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ON THE CRITERION OF THE WELL-POSEDNESS OF THE MODIFIED INITIAL PROBLEM FOR SINGULAR LINEAR ORDINARY DIFFERENTIAL SYSTEMS

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Abstract. Effective necessary and sufficient conditions are established for the well-posedness of the initial problem with weight for linear systems of ordinary differential equations with singularities.

1. Statement of the Problem, Basic Notation and Formulation of Basic Results

Let $[a,b]\subset\mathbb{R}$ be a finite and closed interval, non-degenerate at the point $t_0\in]a,b[$, and $I_{t_0}=[a,b]\setminus\{t_0\}$.

Consider the initial problem with weight for a linear system of ordinary differential equations with singularities

$$\frac{dx}{dt}=P(t)\,x+q(t)\quad\text{for a.a.}\quad t\in I_{t_0},\tag{1.1}$$

$$\lim_{t\to t_0}(\Phi^{-1}(t)\,x(t))=0,\tag{1.2}$$

where $P=(p_{ik})_{i,k=1}^n\in L_{\rm loc}(I_{t_0};\mathbb{R}^{n\times n})$, $q=(q_k)_{k=1}^n\in L_{\rm loc}(I_{t_0};\mathbb{R}^n)$; $\Phi=\operatorname{diag}(\varphi_1,\dots,\varphi_n)$ is a positive diagonal $n\times n$-matrix function, defined continuously on $[a,b]\setminus\{t_0\}$.

Without loss of generality, we can assume that $t_0=a$, or $t_0=b$.

Consider the case $t_0=b$. The case $t_0=a$ will be considered analogously.

Let $I=I_b$=[a,b[.

Along with system (1.1), let us consider the sequence of perturbed singular systems

$$\frac{dx}{dt}=P_m(t)\,x+q_m(t)\quad\text{for a.a.}\quad t\in I\ \ (m=1,2,\dots),\tag{1.3}$$

under condition (1.2), where $P_m=(p_{mik})_{i,k=1}^n\in L_{\rm loc}(I;\mathbb{R}^{n\times n})$, $q_m=(q_{mk})_{k=1}^n\in L_{\rm loc}(I;\mathbb{R}^n)$.

We are interested in establishing the necessary and sufficient conditions under which the unique solvability of problem (1.1), (1.2) guarantees the unique solvability of problem (1.3), (1.2) and the nearness of its solution in the definite sense if the matrix-functions P_m and P and the vector-functions q_m and q are close to each other.

As we know, the question on the necessary and sufficient conditions for the well-posedness was not fully investigated in earlier papers. In [1], only some results are presented. Thus, the problem under consideration is urgent. In our paper, an attempt is made to fill in the existing gaps.

The same problem was previously studied earlier in [6] (see also references therein), where only the sufficient conditions were obtained.

The singularity of system (1.1) is considered in the sense that the matrix P and vector q functions are in general not integrable on each interval including t_0. Moreover, the solution of problem (1.1), (1.2) is not continuous at the point t_0 and, therefore, it cannot be a solution in the classical sense. But its restriction on each interval from I_{t_0} is a solution of system (1.1). In this regard, we recall the following example from [6].

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Let $\alpha > 0$ and $\varepsilon \in]0, \alpha[$. Then $x(t) = |t|^{\varepsilon-\alpha} \operatorname{sgn} t$ is the unique solution of the problem

$$\frac{dx}{dt} = -\frac{\alpha x}{t} + \varepsilon |t|^{\varepsilon-1-\alpha}, \quad \lim_{t\to 0}(t^{\alpha} x(t)) = 0.$$

The function x is not a solution of the equation on the set $\mathbb{R}$, however x is a solution to the above equation only on $\mathbb{R} \setminus \{0\}$.

The questions of the solvability and well-posedness of singular differential problem (1.1), (1.2) have been studied, for example, in [5–7] (see also references therein). As far as we know, the necessary and sufficient conditions for the well-posedness of problem (1.1), (1.2) with singularity have not been studied up to now.

Similar problems for singular impulsive and the so-called generalized orinary differential equations are investigated in [1–4].

The present paper presents the necessary and sufficient conditions for the so-called strongly Φ-well-posedness of problem (1.1), (1.2).

Throughout the paper, we use the following notation and definitions:

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$ matrices $X = (x_{ik})_{i,k=1}^{n,m}$ with the norm $\|X\| = \max\limits_{k=1,\dots,m} \sum\limits_{i=1}^{n} |x_{ik}|$.

$$|X| = (|x_{ik}|)_{i,k=1}^{n,m}, \quad [X]_- = \frac{1}{2}(|X| - X), \quad [X]_+ = \frac{1}{2}(|X| + X).$$

$\mathbb{R}_+^{n\times m} = \big\{(x_{ij})_{i,j=1}^{n,m} : x_{ij} \geq 0 \ \ (i = 1, \dots, n; \ \ j = 1, \dots, m)\big\}$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all column n-vectors $x = (x_i)_{i=1}^{n}$.

$O_{n\times m}$ (or O) is the zero $n \times m$-matrix, 0_n (or 0) is the zero n-vector.

I_n is identity $n \times n$-matrix.

The inequalities between the matrices are understood componentwise.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1} and $r(X)$ are, respectively, the matrix inverse to X and the spectral radius of X.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its components is such.

$C(I;\mathbb{R}^{n\times m})$ is the space of all continuous and bounded matrix-functions $X : I \to \mathbb{R}^{n\times m}$ with the norm $\|X\|_{\infty,I} = \sup\{\|X(t)\| : t \in I\}$.

$C(I;D)$, where $D \subset \mathbb{R}^{n\times m}$, is the set of all continuous and bounded matrix-functions $X : I \to D$.

$C_{\mathrm{loc}}(I;D)$ is the set of all continuous matrix-functions $X : I \to D$.

If $X \in C([a,b];\mathbb{R}^{n\times m})$, then $\|X\|_c = \max\{\|X(t)\| : t \in [a,b]\}$.

$\mathrm{AC}([a,b];D)$ is the set of all absolutely continuous matrix-functions $X : [a,b] \to D$.

$\mathrm{AC}_{\mathrm{loc}}(I;D)$ is the set of all matrix-functions $X : I \to D$ whose restrictions to an arbitrary closed interval $[a,b]$ from I belong to $AC([a,b];D)$.

$L([a,b];\mathbb{R}^{n\times m})$ is the set of all integrable matrix-functions on $[a,b]$.

$L_{\mathrm{loc}}(I;\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : I \to D$ for which the restriction on $[a,c]$ belongs to $L([a,c];\mathbb{R}^{n\times m})$ for every $a, c \in I$.

A vector-function $x \in \mathrm{AC}_{\mathrm{loc}}(I;\mathbb{R}^n)$ is said to be a solution of system (1.1) if

$$x'(t) = P(t)\,x(t) + q(t) \ \text{ for a. a. } \ t \in I.$$

Let $P_* = (p_{*ik})_{i,k=1}^{n} \in L_{\mathrm{loc}}(I;\mathbb{R}^n)$. Then a matrix-function $C_* : I \times I \to \mathbb{R}^{n\times n}$ is said to be the Cauchy matrix of the homogeneous system

$$\frac{dx}{dt} = P_*(t)\,x, \tag{1.4}$$

if for each interval $J \subset I$ and $\tau \in J$, the restriction of the matrix-function $C_*(.,\tau) : I \to \mathbb{R}^{n\times n}$ on J is the fundamental matrix of system (1.4), satisfying the condition

$$C_*(\tau,\tau) = I_n.$$

Therefore, C_* is the Cauchy matrix of system (1.4) if and only if the restriction of C_* on $J \times J$ is the Cauchy matrix of the system in the regular case. Let $X_*(t) \equiv C_*(t,a)$.

Definition 1.1. Problem (1.1), (1.2) is said to be weakly Φ-well-posed if it has the unique solution x_0 and for every sequences of matrix and vector-functions P_m and q_m $(m = 1, 2, \dots)$ such that conditions

$$\lim_{m\to+\infty} \left\| \int\limits_t^b \Phi^{-1}(s)\, |P_m(s) - P(s)|\, \Phi(s) ds \right\| = 0, \tag{1.5}$$

$$\lim_{m\to+\infty} \left\| \int\limits_t^b \Phi^{-1}(s)\, r_m(s) ds \right\| = 0 \tag{1.6}$$

and

$$\lim_{m\to+\infty} \|\Phi^{-1}(t) r_m(t) - \Phi^{-1}(b) r_m(b)\| = 0, \tag{1.7}$$

are fulfilled uniformly on I, where $r_m(t) \equiv \int\limits_a^t |q_m(s) - q(s)| ds$, problem (1.3), (1.2) has the unique solution x_m for each sufficiently large m and the condition

$$\lim_{m\to+\infty} \|\Phi^{-1}(t)\, (x_m(t) - x_0(t))\| = 0 \tag{1.8}$$

holds uniformly on I.

Definition 1.2. Problem (1.1), (1.2) is said to be strongly Φ-well-posed if it has the unique solution x_0 and for every sequences of matrix– and vector-functions P_m and q_m $(m = 1, 2, \dots)$ such that conditions (1.5) and

$$\lim_{m\to+\infty} \left\| \int\limits_t^b \Phi^{-1}(s) |q_m(s) - q(s)| ds \right\| = 0,$$

hold uniformly on I, problem (1.3), (1.2) has the unique solution x_m for each sufficiently large m and condition (1.8) holds uniformly on I.

The case of strongly Φ-well-posedness has been investigated in [6], where only the sufficient conditions guaranteeing this property were obtained.

We establish the necessary and sufficient, as well effective sufficient conditions for the weakly Φ-well-posedness.

Remark 1.1. If problem (1.1), (1.2) is strongly well-posed, then it is also stated weakly well-posed, since due to the formula of integration by parts, we find

$$\int\limits_t^\tau \Phi^{-1}(s) |q_m(s) - q(s)| ds = \left(\Phi^{-1}(s)\, r_m(s)\right)\Big|_t^\tau + \int\limits_t^\tau \Phi^{-1}(s)\, r_m(s) ds$$
$$\text{for } a \le t < \tau < b.$$

Definition 1.3. We say that the sequence (P_m, q_m) $(m = 1, 2, \dots)$ belongs to the set $\mathcal{S}_{P_*}(P, q; \Phi)$, i.e.,

$$\left((P_m, q_m)\right)_{m=1}^{+\infty} \in \mathcal{S}_{P_*}(P, q; \Phi), \tag{1.9}$$

if problem (1.3),(1.2) has the unique solution x_m for each sufficiently large m and condition (1.8) holds uniformly on I.

Let $I(\delta) = [b - \delta, b[$ for every $\delta > 0$.

Theorem 1.1. *Let there exist a matrix-function $P_* \in L_{\mathrm{loc}}(I; \mathbb{R}^{n\times n})$ and constant matrices B_0 and B from $\mathbb{R}_+^{n\times n}$ such that*

$$r(B) < 1 \tag{1.10}$$

and estimates

$$|C_*(t, \tau)| \le \Phi(t)\, B_0\, \Phi^{-1}(\tau) \quad \text{for } b - \delta \le t \le \tau < b \tag{1.11}$$

and

$$\Big|\int\limits_t^b |C_*(t,s)|\,|P(s)-P_*(s)|\,\Phi(s)ds\Big| \le H(t)\,B \ \ for\ \ t\in I(\delta) \tag{1.12}$$

are fulfilled for some $\delta>0$, where C_* is the Cauchy matrix of system (1.4)*. Let, moreover,*

$$\lim_{t\to b}\int\limits_t^b \|\Phi^{-1}(t)\,C_*(t,\tau)q(\tau)\|d\tau = 0. \tag{1.13}$$

Then problem (1.1), (1.2) *is weakly Φ-well-posed with respect to the matrix-function P_*.*

Theorem 1.2. *Let there exist a constant matrix $B=(b_{ik})_{i,k=1}^n\in\mathbb{R}_+^{n\times n}$ such that condition* (1.10) *is satisfied, and the estimates*

$$c_i(t,\tau)\le b_0\frac{\varphi_i(t)}{\varphi_i(\tau)} \ \ for\ \ b-\delta\le t\le\tau<b \ \ (i=1,\dots,n); \tag{1.14}$$

$$\Big|\int\limits_t^b c_i(t,\tau)\varphi_i(\tau)[p_{ii}(\tau)]_-d\tau\Big| \le b_{ii}\,\varphi_i(t) \ \ for\ \ t\in I(\delta) \ \ (i=1,\dots,n) \tag{1.15}$$

and

$$\Big|\int\limits_t^b c_i(t,\tau)\varphi_k(\tau)|p_{ik}(\tau)|d\tau\Big| \le b_{ik}\,h_i(t) \ \ for\ \ t\in I(\delta) \ \ (i\ne k;\ i,k=1,\dots,n) \tag{1.16}$$

are fulfilled for some $b_0>0$ and $\delta>0$, where

$$c_i(t,\tau)=\exp\bigg(-\Big|\int\limits_t^\tau [p_{ii}(s)]_+ds\Big|\bigg). \tag{1.17}$$

Let, moreover,

$$\lim_{t\to b}\int\limits_t^b \frac{c_i(t,\tau)}{\varphi_i(t)}|q_i(\tau)|d\tau = 0 \ \ (i=1,\dots,n). \tag{1.18}$$

Then problem (1.1), (1.2) *is weakly Φ-well-posed with respect to the matrix-function $P_*(t)\equiv$ $\operatorname{diag}([p_{11}(t)]_+,\dots,[p_{nn}(t)]_+)$.*

Corollary 1.1. *Let there exist a constant matrix $B=(b_{ik})_{i,k=1}^n\in\mathbb{R}_+^{n\times n}$ such that the condition* (1.10) *is satisfied, and the estimates*

$$p_{ii}(t)\le\frac{\mu_i}{b-t} \ \ for\ \ a\le t<b \ \ (i=1,\dots,n), \tag{1.19}$$

$$\lim_{\tau\to b-}\int\limits_t^\tau [p_{ii}(s)]_-ds\le b_{ii} \ \ for\ \ t\in I(\delta) \ \ (i=1,\dots,n) \tag{1.20}$$

and

$$\lim_{\tau\to b}\int\limits_t^\tau |p_{ik}(s)|ds\le b_{ik} \ \ for\ \ t\in I(\delta) \ \ (i\ne k;\ i,k=1,\dots,n) \tag{1.21}$$

are fulfilled for some $\mu_i\ge0$ $(i=1,\dots,n)$ and $\delta>0$. Let, moreover,

$$\lim_{t\to b}\int\limits_t^b \frac{1}{(b-\tau)^{\mu_i}}|q_i(s)|ds = 0 \ \ (i=1,\dots,n). \tag{1.22}$$

Then problem (1.1), (1.2) *is weakly Φ-well-posed with respect to matrix-function $P_*(t)\equiv$ $\operatorname{diag}([p_{11}(t)]_+,\dots,[p_{nn}(t)]_+)$, where $\Phi(t)\equiv\operatorname{diag}((b-t)^{\mu_1},\dots,(b-t)^{\mu_n})$.*

Remark 1.2. In the conditions of Corollary 1.1, the solution of problem (1.1), (1.2), where $\Phi(t) \equiv \operatorname{diag}((b-t)^{\mu_1},\dots,(b-t)^{\mu_n})$, belongs to $\mathrm{AC}_{\mathrm{loc}}(I,\mathbb{R}^n)$.

Corollary 1.2. *Let there exist a constant matrix $B=(b_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n\times n}$ such that condition* (1.10) *holds, and estimates* (1.19) *for $\mu_i = 0$ $(i=1,\dots,n)$,*

$$\int\limits_t^b (b-\tau)[p_{ii}(s)]_- ds \le b_{ii}\,(b-t) \ \text{ for } \ t\in I(\delta) \ \ (i=1,\dots,n)$$

and

$$\int\limits_t^b (b-\tau)|p_{ik}(s)| ds \le b_{ik}\,(b-t) \ \text{ for } \ t\in I(\delta) \ \ (i\neq k;\ i,k=1,\dots,n)$$

are fulfilled for some $\delta>0$. Let, moreover,

$$\lim_{t\to b}\frac{1}{b-t}\int\limits_t^b |q_i(s)| ds = 0 \ \ (i=1,\dots,n).$$

Then problem (1.1), (1.2) *is weakly Φ-well-posed with respect to the matrix-function $P_*(t) \equiv \operatorname{diag}([p_{11}(t)]_+,\dots,[p_{nn}(t)]_+)$, where $\Phi(t)\equiv \operatorname{diag}((b-t),\dots,(b-t))$.*

The remark analogous to Remark 1.2 is likewise true for Corollary 1.2 if $\mu_i = 1$ $(i=1,\dots,n)$.

Corollary 1.3. *Let*

$$[p_{ii}(t)]_{\le}\frac{\lambda_i}{b-t} + p_{ii}^*(t) \ \text{ for } \ t\in I \ \ (i=1,\dots,n)$$

hold, where $\lambda_i \ge 0$ $(i=1,\dots,n)$, $p_{ii}^*(t)\in L_{\mathrm{loc}}(I;\mathbb{R}_+)$ $(i=1,\dots,n)$. Let, moreover,

$$\int\limits_t^b (b-\tau)^{\lambda_i-\lambda_k}|p_{ik}(s)| ds < +\infty \ \text{ for } \ t\in I \ \ (i\neq k;\ i,k=1,\dots,n)$$

and

$$\int\limits_t^b (b-\tau)^{\lambda_i}|q_i(s)| ds < +\infty \ \text{ for } \ t\in I \ \ (i=1,\dots,n).$$

Then problem (1.1), (1.2) *is weakly Φ-well-posed with respect to the matrix-function P_* defined as in Corollary* 1.2, *where $\Phi(t)\equiv\operatorname{diag}((b-t)^{-\lambda_1},\dots,(b-t)^{-\lambda_n})$.*

The remark analogous to Remark 1.2 is likewise true for Corollary 1.3 if $\lambda_i = 0$ $(i=1,\dots,n)$.

Theorem 1.3. *Let the conditions of Theorem* 1.1 *be fulfilled and let there exist a sequence of the non-degenerated matrix-functions $H_m\in \mathrm{AC}_{\mathrm{loc}}(I;\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$ such that*

$$\lim_{m\to+\infty}\|\Phi^{-1}(t)H_m^{-1}(t)\Phi(t)-I_n\| = 0, \tag{1.23}$$

$$\lim_{m\to+\infty}\left\|\int\limits_t^b \Phi^{-1}(s)\,|P_m^*(s)-P(s)|\,\Phi(s)ds\right\| = 0, \tag{1.24}$$

$$\lim_{m\to+\infty}\left\|\int\limits_t^b \Phi^{-1}(s)\,r_m^*(s)ds\right\| = 0 \tag{1.25}$$

and

$$\lim_{m\to+\infty}\|\Phi^{-1}(t)r_m^*(t)-\Phi^{-1}(b)r_m^*(b)\| = 0, \tag{1.26}$$

are fulfilled uniformly on I, where $P^*_m(t) \equiv (H'_m(t) + H_m(t)P_m(t))\, H^{-1}_m(t)$, $r^*_m(s) \equiv \int\limits_a^s |q^*_m(\tau) - q(\tau)|d\tau$ *and* $q^*_m(t) \equiv H_m(t)\, q_m(t)$. *Then inclusion* (1.9) *holds.*

Theorem 1.3 has the following form if we assume that $H_m(t) \equiv I_n$ $(m = 1, 2, \dots)$ therein.

Corollary 1.4. *Let the conditions of Theorem* 1.1 *be fulfilled and conditions* (1.5)–(1.7) *hold uniformly on I, where the vector-functions r_m $(m = 1, 2, \dots)$ are defined as in Definition* 1.1. *Then inclusion* (1.9) *holds.*

Theorem 1.4. *Let the conditions of Theorem* 1.1 *be fulfilled and let*

$$\|B_0\|\, \|(I_n - B)^{-1}\| < 1, \tag{1.27}$$

$$\limsup_{t\to b} \left\| \Phi^{-1}(t) \int\limits_t^b |P(s)|\, \Phi(s)ds \right\| < +\infty. \tag{1.28}$$

$$\limsup_{t\to b} \left\| \int\limits_t^b \Phi^{-1}(s)|P_*(s)|\, \Phi(s)ds \right\| < +\infty. \tag{1.29}$$

Then inclusion (1.9) *holds if and only if there exists the sequence of matrix functions* $H_m \in \mathrm{AC}_{\mathrm{loc}}(I; \mathbb{R}^{n\times n})$ $(m = 1, 2, \dots)$ *such that conditions* (1.23)–(1.26) *hold uniformly on I, where the matrix– and vector functions* P^*_m, q^*_m *and* r^*_m $(m = 1, 2, \dots)$ *are defined as in Theorem* 1.3.

Theorem 1.4′. *Let the conditions of Theorem* 1.4 *be fulfilled. Then inclusion* (1.9) *holds if and only if conditions* (1.25), (1.26) *and*

$$\lim_{m\to+\infty} \|\Phi^{-1}(t)(X_m(t) - X_0(t))\| = 0$$

hold uniformly on I, where X_0 and X_m are the fundamental matrices of systems (1.1) *and* (1.3), *respectively, and* $q^*_m(t) \equiv X_0(t)X^{-1}_m(t)\, q_m(t)$ $(m = 1, 2, \dots)$.

Remark 1.3. Condition (1.27) in Theorem 1.4 is essential and it cannot be neglected, i.e., if the condition is violated, then the conclusion of the theorem is not true, in general. Below we present an example.

Let $I = [0, 1[$, $n = 1$, $b = 1$, $B = 0$, $B_0 = 1$, $Phi(t) \equiv 1 - t$;

$$P(t) = P_m(t) = P_*(t) \equiv -(1 - t)^{-1} \ \ (m = 1, 2, \dots);$$

$$q(t) \equiv 0, \ \ q_m(t) \equiv -\frac{1 - t}{m\cos^2(1 - t)} \ \ (m = 1, 2, \dots).$$

Then

$$C_*(t, \tau) \equiv \frac{1 - t}{1 - \tau}, \ \ x_0(t) \equiv 0, \ \ x_m(t) \equiv (1 - t)\frac{\tan(1 - t)}{m} \ \ (m = 1, 2, \dots).$$

So, all the conditions of Theorem 1.4, with the exception of (1.27), are fulfilled for $H_m(t) \equiv I_n$. But condition (1.8) is not fulfilled uniformly on I.

Remark 1.4. The results analogous to Theorems 1.1, 1.2 and Corollary 1.4 are proved in [6] for the strongly well-posed case, as well. However, in the paper under consideration, the necessary and sufficient conditions for the well-posedness in the strong case are not considered.

2. Proofs of Results

We need the following lemma from [6].

Lemma 2.1. *Let the matrix-function* $P_* \in \mathrm{AC}_{\mathrm{loc}}(I, \mathbb{R}^{n\times n})$ *and constant matrices B_0 and B from* $\mathbb{R}^{n\times n}_+$ *be such that conditions* (1.10), (1.11) *and* (1.12) *hold for some* $\delta > 0$, *where C_* is the Cauchy* *matrix of system* (1.4)*. Let, moreover,*

$$\gamma(t)=\sup\bigg\{\bigg\|\int\limits_s^b|\Phi^{-1}(s)C_*(s,\tau)q(\tau)|d\tau\bigg\|:t\le s<b\bigg\}<+\infty\ \ \text{for}\ \ t\in I(\delta).$$

Then each solution $x\in \mathrm{AC}_{\mathrm{loc}}(J,\mathbb{R}^n)$ *of system* (1.1) *admits the estimate*

$$\|\Phi^{-1}(t)x(t)\|\le\rho\left(\|B_0\|\cdot\|\Phi^{-1}(s_0)x(s_0)\|+\gamma(t)\right)\ \ \text{for}\ \ t\in J,\ \ t\le s<b,$$

where $\rho=\|(I_n-B)^{-1}\|$*, and* $J\subset I(\delta)$ *and* $s_0\in J$ *are an arbitrary interval and point.*

Proof of Theorem 1.1. Owing to conditions (1.10)–(1.13), problem (1.1), (1.2) has the unique solution x (see [6, Theorem 2.1]). On the other hand, since I is the finite interval, there exists $\overline{\rho}\in\mathbb{R}_+^{n\times n}$ such that

$$|x(t)|\le\Phi(t)\overline{\rho}\quad\text{for}\quad t\in I. \tag{2.1}$$

It is clear that

$$\rho_1=\sup\{\rho_1(\delta):\delta\in]0,b-a]\}<+\infty, \tag{2.2}$$

where

$$\rho_1(\delta)=\bigg\|\int\limits_a^{b-\delta}\Phi^{-1}(s)|P(s)-P_*(s)|\,\Phi(s)ds\bigg\|.$$

Let B_1 be the $n\times n$-matrix whose every element is equal to 1 and let $\widetilde{B}=B+\eta_0B_0\,B_1$. Then due to (1.10), there exists $\eta_0\in]0,1[$ such that

$$r(\widetilde{B})<1. \tag{2.3}$$

Let $\varepsilon>0$ be an arbitrary fixed number. Then, taking into account (2.2), we find that there exists $\eta\in]0,\eta_0[$ such that

$$\rho_0\big[1+\|B_0\|\,\exp\big((\eta+\rho_1)\|B_0\|\big)\big]<\varepsilon, \tag{2.4}$$

where

$$\rho_0=\eta\,(1+\|\overline{\rho}\|)(1+\|(I_n-\widetilde{B})^{-1}\|\,\|B_0\|).$$

Let $P_m\in L_{\mathrm{loc}}(I;\mathbb{R}^{n\times n})$ and $q_m\in L_{\mathrm{loc}}(I;\mathbb{R}^n)$ $(m=1,2,\dots)$ be arbitrary matrix– and vector-functions satisfying conditions (1.5) and (1.6). First, we have to show that the matrix– and vector-functions P_m and q_m $(m=1,2,\dots)$ satisfy conditions (1.12) and (1.13), as well.

In view of (1.11) and (1.12), we find, without loss of generality, that for every natural m,

$$\begin{aligned}&\bigg|\int\limits_t^\tau|C_*(t,s)||P_m(s)-P_*(s)|\,\Phi(s)ds\bigg|\\ &\le\bigg|\int\limits_t^\tau|C_*(t,s)||P(s)-P_*(s)|\,\Phi(s)ds\bigg|\\ &+\bigg|\int\limits_t^\tau|C_*(t,s)|\,|P_m(s)-P_*(s)|\,\Phi(s)ds\bigg|\\ &\le\Phi(t)B+\Phi(t)B_0\bigg|\int\limits_t^\tau|C_*(t,s)|\,|P_m(s)-P_*(s)|\,\Phi(s)ds\bigg|\\ &\le\Phi(t)\widetilde{B}\ \ \text{for}\ \ a\le t<\tau<b.\end{aligned}$$

Therefore, the matrix-function $\widetilde{B}$ satisfies condition (1.12).

In addition, due to (1.23)–(1.26), there exists a natural m_0 such that for every $m > m_0$,

$$\Big\| \int_t^{b-} \Phi^{-1}(s)\, |P_m(s) - P(s)|\, \Phi(s) ds \Big\| < \eta \ \text{ for } \ t \in I \tag{2.5}$$

and

$$\|\Phi^{-1}(t) r_m(t) - \Phi^{-1}(b-) r_m(b-)\| + \Big\| \int_t^{b-} \Phi^{-1}(s)\, |P_*(s)|\, r_m(s) ds \Big\| < \eta \ \text{ for } \ t \in I, \tag{2.6}$$

where the functions r_m $(m = 1, 2, \dots)$ are defined as in Definition 1.1.

Below, we assume that $m > m_0$ is an arbitrary fixed natural.

Now, using (1.11), we show that

$$\int_t^{\tau} |\Phi^{-1}(t)\, C_*(t,s)(q_m(s) - q(s))| ds \le B_0 \big(\Phi^{-1}(s) r_m(s)\big)\Big|_t^{\tau}$$
$$+ B_0 \int_t^{\tau} \Phi^{-1}(s)\, |P_*(s)|\, |q_m(s) - q(s)| ds \ \text{ for } \ a \le t < \tau < b \tag{2.7}$$

and therefore,

$$\int_t^{\tau} |\Phi^{-1}(t)\, C_*(t,s) q_m(s)| ds \le \int_t^{\tau} |\Phi^{-1}(t)\, C_*(t,s) q(s)| ds$$
$$+ \Phi^{-1}(t) \int_t^{\tau} |C_*(t,s)(q_m(s) - q(s))| ds \le \int_t^{\tau} |\Phi^{-1}(t) C_*(t,s) q(s)| ds$$
$$+ B_0 \int_{\tau}^{t} \Phi^{-1}(s) |q_m(s) - q(s)| ds$$
$$\le \int_t^{\tau} \Phi^{-1}(t) C_*(t,s) P_*(s) q(s) ds + B_0 \big(\Phi^{-1}(s) r_m(s)\big)\Big|_t^{\tau}\Big|$$
$$+ B_0 \int_t^{\tau} \Phi^{-1}(s)\, |P_*(s)|\, |q_m(s) - q(s)| ds \ \text{ for } \ a \le t < \tau < b.$$

From this, in view of conditions (1.6), (1.13) and (1.26), it follows that the vector-function q_m satisfies condition (1.13), as well.

Hence, according to Theorem 2.1 from [6], the last two conditions guarantee the unique solvability of problem (1.3), (1.2).

Let x_m be the unique solution of problem (1.3), (1.2) and, without loss of generality, let

$$z_m(t) \equiv x(t) - x_m(t) \ \text{ and } \ u_m(t) \equiv \|\Phi^{-1}(t) z_m(t)\|$$

for every natural m.

Then z_m will be a solution of the system

$$\frac{dz}{dt} = P_m(t)\, z + \xi_m(t) \tag{2.8}$$

under the condition

$$\lim_{s_0 \to b} (\Phi^{-1}(s_0)\, z(s_0)) = 0,$$

where

$$\xi_m(t) = g_m(t) + (q(t) - q_m(t)), \quad g_m(t) = (P(t) - P_m(t))\, x(t).$$

In view of Lemma 2.1, conditions (2.3) and (2.8) guarantee the estimate

$$u_m(t) \leq \|(I_n - \widetilde{B})^{-1}\|\, \gamma_m(t) \quad \text{for} \quad t \in I(\delta), \tag{2.9}$$

where

$$\gamma_m(t) = \sup\left\{ \left\| \int\limits_s^b |\Phi^{-1}(s) C_*(s,\tau)\xi_m(\tau)|d\tau \right\| : t \leq s < b \right\} \text{ for } t \in I(\delta).$$

In addition, in view of (2.1) and (2.7), we get

$$\begin{aligned}\int\limits_s^{s_0} |\Phi^{-1}(s) C_*(s,\tau)\xi_m(\tau)|d\tau &\leq \int\limits_s^{s_0} \Phi^{-1}(s)|C_*(s,\tau)|\,|g_m(\tau)|d\tau \\ &+ \int\limits_s^{s_0} \Phi^{-1}(s)|C_*(s,\tau)|\,|\xi_m(\tau) - g_m(\tau)|d\tau \\ &\leq \int\limits_s^{s_0} \Phi^{-1}(s)|C_*(s,\tau)|P(\tau) - P_m(\tau)|\,|x(\tau)|d\tau \\ &+ B_0 \int\limits_s^{s_0} \Phi^{-1}(\tau)|q(\tau) - q_m(\tau)|d\tau \text{ for } a \leq s < s_0 < b\end{aligned}$$

and, therefore, due to (1.11), (2.6) and (2.9), we find

$$\gamma_m(t) \leq \eta\,(1 + \|\overline{\rho}\|)\|B_0\|$$

and

$$u_m(t) \leq \eta\,(1 + \|\overline{\rho}\|)\,\|(I_n - \widetilde{B})^{-1}\|\,\|B_0\| < \rho_0 \text{ for } t \in I(\delta). \tag{2.10}$$

Now, consider the case for $t \in [a, b - \delta]$.

Due to (2.8), the vector-function z_m satisfies the system

$$\frac{dz}{dt} = P_*(t)\, z + (P_m(t) - P_*(t))\, z + \xi_m(t).$$

Therefore, using the Cauchy formula, we conclude

$$\begin{aligned}\Phi^{-1}(t) z_m(t) &= \Phi^{-1}(t) C_*(t, b-\delta) z(b-\delta) \\ &+ \int\limits_t^{b-\delta} \Phi^{-1}(t) C_*(t,\tau)(P_m(\tau) - P_*(\tau))\,\Phi(\tau)\cdot(\Phi^{-1}(\tau) z(\tau))d\tau \\ &+ \int\limits_t^{b-\delta} \Phi^{-1}(t) C_*(t,\tau)(P_m(\tau) - P_*(\tau))\, x(\tau)d\tau \\ &+ \int\limits_t^{b-\delta} \Phi^{-1}(t) C_*(t,\tau)(q(\tau) - q_m(\tau))d\tau \text{ for } t \in [a, b-\delta[,\end{aligned}$$

and by (2.7), we have

$$\left\| \int\limits_t^{b-\delta} |\Phi^{-1}(t)\, C_*(t,s)(q_m(s) - q(s))|ds \right\| \leq \|B_0\| \left\| \left(\Phi^{-1}(s) r_m(s)\right)\Big|_t^{b-\delta} \right\|$$

$$+\|B_0\|\left\|\int\limits_t^{b-\delta}\Phi^{-1}(s)\,|P_*(s)|\,|q_m(s)-q(s)|ds\right\|\ \text{ for }\ t\in[a,b-\delta],$$

whence, taking into account (1.11), (2.1) and (2.7), we conclude

$$\|\Phi^{-1}(t)z_m(t)\|\le\|B_0\|\|\Phi^{-1}(t)z_m(b-\delta)\|$$
$$+\|B_0\|\left\|\int\limits_t^{b-\delta}\Phi^{-1}(\tau)|P_m(\tau)-P_*(\tau)|\Phi(\tau)\cdot|\Phi^{-1}(\tau)z(\tau)|d\tau\right\|$$
$$+\|B_0\|\left\|\int\limits_t^{b-\delta}\Phi^{-1}(\tau)|P_m(\tau)-P_*(\tau)|\,|x(\tau)|d\tau\right\|+\|B_0\|\left\|\big(\Phi^{-1}(s)r_m(s)\big)\Big|_t^{b-\delta}\right\|$$
$$+\|B_0\|\left\|\int\limits_t^{b-\delta}\Phi^{-1}(s)\,|P_*(s)|\,|q_m(s)-q(s)|ds\right\|\ \text{ for }\ t\in[a,b-\delta].$$

Thus, due to (2.1), (2.6) and (2.10), we find

$$u_m(t)\le\rho_0\|B_0\|+\|B_0\|\int\limits_t^{b-\delta}\omega_m(\tau)u_m(\tau)d\tau,\ \text{ for }\ t\in[a,b-\delta],$$

where

$$\omega_m(\tau)\equiv\|\Phi^{-1}(\tau)|P_m(\tau)-P_*(\tau)|\Phi(\tau)\|.$$

Therefore, according to the well-known Gronwall's inequality, we get

$$u_m(t)\le\rho_0\|B_0\|\,\exp\left(\|B_0\|\left\|\int\limits_t^{b-\delta}w_m(s)ds\right\|\right)\ \text{ for }\ t\in[a,b-\delta]. \tag{2.11}$$

In addition, by (2.5), we have

$$\|B_0\|\left\|\int\limits_t^{b-\delta}w_m(s)ds\right\|\le\|B_0\|\left\|\int\limits_t^{b-\delta}\|\Phi^{-1}(\tau)|P_m(\tau)-P(\tau)|\Phi(\tau)\|ds\right\|$$
$$+\|B_0\|\left\|\int\limits_t^{b-\delta}\|\Phi^{-1}(\tau)|P(\tau)-P_*(\tau)|\Phi(\tau)\|ds\right\|$$
$$\le(\eta+\rho_1)\|B_0\|\ \text{ for }\ t\in[a,b-\delta].$$

Hence, owing to (2.11), we have

$$u_m(t)\le\rho_0(\|B_0\|\,\exp\big((\eta+\rho_1)\|B_0\|\big)\ \text{ for }\ t\in[a,b-\delta].$$

Relying on the above and (2.11), due to (2.4), we have

$$\|\Phi^{-1}(t)z_m(t)\|<\varepsilon\ \text{ for }\ t\in I.$$

Therefore, estimate (1.8) holds uniformly on I. □

Proof of Theorem 1.2. By the definition of the matrix-function $P_*(t)$, we have $p_{*ik}(t)=0$ if $i\neq k$ $(i,k=1,\dots,n)$.

Consider the case, where $i=k$ $(i=1,\dots,n)$. It is evident that

$$p_{ii}(t)-p_{*ii}(t)=-[p_{ii}(t)]_-\ \text{ for }\ t\in I\ \ (i=1,\dots,n).$$

The Cauchy matrix of system (1.4) has the form

$$C(t,\tau)\equiv\operatorname{diag}(c_1(t,\tau),\dots,c_n(t,\tau)).$$

In addition, due to (1.17), it is evident that

$$c_i(t,\tau) > 0 \ \text{ for } \ t,\tau \in I \ \ (i=1,\dots,n),$$

whence with regard for (1.14), (1.15), (1.16) and (1.18), we can conclude that conditions (1.11), (1.12) and (1.13) of Theorem 1.1 are valid. Hence the theorem immediately follows from Theorem 1.1. □

Proof of Corollary 1.1. In view of (1.19), we have

$$0 < c_i(t,\tau) \le \left|\frac{t-t_0}{\tau-t_0}\right|^{\mu_i} \ \text{ for } \ (t-t_0)(\tau-t_0) > 0 \ \ (i=1,\dots,n). \tag{2.12}$$

So, evidently, the functions

$$h_i(t) = |t-t_i|^{\mu_i} \ \ (i=1,\dots,n) \tag{2.13}$$

satisfy inequalities (1.14), where $b_0 = 1$.

In addition, with regard to (2.12) and (2.13), from (1.20), (1.21) and (1.22), it follows that conditions (1.16), (1.17) and (1.18) hold true. Therefore, according to Theorem 1.2, inclusion (1.9) holds. □

Corollaries 1.2 and 1.3 follow immediately from Theorem 1.2, since under the conditions of these corollaries the conditions of Theorem 1.2 are fulfilled (for the proof, see [5]).

Proof of Theorem 1.3. For each natural m, consider the system

$$\frac{dy}{dt} = P_m^*(t)\cdot y + q_m^*(t) \ \text{ for } \ t \in I. \tag{2.14}$$

Due to (1.10) there exists $\eta_0 \in]0,1[$ such that $r(\widetilde{B}) < 1$, where $\widetilde{B} = B + \eta_0 B_0\, \mathfrak{I}_{n\times n}$.

Let us show that, for each sufficiently large m, the matrix-function P_m^* and the vector-function q_m^* satisfy, respectively, conditions (1.11) and (1.12) for constant matrix $\widetilde{B}$, where C_* is the Cauchy matrix of system (1.4).

Indeed, due to (1.24) we have

$$\left\| \int_t^{b-} \Phi^{-1}(s)|P_m^*(s)-P(s)|\Phi(s)ds \right\| < \eta_0 \ \text{ for } \ t \in I(\delta) \tag{2.15}$$

for each sufficiently large m.

On the other hand, in view of (1.11) and (1.12) we have

$$\begin{aligned} &\left\| \int_t^{b-} |C_*(t,s)|\Phi^{-1}(s)|P_m^*(s)-P(s)|\Phi(s)ds \right\| \\ &\quad \le \left\| \int_t^{b-} |C_*(t,s)|\Phi^{-1}(s)|P_m^*(s)-P(s)|\Phi(s)ds \right\| \\ &\quad + \left\| \int_t^{b-} |C_*(t,s)|\Phi^{-1}(s)|P_m^*(s)-P(s)|\Phi(s)ds \right\| \\ &\quad \le \Phi(t)\, B_0 \left\| \int_t^{b-} \Phi^{-1}(s)|P_m^*(s)-P(s)|\Phi(s)ds \right\| + \Phi(t)B \end{aligned}$$

for each sufficiently large m and, therefore, thanks to (2.15) we conclude that, without loss of generality, for every natural m,

$$\left\| \int_t^{b-} |C_*(t,s)|\Phi^{-1}(s)|P_m^*(s)-P(s)|\Phi(s)ds \right\| \le \Phi(t)\widetilde{B} \ \text{ for } \ t \in I(\delta).$$

Similarly, we show that

$$\lim_{t\to b}\left\|\int\limits_t^{b-} H^{-1}(t)\, C_*(t,\tau)q_m^*(\tau)d\tau\right\| = 0$$

for each natural m.

So, according to Theorem 1.1, system (2.14), under the condition

$$\lim_{t\to b}(\Phi^{-1}(t)\, y(t)) = 0,$$

has the unique solution y_m for every m and

$$\lim_{m\to+\infty}\|\Phi^{-1}(t)\,(y_m(t) - x_0(t))\| = 0 \tag{2.16}$$

uniformly on I (here the value of the left hand equals 0 at the point b).

On the other hand, it is not difficult to verify that x_m is a solution of system (1.3) if and only if the vector-function $y_m(t) = H_m(t)x_m(t)$ is a solution of system (2.14) for each natural m. In addition, by (1.23) and the equality

$$\Phi^{-1}(t)x_m(t) = (\Phi^{-1}(t)H_m^{-1}(t)\Phi(t))\,\Phi^{-1}(t)y_m(t)$$

($m = 1, 2, \dots$), the vector-function x_m satisfy condition (1.2) if and only if the vector-function y_m satisfy the same condition.

So, the vector-functions $x_m(t) = H_m^{-1}(t)\, y_m(t)$ $(m = 1, 2, \dots)$ will be solutions of problems (1.2), (1.3), respectively.

Let us show that that condition (1.8) holds uniformly on I.

We have

$$\begin{aligned}\Phi^{-1}(t)\,(x_m(t) - x_0(t)) &= \Phi^{-1}(t)\,\big(H_m^{-1}(t)y_m(t) - x_0(t)\big)\\ &= \Phi^{-1}(t)\,\big(H_m^{-1}(t)\Phi(t)\,\Phi^{-1}(t)y_m(t) - \Phi(t)\,\Phi^{-1}(t)x_0(t)\big)\\ &+ \Phi^{-1}(t)\,\big(H_m^{-1}(t)\Phi(t)\,\Phi^{-1}(t)x_0(t) - \Phi(t)\,\Phi^{-1}(t)x_0(t)\big)\\ &= \Phi^{-1}(t)H_m^{-1}(t)\Phi(t)\,\big(\Phi^{-1}(t)y_m(t) - \Phi^{-1}(t)x_0(t)\big)\\ &+ \big(\Phi^{-1}(t)(H_m^{-1}(t) - I_n)\Phi(t)\big)\Phi^{-1}(t)x_0(t) \ \text{ for } \ t \in I\end{aligned}$$

and therefore,

$$\begin{aligned}\|\Phi^{-1}(t)\,(x_m(t) - x_0(t))\| &\le \|\Phi^{-1}(t)H_m^{-1}(t)\Phi(t)\|\,\|\Phi^{-1}(t)(y_m(t) - x_0(t))\|\\ &+ \|\Phi^{-1}(t)(H_m^{-1}(t) - I_n)\Phi(t)\|\,\|\Phi^{-1}(t)x_0(t)\| \ \text{ for } \ t \in I,\end{aligned}$$

because the left side of the inequality equals to 0 for $t = b$ (by definition).

From the estimate, due to (1.23) and (2.16), we conclude that (1.8) holds uniformly on I. Hence inclusion (1.9) holds. □

Proof of Theorem 1.4. The sufficiency follows from Theorem 1.3.

Let us show the necessity.

Let $\delta > 0$ be such that the conditions of Lemma 2.1 are fulfilled.

For each $m \in \{0, 1, \dots\}$, let X_m $(X_m(a) = I_n)$ with the columns x_{mj} $(j = 1, \dots, n)$ be a fundamental matrix of system (1.3) (if $m = 0$, then under the system we understand system (1.1) on the interval I).

Due to Lemma 2.1, we have the estimates

$$\|\Phi^{-1}(t)x_{mj}(t)\| \le \rho\|B_0\|\,\|\Phi^{-1}(s_0)x_{mj}(s_0)\| \ \text{ for } \ b-\delta \le t < s_0 < b \quad (j = 1, \dots, n;\ m = 0, 1, \dots), \tag{2.17}$$

where $\rho = \|(I_n - B)^{-1}\|$.

Passing to the limit as $s_0 \to b-$ in the right-hand side of (2.17), we obtain

$$\|\Phi^{-1}(t)x_{mj}(t)\| \le \rho\|B_0\| \limsup_{s_0\to b-} \|\Phi^{-1}(s_0)x_{mj}(s_0)\|$$
$$\text{for } \ b-\delta \le t < b \ \ (j=1,\dots,n;\ m=0,1,\dots).$$

Therefore

$$\limsup_{t\to b} \|\Phi^{-1}(t)x_{mj}(t)\| \le \rho\|B_0\| \limsup_{s_0\to b} \|\Phi^{-1}(s_0)x_{mj}(s_0)\|$$
$$(j=1,\dots,n;\ m=0,1,\dots).$$

From this, in view of (1.27), we have

$$\limsup_{t\to b} \|\Phi^{-1}(t)x_{mj}(t)\| = 0 \ \ (j=1,\dots,n;\ m=0,1,\dots).$$

Hence

$$\lim_{t\to b} \|\Phi^{-1}(t)x_{mj}(t)\| = 0 \ \ (j=1,\dots,n;\ m=0,1,\dots). \tag{2.18}$$

Let $H_m(t) \equiv X_0(t)\,X_m^{-1}(t)$ $(m=0,1,\dots)$. It is evident that $H_m \in \mathrm{AC}_{\rm loc}(I;\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$.

Let us verify conditions (1.24) and (1.25).

Due to the definition of the matrix-function P_m^*, we have

$$P_m^*(t) = \big((X_0(t)X_m^{-1}(t))' + X_0(t)X_m^{-1}(t)P_m(t)\big)\,X_m(t)X_0^{-1}(t) \equiv P(t). \tag{2.19}$$

So, condition (1.24) is valid uniformly on I.

It is clear that the conditions of Corollary 1.4 are fulfilled for the homogeneous systems corresponding to systems (1.1) and (1.3) $(m=1,2,\dots)$, i.e., when $q(t)\equiv 0_n$ and $q_m(t)\equiv 0_n$ $(m=1,2,\dots)$.

Now, taking into account (2.18), owing to Corollary 1.4, we get

$$\lim_{m\to+\infty} \big(\Phi^{-1}(t)X_m(t) - \Phi^{-1}(t)X_0(t)\big) = O_{n\times n} \tag{2.20}$$

uniformly on I. So, condition (1.23) holds.

Moreover, due to (2.20), we have

$$\lim_{m\to+\infty} \|\Phi^{-1}(t)H_m^{-1}(t)\Phi(t) - I_n\| = \lim_{m\to+\infty} \|\Phi^{-1}(t)X_m(t)X_0^{-1}(t)\Phi(t) - I_n\| = 0 \tag{2.21}$$

uniformly on I.

Consider now condition (1.25).

Let x_m $(m=0,1,\dots)$ be the unique solution of problem (1.3), (1.2). Let $y_m(t) \equiv H_m(t)x_m(t)$ $(m=0,1,\dots)$ be, just as in the proof of Theorem 1.3, the solution of system (2.14).

Due to (1.8), we have

$$\lim_{m\to+\infty} \big(\Phi^{-1}(t)x_m(t) - \Phi^{-1}(t)x_0(t)\big) = 0_n \tag{2.22}$$

uniformly on I.

Moreover, due to (2.21), we have

$$\lim_{m\to+\infty} \|\Phi^{-1}(t)H_m(t)\Phi(t) - I_n\| = 0$$

uniformly on I. From this and (2.22), by equalities

$$y_m(t) \equiv \Phi(t)\big(\Phi^{-1}(t)H_m(t)\Phi(t)\big)\,\big(\Phi^{-1}(t)x_m(t)\big) \ \ (m=1,2,\dots),$$

we conclude that the function y_m satisfies condition (1.2) if and only if the function x_m satisfies the same one and, in addition,

$$\lim_{m\to+\infty} \|\Phi^{-1}(t)y_m(t) - \Phi^{-1}(t)x_0(t)\| = 0$$

and

$$\lim_{m\to+\infty} \|\Phi^{-1}(t)z_m(t)\| = 0 \tag{2.23}$$

uniformly on I, where $z_m(t) \equiv y_m(t) - x_0(t)$.

Let

$$f^*_m(t) \equiv \int\limits_a^t q^*_m(\tau)d\tau \quad (m=1,2,\dots) \ \text{ and } \ f(t) \equiv \int\limits_a^t q(\tau)d\tau.$$

Further, due to the formula of integration by parts, we have

$$f^*_m(t) = \int\limits_a^t H_m(\tau)q_m(\tau)d\tau = \int\limits_a^t H_m(\tau)\big(x'_m(\tau) - P_m(\tau)x_m(\tau)\big)d\tau$$

$$= H_m(t)x_m(t) - H_m(a)x_m(a) - \int\limits_a^t \big(H'_m(\tau) + H_m(\tau)H_m(\tau)\big)x_m(\tau)$$

$$\text{for } \ t \in I \ \ (m=1,2,\dots).$$

Hence, due to (2.19),

$$f^*_m(t) \equiv H_m(t)x_m(t) - H_m(a)x_m(a) - \int\limits_a^t P(\tau)\cdot H_m(\tau)\,x_m(\tau)d\tau$$

and

$$f^*_m(t) - f(t) \equiv z_m(t) - \int\limits_a^t P(\tau)\,z_m(\tau)d\tau \quad (m=1,2,\dots).$$

Therefore

$$\Phi^{-1}(t)(f^*_m(t) - f(t)) \equiv \Phi^{-1}(t)z_m(t) - \Phi^{-1}(t)\int\limits_a^t P(s)\cdot\Phi(s)(\Phi^{-1}(s)z_m(s))ds$$

$$(m=1,2,\dots).$$

By this and (1.28), there exists a positive r_0 such that

$$\|\Phi^{-1}(t)(f^*_m(t) - f(t))\| \le \|\Phi^{-1}(t)z_m(t)\| + r_0\|\Phi^{-1}\,z_m\|_\infty$$

$$\text{for } \ t \in I \ \ (m=1,2,\dots).$$

Moreover, in view of (2.23), we conclude that

$$\lim_{m\to+\infty} \|\Phi^{-1}(t)(f^*_m(t) - f(t))\| = 0 \tag{2.24}$$

uniformly on I. So, we find that condition (1.26) holds uniformly on I.

In addition, by (1.29), there exists $r_1 > 0$ such that

$$\left\|\int\limits_t^{b-} \Phi^{-1}(\tau)|P_*(\tau)|\,|f^*_m(\tau) - f(b)|d\tau\right\| \le r_1 \sup\{\|\Phi^{-1}(s)(f^*_m(s) - f(s))\| : s \in I\}$$

$$\text{for } \ t \in I \ \ (m=1,2,\dots).$$

Consequently, due to (2.24), condition (1.25) holds uniformly on I, as well. □

Theorem 1.4′ follows immediately from the proof of the necessity of Theorem 1.4, since we can choose $H_m(t) \equiv X_0(t)X_m^{-1}(t)$ $(m=1,2,\dots)$.

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