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ON SOME PROPERTIES OF THE COEFFICIENTS OF UNIVERSAL SERIES

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Abstract. Some properties of universal series are presented. One of them is the following: for every orthonormal and complete system of functions, there exists a universal series with respect to this system such that the lower limit of the sequence of the moduli of its coefficients is zero, while the upper limit is positive infinity.

Let $\Phi = \{\varphi_n(x)\}_{n=1}^{\infty}$ be a system of measurable and almost everywhere finite functions defined on the closed interval $[a,b]$, and $\mathbb{N}$ be the set of all positive integer numbers.

Definition. A series

$$\sum_{n=1}^{\infty} \alpha_n \varphi_n(x) \tag{1}$$

is called a universal series with respect to Φ in terms of subsequences of partial sums, if for every measurable function $f(x)$ taking at each point of $[a,b]$ a defined value (finite or infinite), there exists a sequence $\{m_k\} \uparrow \infty$ such that

$$\lim_{k\to\infty} \sum_{n=1}^{m_k} \alpha_n \varphi_n(x) = f(x) \quad \text{almost everywhere on } [a,b].$$

Hereafter, for brevity, we will call such a universal series (1) simply a universal series with respect to Φ. D. E. Menshov [2] was the first to establish the existence of universal trigonometric series. A. A. Talalyan (see [1, Theorem 9.2.11]) proved that for any complete orthonormal system on $[a,b]$, there exists a universal series of the form (1) with the property

$$\lim_{n\to\infty} \alpha_n = 0.$$

Below, we formulate several properties of universal series.

Property 1. Let series (1) be a universal series with respect to Φ. Then there exists a universal series $\sum\limits_{n=1}^{\infty} \alpha_n^{(1)} \varphi_n(x)$ such that for some infinite subset $\mathbb{N}^{(1)}$ of $\mathbb{N}$ we have:

$$\alpha_n^{(1)} = \begin{cases} 0, & \text{if } n \in \mathbb{N}^{(1)}, \\ \alpha_n, & \text{if } n \in \mathbb{N}\backslash\mathbb{N}^{(1)}. \end{cases}$$

In particular,

$$\varliminf_{n\to\infty} |\alpha_n^{(1)}| = 0.$$

Property 2. If there exists a universal series with respect to Φ, then for any sequence of non-negative numbers $\{c_n\}_{n=1}^{\infty}$, there exists a universal series $\sum\limits_{n=1}^{\infty} \alpha_n^{(2)} \varphi_n(x)$ such that for some infinite subset $\mathbb{N}^{(2)}$ of $\mathbb{N}$, we have:

$$|\alpha_n^{(2)}| \geq c_n \quad \text{for all } n \in \mathbb{N}^{(2)}.$$

In particular, if $\lim\limits_{n\to\infty} c_n = +\infty$, then

$$\varlimsup_{n\to\infty} |\alpha_n^{(2)}| = +\infty.$$

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Property 3. If there exists a universal series with respect to Φ (in particular, Φ can be any complete orthonormal system on $[a,b]$), then there exists a universal series $\sum\limits_{n=1}^{\infty} \alpha_n^{(3)} \varphi_n(x)$ such that

$$\varliminf_{n\to\infty} |\alpha_n^{(3)}| = 0 \quad \text{and} \quad \varlimsup_{n\to\infty} |\alpha_n^{(3)}| = +\infty.$$

References

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