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TRACE INEQUALITY CRITERIA FOR TRUNCATED POTENTIAL OPERATORS

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Abstract. The necessary and sufficient conditions are given for the Borel measure ν on $\mathbb{R}^n$, ensuring that the trace inequality holds for multilinear truncated potential operators.

1. Preliminaries

Fractional integral operators play a considerable role in harmonic analysis and PDEs. For example, the Riesz potentials

$$I_\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy, \quad 0<\alpha<n,$$

are important for studying Sobolev imbeddings (see, e.g., [7]).

One multilinear variant of the operator I_α is the following operator:

$$\mathcal{I}_\gamma(f_1,\dots,f_m)(x) = \int\limits_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1|+\dots+|x-y_m|)^{mn-\gamma}}\,dy_1\cdots dy_m, \quad 0<\gamma<mn.$$

The operator $\mathcal{I}_\gamma$ is a natural intermediate operator (in m–linear form) between $(I_{\alpha_1}f_1)(I_{\alpha_2}f_2)$ and the bilinear operator $B_{\alpha_1+\alpha_2}(f_1,f_2)$, where

$$I_{\alpha_i} f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha_i}}\,dy, \quad 0<\alpha_i<n, \ \ i=1,2,$$

and

$$B_\alpha(f_1,f_2)(x) = \int\limits_{\mathbb{R}^n} \frac{f_1(x+t)f_2(x-t)}{|t|^{n-\alpha}}\,dt, \quad 0<\alpha<n.$$

The operators $\mathcal{I}_\gamma$ and B_α were introduced and studied in [3–5].

Criteria for the one-weight and trace inequalities for $\mathcal{I}_\gamma$ were established in [8] and [6], respectively.

In 1985, E. Sawyer [9] (see also [1, Chapter 5]) introduced and studied a truncated variant of the Riesz potential

$$J_\alpha f(x) = \int\limits_{|y|<2|x|} \frac{f(y)}{|x-y|^{\,n-\alpha}}\,dy,$$

and established the corresponding trace inequality

$$\|J_\alpha f\|_{L^q_\nu} \le C\|f\|_{L^p(\mathbb{R}^n)}.$$

We present criteria for the trace inequality for the following variants of multilinear truncated potential operators

$$J_\alpha^{(m)}(\vec{f})(x) = \int\limits_{|y_1|<2|x|} \cdots \int\limits_{|y_m|<2|x|} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1|+\dots+|x-y_m|)^{mn-\alpha}}\,dy_1\cdots dy_m,$$

and

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$$K_{\vec{\alpha}}^{(m)} \vec{f}(x) = \prod_{k=1}^{m} \int_{|y_k|<2|x|} \frac{f_k(y_k)}{|x-y_k|^{\,n-\alpha_k}}\, dy_k,$$

where $\vec{\alpha} = (\alpha_1, \dots, \alpha_m)$ and $\vec{f} = (f_1, \dots, f_m)$.

We denote by L^p_ν, $1 < p < \infty$, the classical Lebesgue space with the measure ν given by the standard norm $\|f\|_{L^p_\nu} = \Big(\int\limits_{\mathbb{R}^n} |f(x)|^p d\nu(x) \Big)^{1/p}$. If μ is the Lebesgue measure, then we use the symbol L^p for L^p_ν

2. Main Results

Now, we formulate the main results of this note.

Theorem A. *Let $1 \le \min\{p_1, \dots, p_m\} \le q < \infty$ and $n(m - 1/p'_{i_0}) < \alpha < mn$ for some i_0, where $p'_{i_0} = p_{i_0}/(p_{i_0} - 1)$. Then the following are equivalent:*

(i) $\|J_\alpha^{(m)}(\vec{f})\|_{L^q_\nu} \le C \prod_{k=1}^m \|f_k\|_{L^{p_k}}$;

(ii) $B_1 := \sup_{t>0} \bigg(\int_{|x|>t} |x|^{(\alpha - mn)q} d\nu(x) \bigg)^{1/q} t^{n(m-1/p)} < \infty$;

(iii) $B_2 := \sup_{k \in \mathbb{Z}} \Big(\nu(B(0, 2^{k+1}) \setminus B(0, 2^k)) \Big)^{1/q} 2^{k(\alpha - n/p)} < \infty$,

with $1/p = \sum_{k=1}^m 1/p_k$.

Theorem B. *Let $1 < \min\{p_1, \dots, p_m\} \le q < \infty$ and $\frac{n}{p_k} < \alpha_k < n$. Then the following are equivalent:*

(i) $\|K_{\vec{\alpha}}^{(m)}(\vec{f})\|_{L^q_\nu} \le C \prod_{k=1}^m \|f_k\|_{L^{p_k}}$;

(ii) $B_1 < \infty$, *with* $\alpha = \sum_{k=1}^m \alpha_k$;

(iii) $B_2 < \infty$, *with* $\alpha = \sum_{k=1}^m \alpha_k$,

where B_1 and B_2 are defined in Theorem A.

Finally, we mention that similar results for multilinear Riemann–Liouville operators were established in [2].

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