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INITIAL VALUE PROBLEM FOR FIRST ORDER ADVANCED DIFFERENTIAL EQUATIONS

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Abstract. For first order advanced differential equations, sufficient conditions for the solvability and unique solvability of the Cauchy initial value problem are established.

In the present paper, on a finite interval $[a,b]$ we consider the first order advanced differential equation

$$u'(t) = f(t, u(\tau(t))) \tag{1}$$

with the initial condition

$$u(a) = c_0. \tag{2}$$

Here $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a function from the Carathéodory space, $\tau : [a,b] \to [a,b]$ is a measurable function such that

$$a \le t \le \tau(t) \le b \ \text{ for } \ a \le t \le b,$$

and c_0 is a real constant.

Everywhere below we use the following notation and definitions.

$\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ = [0, +\infty[$, $L([a,b])$ is the space of Lebesgue integrable real functions defined on $[a,b]$,

$$f^*(t,x) = \max\big\{|f(t,y)| : \ 0 \le |y| \le x\big\} \ \text{ for } \ t \in [a,b], \ \ x \in \mathbb{R}_+.$$

We say that the function $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ belongs to the Carathéodory space if $f(\cdot, x) : [a,b] \to \mathbb{R}$ is measurable for any $x \in \mathbb{R}$, $f(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for almost all $t \in [a,b]$, and

$$f^*(\cdot, x) \in L([a,b]) \ \text{ for } \ x \in \mathbb{R}_+.$$

f is said to be locally Lipschitz in the second argument if for any positive number r there exists a nonnegative function $\ell_r \in L([a,b])$ such that

$$|f(t,x) - f(t,y)| \le \ell_r(t)|x - y| \ \text{ for } \ t \in [a,b], \ \ -r \le x, \, y \le r.$$

From the Schauder principle, it immediately follow the following propositions.

Proposition 1. *If*

$$\limsup_{x \to +\infty} \int_a^b \frac{f^*(t,x)}{x} \, dt < 1, \tag{3}$$

then problem (1), (2) *has at least one solution.*

Proposition 2. *Let the function f satisfy the Lipschitz condition in the second argument*

$$|f(t,x) - f(t,y)| \le \ell(t)|x - y| \ \text{ for } \ t \in [a,b], \ \ x, \, y \in \mathbb{R},$$

where $\ell \in L([a,b])$, and

$$\int_a^b \ell(t) \, dt < 1. \tag{4}$$

Then problem (1), (2) *has a unique solution.*

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As an example, we consider the equation

$$u'(t) = p(t)u(\tau(t)) + q(t), \tag{5}$$

where $p, q \in L([a,b])$.

If

$$\int_a^b |p(t)|\,dt < 1, \tag{6}$$

then, by Proposition 2, problem (5), (2) has a unique solution.

On the other hand, it is easy to see that if

$$p(t) \geq 0, \quad q(t) \geq 0, \quad \tau(t) = b \text{ for } t \in [a,b],$$
$$\int_a^b p(t)\,dt \geq 1, \quad c_0 > 0,$$

then problem (5), (2) has no solution.

Consequently, the strict inequalities (3) and (4) in Propositions 1 and 2 are unimprovable and they cannot be replaced by nonstrict ones.

An important particular case of (1) is the differential equation

$$u'(t) = p(t)g(u(\tau(t))), \tag{7}$$

where $p \in L([a,b])$, and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function.

Propositions 1 and 2 yield

Corollary 1. *Let inequality* (6) *be satisfied. If, moreover,*

$$\limsup_{|x| \to +\infty} \frac{|g(x)|}{|x|} < 1, \tag{8}$$

then problem (7), (2) *has at least one solution. And if*

$$|g(x) - g(y)| \leq |x - y| \text{ for } x \in \mathbb{R}, \quad y \in \mathbb{R},$$

then that problem has a unique solution.

On the other hand, the following statement is true.

Proposition 3. *Let*

$$p(t) \geq 0 \text{ for } t \in [a,b], \quad g(x) > 0 \text{ for } |x| > 0, \quad c_0 > 0,$$

and

$$\int_{c_0}^{+\infty} \frac{dx}{g(x)} < \int_a^b p(t)\,dt.$$

Then problem (7), (2) *has no solution.*

According to Proposition 3, condition (8) in Corollary 1 cannot be replaced by the condition

$$\limsup_{|x| \to +\infty} \frac{|g(x)|}{|x|^{1+\varepsilon}} \leq 1,$$

no matter how small $\varepsilon > 0$ is.

In the case, where equation (1) (equation (7)) is superlinear, i.e. when

$$\lim_{x \to +\infty} \int_a^b \frac{f^*(t,x)}{x}\,dt = +\infty \quad \left(\lim_{|x| \to +\infty} \frac{|g(x)|}{|x|} = +\infty \right),$$

the question on the solvability of problem (1), (2) (problem (7), (2)) still remains unstudied (see, [1–6] and the references therein). The results below to some extent fill the existing gap.

Below everywhere a solution u of problem (1), (2) is said to be nonnegative (positive) if it is nonnegative (positive) on $[a, b]$.

Theorem 1. *If the conditions*

$$f(t, 0) = 0, \quad f(t, x) \le 0 \ \text{ for } \ t \in [a, b], \quad x \in \mathbb{R}_+ \tag{9}$$

hold, then problem (1), (2) *has at least one nonnegative solution. And if there exists a positive constant* ε *and a positive function* $\ell \in L([a, b])$ *such that along with* (9) *the condition*

$$|f(t, x)| \le \ell(t)x \ \text{ for } \ t \in [a, b], \quad 0 \le x \le \varepsilon \tag{10}$$

holds, then that solution is positive.

Theorem 1 yields

Corollary 2. *If the conditions*

$$p(t) \le 0 \ \text{ for } \ t \in [a, b], \quad g(0) = 0, \quad g(x) \ge 0 \tag{11}$$

hold, then problem (7), (2) *has at least one nonnegative solution. And if along with* (10) *the condition*

$$\limsup_{x \to 0+} \frac{g(x)}{x} < +\infty \tag{12}$$

is satisfied, then that solution is positive.

Remark 1. In Theorem 1 (in Corollary 2) restriction (10) (restriction (12)) is imposed on the function f (on the function g) in order to guarantee the positiveness of every nonnegative solution of problem (1), (2) (of problem (7), (2)). This restriction is essential and it cannot be omitted. Indeed, let

$$\tau(t) \equiv t, \quad t_0 \in\,]a, b[\,, \quad c_0 = (t_0 - a)^2,$$

and

$$f(t, x) = -2|x|^{\frac{1}{2}} \quad \Big(p(t) \equiv -2, \quad g(x) = |x|^{\frac{1}{2}}\Big).$$

Then conditions (9) (conditions (11)) are fulfilled but condition (10) (condition (12)) does not hold. On the other hand, in this case both problem (1), (2) and problem (7), (2) have a unique solution

$$u(t) = \begin{cases} (t_0 - t)^2 & \text{for } \ a \le t \le t_0, \\ 0 & \text{for } \ t_0 < t \le b, \end{cases}$$

and, consequently, they do not have a positive solution.

Theorem 2 below and its corollary contain sufficient conditions for the unique solvability of problems (1), (2) and (7), (2).

Theorem 2. *If conditions* (9) *hold and the function* f *is locally Lipschitz and nonincreasing in the second argument, then problem* (1), (2) *has a unique positive solution.*

Corollary 3. *If conditions* (11) *hold and the function* g *is locally Lipschitz and nonincreasing, then problem* (7), (2) *has a unique positive solution.*

Remark 2. Under the conditions of Theorem 2, the function f may have an arbitrary order of growth with respect to the second argument at infinity. For example, the function

$$f(t, x) = p(t)(e_n(x) - e_n(0)),$$

where $p \in L([a, b])$ is a nonpositive function, n is any natural number,

$$e_1(x) = \exp(x), \quad e_{k+1}(x) = \exp(e_k(x)), \quad k = 1, 2, \dots,$$

satisfies the conditions of Theorem 2.

Sketch of the Proof of Theorem 1. For every natural n we introduce the strictly advanced differential equation

$$u'(t) = f_n(t, u(\tau_n(t))), \tag{13}$$

where

$$\tau_n(t) = \begin{cases} \tau(t) + \frac{1}{n} & \text{for } a \le t \le b, \\ b_n & \text{for } b < t \le b_n, \end{cases}$$

$$b_n = b + \frac{1}{n},$$

$$f_n(t, x) = \begin{cases} f(t, x) & \text{for } a \le t \le b, \\ 0 & \text{for } b < t \le b_n. \end{cases}$$

We prove:

1. If the conditions of Theorem 1 are satisfied, then for any positive number x Eq. (13) has a unique solution $u_n(\cdot; x)$, defined on $[a, b_n]$, such that $u_n(b_n; x) = x$;

2. For every n there exists $x_n > 0$ such that $u_n(a; x_n) = c_0$;

3. From the sequence $(u_n(\cdot; x_n))_{n=1}^{+\infty}$ we can get a uniformly converging on $[a, b]$ subsequence $(u_{n_k}(\cdot; x_{n_k}))_{k=1}^{+\infty}$ whose limit is a solution of $(1), (2)$. □

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