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A MAZURKIEWICZ SET CONTAINING THE GRAPH OF AN ABSOLUTELY NONMEASURABLE FUNCTION

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Abstract. Assuming Martin's axiom (**MA**), it is proved that there exists a Mazurkiewicz set in the plane $\mathbf{R}^2$ containing the graph of some absolutely nonmeasurable function.

Throughout this short communication, the following notation will be used.

$\mathbf{R}$ is the set of all real numbers (the real line);

$\mathbf{c}$ is the cardinality of the continuum (i.e., $\mathbf{c} = \mathrm{card}(\mathbf{R})$);

ω is the least infinite cardinal number and ω_1 is the least uncountable cardinal number;

λ is the standard Lebesgue measure on $\mathbf{R}$;

$\mathbf{R}^2$ is the Euclidean plane (i.e., $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$);

λ_2 is the Lebesgue two-dimensional measure on $\mathbf{R}^2$ (i.e., λ_2 is the completion of the product measure $\lambda \otimes \lambda$).

$\mathcal{M}_0(\mathbf{R})$ is the class of the completions of all nonzero σ-finite Borel measures on $\mathbf{R}$ that vanish at every singleton in $\mathbf{R}$.

$\mathcal{M}(\mathbf{R})$ is the class of all nonzero σ-finite measures on $\mathbf{R}$ that vanish at every singleton in $\mathbf{R}$.

ZF & **DC** is an abbreviation of **ZF** set theory with the Axiom of Dependent Choice (see, e.g., [5]).

MA is an abbreviation of Martin's Axiom (see again [5]).

In the definition below, a topological space E is assumed to have the following property: each singleton in E is a Borel subspace of E.

A subset Z of a space E is called absolute null if $\mu^*(Z) = 0$ for every nonzero σ-finite Borel measure μ on E vanishing at all singletons in E (here, μ^* denotes, as usual, the outer measure produced by μ).

Obviously, any absolute null subset Z of E is simultaneously totally imperfect in E (i.e., Z does not contain a subspace homeomorphic to the Cantor space $\{0,1\}^\omega$). For various properties of totally imperfect sets, see [4, 6, 11, 12, 14–16]. The standard representatives of such sets are widely known Bernstein subsets of $\mathbf{R}$ (see especially [12, 14, 15]).

A subset A of $\mathbf{R}$ is called a generalized Luzin set if $\mathrm{card}(A) = \mathbf{c}$ and, for every first category set $X \subset \mathbf{R}$, one has $\mathrm{card}(X \cap A) < \mathbf{c}$.

A subset B of $\mathbf{R}$ is called a generalized Sierpiński set if $\mathrm{card}(B) = \mathbf{c}$ and for every λ-measure zero set $Y \subset \mathbf{R}$, one has $\mathrm{card}(Y \cap B) < \mathbf{c}$.

Observe that all generalized Luzin sets and all generalized Sierpiński sets are totally imperfect in $\mathbf{R}$. Also, as is well-known, Martin's axiom implies that:

(i) there exist generalized Luzin subsets of $\mathbf{R}$ and generalized Sierpiński subsets of $\mathbf{R}$;

(ii) any generalized Luzin set in $\mathbf{R}$ is absolute null;

(iii) no generalized Sierpiński set in $\mathbf{R}$ is absolute null.

Generalized Luzin sets and generalized Sierpiński sets are natural representatives of the so-called thin sets in topological spaces (see, e.g., [12, 14, 15]). There are many examples of totally imperfect subsets of $\mathbf{R}$ that differ from generalized Luzin sets and generalized Sierpiński sets (for instance, no Bernstein set can be a generalized Luzin set or a generalized Sierpiński set).

We say that a function $f : \mathbf{R} \to \mathbf{R}$ is absolutely nonmeasurable with respect to the class $\mathcal{M}_0(\mathbf{R})$ ($\mathcal{M}(\mathbf{R})$) if f is nonmeasurable with respect to any measure belonging to $\mathcal{M}_0(\mathbf{R})$ (to $\mathcal{M}(\mathbf{R})$).

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Remark 1. Within the framework of **ZF** & **DC** theory, it can be proved that a function $f : \mathbf{R} \to \mathbf{R}$ is absolutely nonmeasurable with respect to $\mathcal{M}_0(\mathbf{R})$ if and only if the graph of f is totally imperfect in the plane $\mathbf{R}^2$ (see, e.g., [9]).

Lemma 1. *In* **ZF** & **DC** *theory, a function* $f : \mathbf{R} \to \mathbf{R}$ *is absolutely nonmeasurable with respect to* $\mathcal{M}(\mathbf{R})$ *if and only if the range of* f *is an absolute null subset of* $\mathbf{R}$ *and the sets* $f^{-1}(r)$ *are at most countable for all* $r \in \mathbf{R}$.

For the proof of Lemma 1, see e.g. [6].

Remark 2. Lemma 1 implies that the existence of absolutely nonmeasurable functions with respect to $\mathcal{M}(\mathbf{R})$ cannot be established within **ZFC** set theory. Indeed, there are models of **ZFC** theory in which $\mathbf{c} > \omega_1$ and the cardinality of any absolute null set in $\mathbf{R}$ (or in $\mathbf{R}^2$) does not exceed ω_1 (for further details, see [5]).

Lemma 2. *In* **ZF** & **DC** *theory, let* E_1 *and* E_2 *be two topological spaces and let* X *be an absolute null subset of* E_1. *Suppose also that for every point* $x \in X$, *a nonempty subset* Y_x *of* E_2 *is given, which is absolute null in* E_2.

Then the set $Z = \cup\{\{x\} \times Y_x : x \in X\}$ *is absolute null in the topological product space* $E_1 \times E_2$.

Remark 3. It follows from Lemmas 1 and 2 that if a function $g : \mathbf{R} \to \mathbf{R}$ is absolutely nonmeasurable with respect to $\mathcal{M}(\mathbf{R})$, then the graph of g is absolute null in $\mathbf{R}^2$ and hence is totally imperfect in $\mathbf{R}^2$.

Theorem 1. *In* **ZF** & **DC** *theory, the following three assertions are equivalent:*

(1) *there exists an absolute null subset of* $\mathbf{R}$ *whose cardinality is* $\mathbf{c}$;

(2) *there exists a function from* $\mathbf{R}$ *into* $\mathbf{R}$, *absolutely nonmeasurable with respect to the class* $\mathcal{M}(\mathbf{R})$;

(3) *there exists a partition of* $\mathbf{R}$ *into continuum many absolute null sets, all of which are of cardinality* $\mathbf{c}$.

A subset Z of the plane $\mathbf{R}^2$ is called a Mazurkiewicz set if every straight line in $\mathbf{R}^2$ meets Z at exactly two points (see [13]).

Various properties of Mazurkiewicz type sets are discussed in [1–3,7,8,10] and in some other works.

A function $f : \mathbf{R} \to \mathbf{R}$ is called a Sierpiński–Zygmund function if for each set $X \subset \mathbf{R}$ with $\mathrm{card}(X) = \mathbf{c}$, the restricted function $f|X$ is not continuous on X (see [17]).

Remark 4. It is easy to see that any Sierpiński–Zygmund function $f : \mathbf{R} \to \mathbf{R}$ has the following additional property:

For each set $X \subset \mathbf{R}$ with $\mathrm{card}(X) = \mathbf{c}$, the restricted function $f|X$ is not monotone on X.

On the other hand, under **MA**, there exists a function $g : \mathbf{R} \to \mathbf{R}$ such that:

(a) for each set $X \subset \mathbf{R}$ with $\mathrm{card}(X) = \mathbf{c}$, the restricted function $g|X$ is not monotone on X;

(b) g is not a Sierpiński–Zygmund function.

It is not hard to prove that every Sierpiński–Zygmund function is absolutely nonmeasurable with respect to the class $\mathcal{M}_0(\mathbf{R})$ (see, e.g., [9]).

It was shown in [8] that there exists a Mazurkiewicz set in $\mathbf{R}^2$ containing the graph of some Sierpiński–Zygmund function. Since any Sierpiński–Zygmund function is absolutely nonmeasurable with respect to the class $\mathcal{M}_0(\mathbf{R})$, it follows from the above-mentioned result that there exists a Mazurkiewicz set in $\mathbf{R}^2$ containing the graph of an absolutely nonmeasurable function with respect to the same class $\mathcal{M}_0(\mathbf{R})$. Further, since the proper inclusion

$$\mathcal{M}_0(\mathbf{R}) \subset \mathcal{M}(\mathbf{R})$$

holds, a natural question arises whether there exists a Mazurkiewicz set in $\mathbf{R}^2$ containing the graph of some function that is absolutely nonmeasurable with respect to $\mathcal{M}(\mathbf{R})$. Certainly, the existence of a Mazurkiewicz set with this property needs additional set-theoretical assumptions (cf. Remark 2).

Theorem 2. *Under* **MA**, *there exists a subset* Z *of the plane* $\mathbf{R}^2$ *such that:*

(1) Z *is a Mazurkiewicz set in* $\mathbf{R}^2$;

(2) *there is a function* $f : \mathbf{R} \to \mathbf{R}$ *whose graph is entirely contained in* Z *and for which* $\mathrm{ran}(f)$ *is a generalized Luzin subset of* $\mathbf{R}$.

It follows from (ii) and Lemma 1 that the function f of Theorem 2 is absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$.

Remark 5. Using Lemma 1, it is not hard to see that no Mazurkiewicz set can be represented as a union of the graphs of two functions $g : \mathbf{R} \to \mathbf{R}$ and $h : \mathbf{R} \to \mathbf{R}$ such that both g and h are absolutely nonmeasurable with respect to $\mathcal{M}(\mathbf{R})$.

Moreover, suppose that a subset Z of the plane $\mathbf{R}^2$ satisfies the following two conditions:

(a) $\mathrm{pr}_2(Z)$ is not an absolute null set in $\mathbf{R}$;

(b) Z admits a representation in the form $Z = \cup\{g_k : k \in K\}$, where $\mathrm{card}(K) \leq \omega$ and all g_k $(k \in K)$ are the graphs of some functions acting from $\mathbf{R}$ into $\mathbf{R}$.

Then at least one g_k is not absolutely nonmeasurable with respect to $\mathcal{M}(\mathbf{R})$.

Assuming Martin's axiom, it can be proved that there exists a Mazurkiewicz set no uniformization of which is absolutely nonmeasurable with respect to $\mathcal{M}(\mathbf{R})$. More precisely, the next statement holds true.

Theorem 3. *Under* **MA**, *there exists a subset T of the plane $\mathbf{R}^2$ such that:*

(1) *T is a Mazurkiewicz set in $\mathbf{R}^2$;*

(2) *every function $f : \mathbf{R} \to \mathbf{R}$ whose graph is contained in T has the property that* $\mathrm{ran}(f)$ *contains a generalized Sierpiński subset of* $\mathbf{R}$.

It follows from (iii), Lemma 1, and (2) of Theorem 3 that if the graph of a function $g : \mathbf{R} \to \mathbf{R}$ is entirely contained in T, then g cannot be absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$.

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