\boldsymbol{s}_n^m)\|_n+\|\boldsymbol{\psi}_n^{m,m-1}\|_n\right).$$

Applying to it (5.1), (5.9) and (5.10), we obtain

$$\sum_{i=0}^{1}\left((-1)^i-\tau_m n\chi^{m,m-1}\right)\|\Delta\boldsymbol{s}_n^{m-i}\|_n\le\tau_m\|\boldsymbol{\psi}_n^{m,m-1}\|_n.\tag{5.26}$$

Here,

$$\chi^{m,m-1} = \frac{1}{2}\max\Big(1, 2^{-1}\sum_{i=0}^{1}\max_{0\le z\le z_{i+1}}|\varphi(z)| + (2^{-\frac{3}{2}}+2^{-2})$$
$$\times \max_{0\le z\le \max(z_1,z_2)}|\varphi'(z)|\sum_{i=0}^{1}\left(\|\boldsymbol{v}_n(t_{m-i})\|_n^2 + \|\boldsymbol{v}_n^{m-i}\|_n^2\right)\Big),$$
$$z_1 = \max\left(\|\boldsymbol{v}_n(t_{m-j})\|_n^2,\ \ j=0,1\right),\ \ z_2 = \max\Big(\|\boldsymbol{v}_n^{m-j}\|_n^2,\ \ j=0,1\Big).$$

If in the expression for $\chi^{m,m-1}$ we use estimates (3.16), (5.8), and notation (3.20), then (5.26) can be replaced by the inequality

$$\sum_{i=0}^{1}\left((-1)^i - \tau_m n\chi\right)\|\Delta\boldsymbol{s}_n^{m-i}\|_n \le \tau_m\|\boldsymbol{\psi}_n^{m,m-1}\|_n, \tag{5.27}$$

where

$$\chi = \frac{1}{2}\max\Big(1, \sum_{l=0}^{1}((1+\sqrt{2})\,z_*)^l\varphi_l\Big). \tag{5.28}$$

Since (5.21), (5.22) and (5.28) imply $1-\tau_m n\chi \ge \sigma > 0$, it follows from (5.17) and (5.27) that

$$\|\Delta\boldsymbol{s}_n^m\|_n \le \Big(1 + 2\,\frac{\tau_m n\chi}{\sigma}\Big)\|\Delta\boldsymbol{s}_n^{m-1}\|_n + \frac{1}{\sigma}\,\varkappa^{m,m-1}\tau_m^{p+1}. \tag{5.29}$$

Observing that (5.29) is fulfilled for $m=1,2,\dots,m_0$, $m_0>1$, and using (5.2), we obtain

$$\|\Delta\boldsymbol{s}_n^{m_0}\|_n \le \frac{1}{\sigma}\max_{1\le m\le m_0}\varkappa^{m,m-1}\Big[\sum_{m=1}^{m_0-1}\prod_{i=m+1}^{m_0}\Big(1+2\,\frac{\tau_i n\chi}{\sigma}\Big)\tau_m^{p+1}\Big]. \tag{5.30}$$

Applying (5.28), (5.30) and the inequalities $\tau_m \le \max_l \tau_l \le \frac{t_{m_0}}{m_0}\max\frac{\tau_i}{\tau_j}$, where $1\le m\le m_0$, $l,i,j=1,2,\dots,m_0$, we conclude that (5.23) is true, and the coefficient $c_3(t_{m_0})$ and the parameter λ are defined by (5.24) and (5.25). $\square$

Let us introduce into consideration the single-index parameters $\nu_1 = \pi\varphi_0^1\varphi_1$, $\nu_2 = (\varphi_0^1)^{\frac{1}{2}}\max\limits_{l=1,2}\varphi_0^l$ and the double-index parameters $\nu_{11} = 3\varphi_0(\varphi_0^1)^{\frac{1}{2}}$, $\nu_{12} = 4,5(\varphi_0^1)^{\frac{3}{2}}$, $\nu_{21} = 2\varphi_0$, $\nu_{22} = 0,5\varphi_1^{-1}(\varphi_0^1)^2\sum_{l=1}^{2}(2l+1)\varphi_l$. Using (3.24)–(3.27) and (5.20) together with notation (5.28) in (5.24), we come to a conclusion that

$$\begin{aligned}
&\text{for } p=1,\ \ c_3(t_{m_0}) = 2\sqrt{2}\chi\\
&\quad\times\Big(\sum_{i=1}^{2}\nu_i(H(t_{m_0}))^{i-\frac{1}{2}}\prod_{j=1}^{i}\zeta(2(p+s+i)+j-4)\Big),\\
&\text{for } p=2,\ \ c_3(t_{m_0}) = \frac{\sqrt{2}\pi}{3}\,\chi H(t_{m_0})\Big(\frac{5}{\pi}\,\varphi_0^1\zeta(2(p+s)-3)\\
&\quad+\varphi_1 H^{\frac{1}{2}}(t_{m_0})\sum_{i,j=1}^{2}\nu_{ij}\left(\zeta(2(p+s)+2i-3)\right)^{\frac{1}{2}}\zeta(2(p+s)+j)\Big).
\end{aligned} \tag{5.31}$$

6. The Iteration Process Error

6.1. **Some inequalities and definitions.** In this subsection, we consider equation (2.16). Let us estimate the norm of the vector $\boldsymbol{v}_{n,R}^m$ which forms the nonlinearity. As will be seen, the estimate depends on the iteration results on the preceding $(m-1)$th level. Denote

$$z_{n,R}^{m-1,L} = \frac{1}{\alpha}\left(\|\boldsymbol{u}_{n,R}^{m-1,L}\|_n^2 + \phi(\|\boldsymbol{v}_{n,R}^{m-1,L}\|_n^2)\right), \tag{6.1}$$
$$n=1,2,\dots,\quad m=1,2,\dots,M.$$

In particular, for $m = 1$, it follows from this relation and (2.9), (2.14), (2.15) and (2.17) that $z_{n,R}^{0,L}$ is calculated by (3.12).

Lemma 6.1. *We have an a priori estimate*

$$|\boldsymbol{v}_{n,R}^{m}\|_n^2 \le z_{n,R}^{m-1,L}, \qquad n = 1, 2, \dots, \quad m = 1, 2, \dots, M. \tag{6.2}$$

Proof. Equation (2.16), whose solvability will be shown below, is a particular case of equation (5.5). Hence, using (5.6), we obtain the equality

$$\|\boldsymbol{u}_{n,R}^{m}\|_n^2 + \phi(\|\boldsymbol{v}_{n,R}^{m}\|_n^2) = \|\boldsymbol{u}_{n,R}^{m-1,L}\|_n^2 + \phi(\|\boldsymbol{v}_{n,R}^{m-1,L}\|_n^2),$$

which together with (3.11) and (6.1) gives (6.2). □

Further, we will also need to estimate the norm of the vector $\boldsymbol{v}_{n,R}^{m-1,L}$ by means of $z_{n,R}^{m-1,L}$. Using (3.11) and (6.1), we obtain the needed inequality

$$\|\boldsymbol{v}_{n,R}^{m-1,L}\|_n^2 \le z_{n,R}^{m-1,L}, \qquad n = 1, 2, \dots, \quad m = 1, 2, \dots, M. \tag{6.3}$$

Now, let us introduce some definitions. The iteration process (2.18) enables us to find an approximate value of the vector $\boldsymbol{s}_n^m$. We define the error of the kth step of process (2.18) on the mth level, i.e., for $t = t_m$, as the vector

$$\Delta \boldsymbol{s}_{n,R}^{m,k} = \boldsymbol{s}_n^m - \boldsymbol{s}_{n,R}^{m,k}. \tag{6.4}$$

To estimate the norm of $\Delta \boldsymbol{s}_{n,R}^{m,k}$, we have to introduce the values h_m, ∇_m and q_m for each mth time level, $m = 1, 2, \dots, M$. The first of these values depends on the last approximation on the $(m-1)$th level. The next value depends on the constants z_* and $z_{n,R}^{m-1,L}$ from (3.1) and (6.1). As to q_m, it is defined by means of simple inequalities. To be more exact, assume that

$$h_m = \left(\|\boldsymbol{u}_{n,R}^{m-1,L}\|_n + \varphi(\|\boldsymbol{v}_{n,R}^{m-1,L}\|_n^2)\,\|\boldsymbol{v}_{n,R}^{m-1,L}\|_n\right), \tag{6.5}$$

$$\nabla_m = \frac{1}{2}\max\left(1, \sum_{l=0}^{1}\left[\left(\frac{1}{\sqrt{2}} + \frac{1}{2}\right)(z_* + z_{n,R}^{m-1,L})\right]^l \max_z \left|\frac{d^l\varphi}{dz^l}(z)\right|\right), \tag{6.6}$$

$$0 \le z \le \max\left(z_*, z_{n,R}^{m-1,L}\right),$$

$$0 < q_m < 1. \tag{6.7}$$

Also, for the step τ_m, consider the inequalities

$$\tau_m < \frac{1}{n\nabla_m}, \tag{6.8}$$

$$\frac{\tau_m n}{2}\max\left(1, \sum_{l=0}^{1}\left[\left(\frac{1}{\sqrt{2}} + \frac{1}{2}\right)(z_{n,R}^{m-1,L} + z_{n,R}^{m,m-1})\right]^l \max_z \left|\frac{d^l\varphi}{dz^l}(z)\right|\right) \le q_m, \tag{6.9}$$

$$0 \le z \le \max\left(z_{n,R}^{m-1,L}, z_{n,R}^{m,m-1}\right). \tag{6.10}$$

In (6.9) and (6.10),

$$z_{n,R}^{m,m-1} = \left(\|\boldsymbol{s}_{n,R}^{m-1,L}\|_n + \tau_m n h_m \frac{1}{1 - q_m}\right)^2. \tag{6.11}$$

From (3.1), (6.1) and (6.5)–(6.11) it follows that h_m, ∇_m, q_m and, which is important, τ_m, can be found prior to performing the iteration on the mth level.

Finally, we introduce one more notation. We denote by m_L the number of iterations performed on the mth level, $m = 1, 2, \dots, M$. Therefore m_L is the number of the iteration that yields $\boldsymbol{s}_{n,R}^{m,L}$.

6.2. **The estimate of the iteration process error.** To estimate the error of the iteration process (2.18), we use the inequalities and definitions of the preceding subsection.

Lemma 6.2. *If conditions* (6.7)–(6.9) *are fulfilled for each mth level,* $m = 1, 2, \dots, m_0$, $1 < m_0 \le M$, *except for the first level for condition* (6.8), *then the error of the kth step of process* (2.18) *on the* m_0*th level satisfies the inequality*

$$\|\Delta \boldsymbol{s}_{n,R}^{m_0,k}\|_n \le \sum_{m=1}^{m_0-1} \frac{q_m^{m_L}}{1-q_m} \big\|\boldsymbol{s}_{n,R}^{m,1} - \boldsymbol{s}_{n,R}^{m-1,L}\big\|_n \prod_{i=m+1}^{m_0} \frac{1+\tau_i n \nabla_i}{1-\tau_i n \nabla_i}$$
$$+ \frac{q_{m_0}^k}{1-q_{m_0}} \big\|\boldsymbol{s}_{n,R}^{m_0,1} - \boldsymbol{s}_{n,R}^{m_0-1,L}\big\|_n, ^1 \quad k = 1, 2, \dots . \tag{6.12}$$

Proof. We use the inequality

$$\big\|\Delta \boldsymbol{s}_{n,R}^{m,k}\big\|_n \le \big\|\boldsymbol{s}_n^m - \boldsymbol{s}_{n,R}^m\big\|_n + \big\|\boldsymbol{s}_{n,R}^m - \boldsymbol{s}_{n,R}^{m,k}\big\|_n, \tag{6.13}$$
$$m = 1, 2, \dots, m_0, \quad k = 1, 2, \dots ,$$

obtained from (6.4) and estimate each summand in the right-hand part of (6.13).

Consider the first summand. Since by virtue of (2.17) for $m = 1$ equation (2.16) does not differ from equation in (2.14), we have

$$\|\boldsymbol{s}_n^1 - \boldsymbol{s}_{n,R}^1\|_n = 0. \tag{6.14}$$

Let $m > 1$. For simplicity, we use the notation

$$\boldsymbol{v}_{n,R}^{m-1} = \boldsymbol{v}_{n,R}^{m-1,L}, \quad \boldsymbol{s}_{n,R}^{m-1} = \boldsymbol{s}_{n,R}^{m-1,L}. \tag{6.15}$$

Subtracting (2.16) from the first equality in (2.14), we get

$$\|\boldsymbol{s}_n^m - \boldsymbol{s}_{n,R}^m\|_n \le \|\boldsymbol{s}_n^{m-1} - \boldsymbol{s}_{n,R}^{m-1}\|_n$$
$$+ \tau_m \big\|N_n(\boldsymbol{s}_n^{m-1}, \boldsymbol{s}_n^m) - N_n(\boldsymbol{s}_{n,R}^{m-1}, \boldsymbol{s}_{n,R}^m)\big\|_n. \tag{6.16}$$

Applying (5.9) and (5.10) to $\|N_n(\boldsymbol{s}_n^{m-1}, \boldsymbol{s}_n^m) - N_n(\boldsymbol{s}_{n,R}^{m-1}, \boldsymbol{s}_{n,R}^m)\|_n$, we obtain

$$\big\|N_n(\boldsymbol{s}_n^{m-1}, \boldsymbol{s}_n^m) - N_n(\boldsymbol{s}_{n,R}^{m-1}, \boldsymbol{s}_{n,R}^m)\big\|_n \le \frac{1}{2} n \max\Bigg(1, \frac{1}{2}\sum_{i=1}^{2} \max_{0\le z\le z_i} |\varphi(z)|$$
$$+\frac{1}{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{2}\right) \max_{0\le z\le z_3} |\varphi'(z)| \sum_{i=0}^{1} (\|\boldsymbol{v}_n^{m-i}\|_n^2 + \|\boldsymbol{v}_{n,R}^{m-i}\|_n^2)\Bigg) \sum_{i=0}^{1} \|\boldsymbol{s}_n^{m-i} - \boldsymbol{s}_{n,R}^{m-i}\|_n,$$
$$z_1 = \max\big(\|\boldsymbol{v}_n^{m-j}\|_n^2,\ j = 0, 1\big),$$
$$z_2 = \max\Big(\|\boldsymbol{v}_{n,R}^{m-j}\|_n^2,\ j = 0, 1\Big), \quad z_3 = \max(z_1, z_2).$$

This inequality together with (5.8), (6.2), (6.3) and (6.6) lead to the relation $\big\|N_n(\boldsymbol{s}_n^{m-1}, \boldsymbol{s}_n^m) - N_n(\boldsymbol{s}_{n,R}^{m-1}, \boldsymbol{s}_{n,R}^m)\big\|_n \le n\nabla_m \sum_{i=0}^{1} \|\boldsymbol{s}_n^{m-i} - \boldsymbol{s}_{n,R}^{m-i}\|_n$. Using this relation and (6.8), (6.15), (6.16), we conclude that the first summand in the right-hand side of (6.13) is estimated as follows:

$$\|\boldsymbol{s}_n^m - \boldsymbol{s}_{n,R}^m\|_n \le \frac{1+\tau_m n \nabla_m}{1-\tau_m n \nabla_m} \big\|\boldsymbol{s}_n^{m-1} - \boldsymbol{s}_{n,R}^{m-1,L}\big\|_n, \quad m = 2, 3, \dots, m_0 . \tag{6.17}$$

Let us estimate the second summand in (6.13). From (2.18), we have $\boldsymbol{s}_{n,R}^{m,k+1} - \boldsymbol{s}_{n,R}^{m,k} = \tau_m\big(N_n(\boldsymbol{s}_{n,R}^{m-1,L}, \boldsymbol{s}_{n,R}^{m,k}) - N_n(\boldsymbol{s}_{n,R}^{m-1,L}, \boldsymbol{s}_{n,R}^{m,k-1})\big)$, $k = 1, 2, \dots$. Applying (5.9), (5.10) and (6.3) to this

[1]For simplicity, here and in some formulas below, we use certain conventions in the notation: firstly, when the index m takes values $1, 2, \dots, m_0-1$, it is assumed that the parameter m_L becomes respectively equal to $1_L, 2_L, \dots, (m_0-1)_L$ and, secondly, if in the summation symbol $\sum$ the value of the subscript is greater than that of the superscript, the summation operation should be omitted.

relation and also taking into account the inequality $\|\boldsymbol{v}_{n,R}^{m,k-l}\|_n \le \|\boldsymbol{s}_{n,R}^{m,k-l}\|_n$, $l=0,1$ obtained from (2.19) and (4.8), we find that

$$\big\|\boldsymbol{s}_{n,R}^{m,k+1}-\boldsymbol{s}_{n,R}^{m,k}\big\|_n \le \frac{1}{2}\,\tau_m n \max\bigg\{1,\sum_{l=0}^{1}\bigg[\bigg(\frac{1}{\sqrt{2}}+\frac{1}{2}\bigg)\bigg(z_{n,R}^{m-1,L} + \frac{1}{2}\sum_{i=0}^{1}\|\boldsymbol{s}_{n,R}^{m,k-i}\|_n^2\bigg)\bigg]^l \max_z\bigg|\frac{d^l\varphi}{dz^l}(z)\bigg|\bigg\}\|\boldsymbol{s}_{n,R}^{m,k}-\boldsymbol{s}_{n,R}^{m,k-1}\|_n, \tag{6.18}$$

$$0\le z\le \max\Big(z_{n,R}^{m-1,L},\|\boldsymbol{s}_{n,R}^{m,k-i}\|_n^2,\ \ i=0,1\Big).$$

Further, we apply (2.11), (2.12), (2.15) and (2.20) to (2.18) and also take into account (4.8), (5.14) and (6.5). This results in

$$(1-l)\|\boldsymbol{s}_{n,R}^{m,0}\|_n,\ \ \|\boldsymbol{s}_{n,R}^{m,1}-l\boldsymbol{s}_{n,R}^{m,0}\|_n \le \xi_{lm}, \tag{6.19}$$

where $\xi_{lm}=(1-l)\|\boldsymbol{s}_{n,R}^{m-1,L}\|_n+\tau_m n h_m,\quad l=0,1$. The parameters ξ_{0m} and ξ_{1m} allow us to rewrite condition (6.9) as

$$\frac{\tau_m n}{2}\ \max\bigg\{1,\sum_{l=0}^{1}\bigg[\bigg(\frac{1}{\sqrt{2}}+\frac{1}{2}\bigg)\big(z_{n,R}^{m-1,L}+\xi_m\big)\bigg]^l\max_z\bigg|\frac{d^l\varphi}{dz^l}(z)\bigg|\bigg\}\le q_m, \tag{6.20}$$

where $0\le z\le \max\Big(z_{n,R}^{m-1,L},\xi_m\Big)$, $\quad \xi_m=\Big(\xi_{0m}+\xi_{1m}\frac{q_m}{1-q_m}\Big)^2$.

By (6.18)–(6.20), we have

$$\big\|\boldsymbol{s}_{n,R}^{m,2}-\boldsymbol{s}_{n,R}^{m,1}\big\|_n\le q_m\big\|\boldsymbol{s}_{n,R}^{m,1}-\boldsymbol{s}_{n,R}^{m,0}\big\|_n\le \xi_{1m}q_m. \tag{6.21}$$

This and (6.19) imply

$$\|\boldsymbol{s}_{n,R}^{m,2}\|_n\le \xi_{0m}+\xi_{1m}q_m. \tag{6.22}$$

Now, assume that for $k=2,3,\dots,l$m we have the inequalities

$$\big\|\boldsymbol{s}_{n,R}^{m,k}-\boldsymbol{s}_{n,R}^{m,k-1}\big\|_n\le q_m\big\|\boldsymbol{s}_{n,R}^{m,k-1}-\boldsymbol{s}_{n,R}^{m,k-2}\big\|_n, \tag{6.23}$$

$$\|\boldsymbol{s}_{n,R}^{m,k}\|_n\le \xi_{0m}+\xi_{1m}\sum_{i=1}^{k-1}q_m^i. \tag{6.24}$$

Substituting the estimate $\|\boldsymbol{s}_{n,R}^{m,l-j}\|_n\le \xi_{0m}+\xi_{1m}\frac{q_m}{1-q_m},\quad j=0,1$, obtained from (6.7), (6.19) and (6.24) into (6.18) for $k=l$ and using (6.20), we find

$$\|\boldsymbol{s}_{n,R}^{m,l+1}-\boldsymbol{s}_{n,R}^{m,l}\|_n\le q_m\|\boldsymbol{s}_{n,R}^{m,l}-\boldsymbol{s}_{n,R}^{m,l-1}\|_n. \tag{6.25}$$

This inequality, (6.19) and (6.23) lead to the estimate $\|\boldsymbol{s}_{n,R}^{m,l+1}-\boldsymbol{s}_{n,R}^{m,l}\|_n\le\xi_{1m}q_m^l$ which, together with (6.24) for $k=l$, gives

$$\|\boldsymbol{s}_{n,R}^{m,l+1}\|_n\le\|\boldsymbol{s}_{n,R}^{m,l}\|_n+\|\boldsymbol{s}_{n,R}^{m,l+1}-\boldsymbol{s}_{n,R}^{m,l}\|_n\le\xi_{0m}+\xi_{1m}\sum_{i=1}^{l}q_m^i. \tag{6.26}$$

By (6.25) and (6.26), we conclude that (6.23) and (6.24) are fulfilled for $k=l+1$. This fact, together with (6.21) and (6.22), means that (6.23) and (6.24) hold for arbitrary $k>1$. From (6.23) follows

$$\big\|\boldsymbol{s}_{n,R}^{m,k}-\boldsymbol{s}_{n,R}^{m,k-1}\big\|_n\le q_m^{k-1}\big\|\boldsymbol{s}_{n,R}^{m,1}-\boldsymbol{s}_{n,R}^{m,0}\big\|_n,\quad k=1,2,\dots. \tag{6.27}$$

Let us verify that the sequence $(\boldsymbol{s}_{n,R}^{m,k})_{k=0}^\infty$ is fundamental. On the strength of (6.27), for any $l>0$, we have

$$\big\|\boldsymbol{s}_{n,R}^{m,k+l}-\boldsymbol{s}_{n,R}^{m,k}\big\|_n\le\sum_{i=1}^{l}\big\|\boldsymbol{s}_{n,R}^{m,k+i}-\boldsymbol{s}_{n,R}^{m,k+i-1}\big\|_n\le\big\|\boldsymbol{s}_{n,R}^{m,1}-\boldsymbol{s}_{n,R}^{m,0}\big\|_n\sum_{i=1}^{l}q_m^{k+i-1}.$$

Therefore

$$\big\|\boldsymbol{s}_{n,R}^{m,k+l}-\boldsymbol{s}_{n,R}^{m,k}\big\|_n \le \frac{q_m^k}{1-q_m}\big\|\boldsymbol{s}_{n,R}^{m,1}-\boldsymbol{s}_{n,R}^{m,0}\big\|_n, \qquad k=0,1,\dots, \quad l=1,2,\dots. \tag{6.28}$$

For any l, the right-hand side of this inequality tends to zero as $k\to\infty$. Hence the sequence $(\boldsymbol{s}_{n,R}^{m,k})_{k=0}^{\infty}$ is fundamental and has $\lim \boldsymbol{s}_{n,R}^{m,k}=\boldsymbol{s}_{n,R}^{m}$. To pass to the limit in (2.18) as $k\to\infty$, we use the continuity property of the matrix N_n. Then it is obvious that $\boldsymbol{s}_{n,R}^{m}$ is a solution of equation (2.16). Passing to the limit in (6.28) as $l\to\infty$, we obtain the inequality $\|\boldsymbol{s}_{n,R}^{m}-\boldsymbol{s}_{n,R}^{m,k}\|_n \le \frac{q_m^k}{1-q_m}\|\boldsymbol{s}_{n,R}^{m,1}-\boldsymbol{s}_{n,R}^{m,0}\|_n$, which together with inequalities (6.13), (6.17) and relation (6.14) imply the estimates $\|\Delta\boldsymbol{s}_{n,R}^{1,k}\|_n \le \frac{q_1^k}{1-q_1}\big\|\boldsymbol{s}_{n,R}^{1,1}-\boldsymbol{s}_{n,R}^{1,0}\big\|_n$, $\|\Delta\boldsymbol{s}_{n,R}^{m,k}\|_n \le \frac{1+\tau_m n\dot{\nabla}_m}{1-\tau_m n\nabla_m}\times\big\|\boldsymbol{s}_{n}^{m-1}-\boldsymbol{s}_{n,R}^{m-1,L}\big\|_n+\frac{q_m^k}{1-q_m}\big\|\boldsymbol{s}_{n,R}^{m,1}-\boldsymbol{s}_{n,R}^{m,0}\big\|_n$, $m=2,3,\dots,m_0$. Using these formulas and also taking into account (2.20), (6.4) and the definition of m_L, we get (6.12). □

By (2.5), (2.15), (2.19), (2.21) and (4.8), we rewrite (6.12) in the form needed in what follows:

$$\|\Delta\boldsymbol{s}_{n,R}^{m_0,k}\|_n \le \sum_{m=1}^{m_0-1} d_m q_m^{m_L}+d_{m_0}q_{m_0}^k, \tag{6.29}$$

where

$$d_m=\frac{1}{1-q_m}\Big\{\sum_{i=1}^{n}\big[(u_{ni,R}^{m,1}-u_{ni,R}^{m-1,L})^2+(v_{ni,R}^{m,1}-v_{ni,R}^{m-1,L})^2\big]\Big\}^{\frac12}e_m, \qquad m=1,2,\dots,m_0,\quad m_0>1,$$
$$e_m=\prod_{j=m+1}^{m_0}\frac{1+\tau_j n\nabla_j}{1-\tau_j n\nabla_j},\quad m=1,2,\dots,m_0-1,\quad e_{m_0}=1. \tag{6.30}$$

7. The Total Error of the Algorithm

7.1. **Definition of the total error of the algorithm.** We calculate the components $u_{ni,R}^{m,k}$ and $v_{ni,R}^{m,k}$ by formulas (2.22). Then, for the chosen n and for $t=t_m$, the series $\sum_{i=1}^{n} w_{ni,R}^{m,k}\sin ix$, where

$$w_{ni,R}^{m,k}=\frac{1}{i}\,v_{ni,R}^{m,k}, \tag{7.1}$$

gives, at the kth iteration step, an approximate value of the exact solution $w(x,t_m)$ of problem (1.2), (1.3). Therefore the total error of algorithm (2.22) can be characterized by the difference

$$\Delta w_{n,R}^{m,k}(x)=w(x,t_m)-\sum_{i=1}^{n}w_{ni,R}^{m,k}\sin ix. \tag{7.2}$$

7.2. **Decomposition of the estimate into components.** To estimate error (7.2) for $t=t_{m_0}$, $1\le m_0\le M$, we have to majorize it by the errors for which the estimates have already been found. First of all, note that $\Delta w_{n,R}^{m_0,k}(x)$ is estimated through $\Delta_x w_{n,R}^{m_0,k}(x)=w_x(x,t_{m_0})-\sum\limits_{i=1}^{n}v_{ni,R}^{m_0,k}\cos ix$. Indeed, by virtue of (1.3), (7.1) and (7.2) we have $\Delta w_{n,R}^{m_0,k}(x)=\int_0^x\Delta_x w_{n,R}^{m_0,k}(\xi)d\xi$. Hence, denoting by $\|\cdot\|_{L^2(0,\pi)}$ the norm in the space $L^2(0,\pi)$, we get $\big(\Delta w_{n,R}^{m_0,k}(x)\big)^2\le x\int_0^x\big(\Delta_x w_{n,R}^{m_0,k}(\xi)\big)^2d\xi\le x\big\|\Delta_x w_{n,R}^{m_0,k}(x)\big\|^2_{L^2(0,\pi)}$ and therefore $\|\Delta w_{n,R}^{m_0,k}(x)\|_{L^2(0,\pi)}\le\frac{\pi}{\sqrt2}\|\Delta_x w_{n,R}^{m_0,k}(x)\|_{L^2(0,\pi)}$. If in this inequality we use (2.5), (2.19), (2.21), (3.3), (3.7), (3.8) and finally (4.1), (4.8), (7.1), then we obtain

$$\|\Delta w_{n,R}^{m_0,k}(x)\|_{L^2(0,\pi)}\le\pi\bigg[\bigg(\Big\|\sum_{i=1}^{n}(w_i'(t_{m_0})-u_{ni,R}^{m_0,k})\sin ix\Big\|^2_{L^2(0,\pi)} +\Big\|\sum_{i=1}^{n}(iw_i(t_{m_0})-v_{ni,R}^{m_0,k})\cos ix\Big\|^2_{L^2(0,\pi)}\bigg)^{\frac12}$$

$$+\left(\left\|\sum_{i=n+1}^{\infty} w_i'(t_{m_0})\sin ix\right\|^2_{L^2(0,\pi)} + \left\|\sum_{i=n+1}^{\infty} iw_i(t_{m_0})\cos ix\right\|^2_{L^2(0,\pi)}\right)^{\frac{1}{2}}\Bigg]$$

$$= \pi\Bigg(\|p_n\boldsymbol{s}(t_{m_0}) - \boldsymbol{s}_{n,R}^{m_0,k}\|_n + \left(\frac{\pi}{2}\sum_{l=0}^{1}\sum_{i=n+1}^{\infty}\left(i^l\frac{d^{1-l}w_i}{dt^{1-l}}(t_{m_0})\right)^2\right)^{\frac{1}{2}}\Bigg). \tag{7.3}$$

Let us estimate the first summand in the right-hand side of the equality in (7.3). Recalling definitions (4.2), (5.1), (6.4), we write

$$\|\Delta w_{n,R}^{m_0,k}(x)\|_{L^2(0,\pi)} \le \pi\Bigg(\|\Delta\boldsymbol{s}_n(t_{m_0})\|_n + \|\Delta\boldsymbol{s}_n^{m_0}\|_n$$
$$+ \|\Delta\boldsymbol{s}_{n,R}^{m_0,k}\|_n + \left(\frac{\pi}{2}\sum_{l=0}^{1}\sum_{i=n+1}^{\infty}\left(i^l\frac{d^{1-l}w_i}{dt^{1-l}}(t_{m_0})\right)^2\right)^{\frac{1}{2}}\Bigg). \tag{7.4}$$

7.3. **The main result – the total error of the algorithm.** In Lemmas 5.5 and 6.2, we have imposed restrictions (5.21), (6.8) and (6.9) on the grid step τ_m. For their fulfillment, in view of (5.22) and (6.7), it suffices for the inequality

$$\tau_m\max\Bigg[1,\sum_{l=0}^{1}\Bigg(\left(\frac{1}{\sqrt{2}}+\frac{1}{2}\right)\big(\max(z_*,z_{n,R}^{m-1,L})$$
$$+\max(z_*,z_{n,R}^{m,m-1})\big)\Bigg)^l\max_z\left|\frac{d^l\varphi}{dz^l}(z)\right|\Bigg] \le \frac{2q_m(1-\sigma)}{n} \tag{7.5}$$

to be valid, where

$$0 \le z \le \max\big(z_*, z_{n,R}^{m-1,L}, z_{n,R}^{m,m-1}\big). \tag{7.6}$$

The formulas for defining the values z_*, $z_{n,R}^{m-1,L}$ and $z_{n,R}^{m,m-1}$ in inequalities (7.5) and (7.6) have been presented above in a scattered way, but since they play an essential role in our discussion, we give them anew, this time in the combined form. Thus,

$$z_* = \frac{1}{\alpha}\left(\|w^1(x)\|^2_{L^2(0,\pi)} + \phi(\|w^{0\prime}(x)\|^2_{L^2(0,\pi)})\right),\quad z_{n,R}^{m-1,L} = \frac{1}{\alpha}(\|\boldsymbol{u}_{n,R}^{m-1,L}\|_n^2$$
$$+\phi(\|\boldsymbol{v}_{n,R}^{m-1,L}\|_n^2)),\quad z_{n,R}^{m,m-1} = \left(\|\boldsymbol{s}_{n,R}^{m-1,L}\|_n + \tau_m n h_m\frac{1}{1-q_m}\right)^2. \tag{7.7}$$

Further, let us replace (3.2) by an estimate, more convenient for calculation. Applying (3.21) and (3.22) to (3.2), we write

$$0 < T < \frac{1}{2\omega^2 N}\min(1,\alpha)\big(\zeta(2(p+s+1))\big)^{-1}. \tag{7.8}$$

Inequalities (7.5), (7.6) imply that the step τ_m satisfying (7.5) can be found before performing the counting operation by relations (2.22) for the given m. In fact, the problem consists in determining the step τ_m and the value q_m by (7.5)–(7.7) so that the calculation on the mth level by formulas (2.22) can be continued. Two approaches are possible here: either, in view of (6.7), we set the parameter q_m determining the error decrease rate on the mth level and use it to find the value of τ_m satisfying (7.5), or vice versa. As follows from (1.6) and (7.5)–(7.7), in the first case, for any q_m, there always exists a sufficiently small τ_m such that (7.5) is fulfilled. In the second case, it should be kept in mind that not every τ_m corresponds to an admissible value of q_m. When this happens, τ_m should be reduces.

Now let us formulate the main result.

Theorem 7.1. *Suppose that restrictions* (1.4)–(1.7) *are fulfilled, thereby ensuring the existence of a local solution of problem* (1.2), (1.3), *i.e., of a solution for* T *satisfying* (7.8) [8]. *Choose a value* σ *such that* $0<\sigma<1$. *Assume for each* $m=1,2,\dots,m_0$, $1\le m_0\le M$ *that the step* τ_m *is such that for* $0<q_m<1$ *it satisfies inequality* (7.5), *where the values* z_*, $z_{n,R}^{m-1,L}$ *and* $z_{n,R}^{m,m-1}$ *are defined by formulas* (7.7).

Then, with the chosen n *and for* $t = t_{m_0}$, *the total error of algorithm* (2.22) *is estimated at the* k*th iteration step,* $k = 1, 2, \dots$, *by*

$$\Big\| w(x, t_{m_0}) - \sum_{i=1}^{n} \frac{1}{i} v_{ni,R}^{m_0,k} \sin ix \Big\|_{L^2(0,\pi)} \le \pi \Big\{ \sum_{l=1}^{2} c_l(t_{m_0}) \Big(\frac{1}{n^{p+s+1}} \Big)^l$$
$$+ c_3(t_{m_0}) \frac{1}{n} \Big[\Big(1 + \frac{1}{m_0} \lambda n \Big)^{m_0} - 1 \Big] \max_{1 \le m \le m_0} \tau_m^p + \sum_{m=1}^{m_0-1} d_m q_m^{m_L} + d_{m_0} q_{m_0}^k \Big\}, \tag{7.9}$$

where $c_l(t_{m_0})$ *and* d_m, $l = 1, 2, 3$, $m = 1, 2, \dots, m_0$, *are the coefficients defined by* (3.32), (4.10), (5.31) *and* (6.30), λ *is the parameter defined by* (5.25), *and, finally,* m_L *denotes the number of iterations performed on the* m*th level,* $m = 1, 2, \dots, m_0 - 1$.

Proof. Let us consider relation (7.4). To estimate the summands $\|\Delta \boldsymbol{s}_n(t_{m_0})\|_n$, $\|\Delta \boldsymbol{s}_n^{m_0}\|_n$, $\|\Delta \boldsymbol{s}_{n,R}^{m_0,k}\|_n$ and $\Big(\frac{\pi}{2} \sum_{l=0}^{1} \sum_{i=n+1}^{\infty} \le \Big(i^l \frac{d^{1-l} w_i}{dt^{1-l}}(t_{m_0}) \Big)^2 \Big)^{\frac{1}{2}}$ in (7.4) we respectively apply inequalities (4.9), (5.23), (6.29) and (3.28). Recall also (7.1) and (7.2). The result is (7.9). □

We conclude the paper by making a few comments.

1. The simple way of finding τ_m by the given q_m, $m = 1, 2, \dots, m_0$, $1 \le m_0 \le M$, is to calculate the majorant of $z_{n,R}^{m,m-1}$ after replacing, on the basis of (7.5), $\tau_m n$ by $2q_m(1-\sigma)$ in the third equality of (7.7). Then we use $z_{n,R}^{m,m-1}$ in (7.6) and (7.5). As a result, we obtain a linear inequality with respect to τ_m. However, this technique gives, in general, a smaller upper bound of possible τ_m than it actually is.

2. If the value of an approximate solution $\sum_{i=1}^{n} w_{ni,R}^{m,k} \sin ix$, where $w_{ni,R}^{m,k}$ is defined by (7.1), is compared not with $w(x, t_m)$, as is done in subsection 7.1, but with $\sum_{i=1}^{n} w_i(t_m) \sin ix$, i.e., with the truncation of series (3.3) for $t = t_m$, then by analogy with (7.2), the total error of algorithm (2.22) is defined by the relation $\Delta w_{n,R}^{m,n}(x) = \sum_{i=1}^{n} (w_i(t_m) - w_{ni,R}^{m,k}) \sin ix$. Estimate (7.9) remains valid for this difference with only one correction. The coefficient $c_1(t_{m_0})$ should be set equal to zero.

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