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THE EFFECT OF LOWER TERMS ON THE WELL-POSEDNESS OF THE CHARACTERISTIC DIRICHLET PROBLEM FOR A THIRD-ORDER LINEAR HYPERBOLIC EQUATION

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On the plane of independent variables x and t, let us consider a strictly hyperbolic third-order equation with constant coefficients of the form

$$\sum_{i=0}^{3} a_i \frac{\partial^3 u}{\partial t^i \partial x^{3-i}} + \sum_{i+j=2} b_{ij} \frac{\partial^2 u}{\partial t^i \partial x^j} + cu = F(x,t), \tag{1}$$

where F is a given and u is the unknown real functions. Without loss of generality, we assume that $a_3 = 1$. Strictly hyperbolicity of equation (1) implies that the corresponding characteristic equation

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

has only simple real roots λ_1, λ_2, λ_3, which correspond to the families of characteristics lines defined by the equalities $x + \lambda_i t = \text{const}$, $i = 1, 2, 3$, and without loss of generality, we assume that $\lambda_3 < \lambda_2 < \lambda_1$.

Introducing the notation $L_i := \frac{\partial}{\partial t} - \lambda_i \frac{\partial}{\partial x}$, $i = 1, 2, 3$, equation (1) can be rewritten in the form

$$L_1L_2L_3u + \alpha L_1L_2u + \beta L_2L_3u + \gamma L_1L_3u + cu = F, \tag{2}$$

where the values α, β and γ are uniquely determined by the following formulas:

$$\alpha = \frac{b_{20}\lambda_3^2 + b_{11}\lambda_3 + b_{02}}{(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_3)}, \quad \beta = \frac{b_{20}\lambda_1^2 + b_{11}\lambda_1 + b_{02}}{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)}, \quad \gamma = \frac{b_{20}\lambda_2^2 + b_{11}\lambda_2 + b_{02}}{(\lambda_3 - \lambda_2)(\lambda_1 - \lambda_2)}.$$

By D we denote the parallelogram bounded by the characteristic lines: $x + \lambda_1 t = 0$, $x + \lambda_1 t = c_1$ and $x + \lambda_2 t = 0$, $x + \lambda_2 t = c_2$, where $c_i = \text{const} > 0$, $i = 1, 2$, are the given numbers.

In the domain D, for equation (1), we consider the characteristic Dirichlet problem in the following formulation: find in the domain D a solution u of equation (1) satisfies the boundary condition

$$u\big|_{\partial D} = f, \tag{3}$$

where f is the real function given on the boundary of the domain D. In this direction, the works [1–7] are worth mentioning.

Remark. When considering problem (1), (3), we seek the solution u in the class $C^3(\overline{D})$ under the fulfillment of the corresponding smoothness and consistency conditions for the data of this problem.

Theorem. *Let all coefficients in the left-hand side of equation* (2) *be equal to zero, except for α, then problem* (1), (3) *is well-posed in the characteristic parallelogram D. If the coefficient α is also equal to zero, then problem* (1), (3) *is ill-posed. In particular, the corresponding to* (1), (3) *homogeneous problem has an infinite number of linearly independent solutions.*

Proof. Introducing the characteristic variables

$$\xi = \frac{x + \lambda_1 t}{\lambda_1 - \lambda_3}, \quad \eta = \frac{x + \lambda_2 t}{\lambda_2 - \lambda_3} \tag{4}$$

under the conditions of the theorem equation (2) will be written in the form

$$\Big(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} + \alpha\Big)\widetilde{u}_{\xi\eta} = \widetilde{F}, \tag{5}$$

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where $\widetilde{u}(\xi,\eta) := u(x,t)$, $\widetilde{F}(\xi,\eta) := (\lambda_1-\lambda_2)^{-2}(\lambda_1-\lambda_3)(\lambda_3-\lambda_2)F(x,t)$, and the domain D will be transformed into the characteristic rectangle $\widetilde{D}: 0<\xi<c_1(\lambda_1-\lambda_3)^{-1}$, $0<\eta<c_2(\lambda_2-\lambda_3)^{-1}$.

Below, for the simplicity of presentation, we assume that $\widetilde{D}$ is a square, i.e., $\widetilde{D}: 0<\xi,\eta<l$, where $l := c_1(\lambda_1-\lambda_3)^{-1} = c_2(\lambda_2-\lambda_3)^{-1}$.

Under the above assumptions, the characteristic Dirichlet boundary condition (3) can be written as

$$\widetilde{u}(\xi,0)=f_1(\xi),\quad \widetilde{u}(0,\eta)=f_2(\eta),\quad 0\le\xi,\eta\le l, \tag{6}$$

$$\widetilde{u}(l,\eta)=f_3(\eta),\quad \widetilde{u}(\xi,l)=f_4(\xi),\quad 0\le\xi,\eta\le l, \tag{7}$$

where the functions f_i, $i=1,2,3,4$, satisfy the following conditions of smoothness $f_i\in C^3([0,l])$, $i=1,2,3,4$, and consistency

$$f_1(0)=f_2(0),\quad f_2(l)=f_4(0),\quad f_3(0)=f_1(l),\quad f_3(l)=f_4(l).$$

First, for equation (5), we consider the auxiliary Goursat problem with the boundary conditions on the characteristic segments $\xi=0$ and $\eta=0$, when, in addition to conditions (6), the following conditions must be satisfied:

$$\widetilde{u}_\eta(\xi,0)=\varphi(\xi),\quad \widetilde{u}_\xi(0,\eta)=\psi(\eta),\quad 0\le\xi,\quad \eta\le l, \tag{8}$$

with a subsequent selection of the functions φ and ψ such that condition (7) is satisfied. It is assumed that the following smoothness conditions $\varphi,\psi\in C^2([0,l])$ and consistency

$$f_1(0)=f_2(0),\quad \psi(0)=f_1'(0),\quad \varphi(0)=f_2'(0),\quad \varphi'(0)=\psi'(0),$$

are satisfied.

We introduce the following notation:

$$v := \widetilde{u}_{\xi\eta}.$$

Then, taking into account equation (5) and condition (8) for the function v, we obtain the following boundary value problem:

$$\Big(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}+\alpha\Big)v=\widetilde{F},\quad (\xi,\eta)\in\widetilde{D}, \tag{9}$$

$$v(\xi,0)=\varphi'(\xi),\quad v(0,\eta)=\psi'(\eta),\quad 0\le\xi,\eta\le l. \tag{10}$$

By direct integration, the solution of problem (9), (10) is determined by the formula

$$v(\xi,\eta)=\begin{cases}\exp(-\alpha\eta)\varphi'(\xi-\eta)+F_1(\xi,\eta), & \eta\le\xi,\\ \exp(-\alpha\xi)\psi'(\eta-\xi)+F_2(\xi,\eta), & \eta\ge\xi,\end{cases}$$

where

$$F_1(\xi,\eta) := \exp(-\alpha\xi)\int\limits_{\xi-\eta}^{\xi}\widetilde{F}(\xi_1,\xi_1+\eta-\xi)\exp(\alpha\xi_1)d\xi_1,\quad \eta\le\xi,$$

$$F_2(\xi,\eta) := \exp(-\alpha\eta)\int\limits_{\eta-\xi}^{\eta}\widetilde{F}(\eta_1+\xi-\eta,\eta_1)\exp(\alpha\eta_1)d\eta_1,\quad \eta\ge\xi.$$

The next step is the solution of the Goursat problem for the equation

$$\widetilde{u}_{\xi\eta}=v,\quad 0\le\xi,\quad \eta\le l,$$

with the characteristic data (6). The solution is given by the formula

$$\widetilde{u}(\xi,\eta)=\begin{cases} f_1(\xi)+f_2(\eta)-f_2(0)+\int\limits_0^{\eta}d\xi_1\int\limits_{\xi_1}^{\eta}\exp(-\alpha\xi_1)\psi'(\eta_1-\xi_1)d\eta_1 \\ \qquad +\int\limits_0^{\eta}d\eta_1\int\limits_{\eta_1}^{\xi}\exp(-\alpha\eta_1)\varphi'(\xi_1-\eta_1)d\xi_1+F_3(\xi,\eta), \quad \eta\le\xi, \\ f_1(\xi)+f_2(\eta)-f_2(0)+\int\limits_0^{\xi}d\xi_1\int\limits_{\xi_1}^{\eta}\exp(-\alpha\xi_1)\psi'(\eta_1-\xi_1)d\eta_1 \\ \qquad +\int\limits_0^{\xi}d\xi_1\int_0^{\xi_1}\exp(-\alpha\eta_1)\varphi'(\xi_1-\eta_1)d\eta_1+F_4(\xi,\eta), \quad \eta\ge\xi, \end{cases} \tag{11}$$

where

$$F_3(\xi,\eta):=\int\limits_0^{\eta}d\xi_1\int\limits_{\xi_1}^{\eta}F_2(\xi_1,\eta_1)d\eta_1+\int\limits_0^{\eta}d\eta_1\int\limits_{\eta_1}^{\xi}F_1(\xi_1,\eta_1)d\xi_1, \quad \eta\le\xi,$$

$$F_4(\xi,\eta):=\int\limits_0^{\xi}d\xi_1\int\limits_{\xi_1}^{\eta}F_2(\xi_1,\eta_1)d\eta_1+\int\limits_0^{\xi}d\xi_1\int\limits_0^{\xi_1}F_1(\xi_1,\eta_1)d\eta_1, \quad \eta\ge\xi.$$

Using (11), we satisfy conditions (7) and obtain

$$\begin{aligned} & f_1(l)+f_2(\eta)-f_2(0)+\int\limits_0^{\eta}d\xi_1\int\limits_{\xi_1}^{\eta}\exp(-\alpha\xi_1)\psi'(\eta_1-\xi_1)d\eta_1 \\ & +\int\limits_0^{\eta}d\eta_1\int\limits_{\eta_1}^{l}\exp(-\alpha\eta_1)\varphi'(\xi_1-\eta_1)d\xi_1+F_3(l,\eta)=f_3(\eta), \quad 0\le\eta\le l, \\ & f_1(\xi)+f_2(l)-f_2(0)+\int\limits_0^{\xi}d\xi_1\int\limits_{\xi_1}^{l}\exp(-\alpha\xi_1)\psi'(\eta_1-\xi_1)d\eta_1 \\ & +\int\limits_0^{\xi}d\xi_1\int\limits_0^{\xi_1}\exp(-\alpha\eta_1)\varphi'(\xi_1-\eta_1)d\eta_1+F_4(\xi,l)=f_4(\xi), \quad 0\le\xi\le l. \end{aligned} \tag{12}$$

Introducing a single parametrization in equation (12), we get

$$\begin{aligned} & \int\limits_0^{\xi}d\xi_1\int\limits_{\xi_1}^{\xi}\exp(-\alpha\xi_1)\psi'(\eta_1-\xi_1)d\eta_1 \\ & +\int\limits_0^{\xi}d\eta_1\int\limits_{\eta_1}^{l}\exp(-\alpha\eta_1)\varphi'(\xi_1-\eta_1)d\xi_1=h_1(\xi), \quad 0\le\xi\le l, \\ & \int\limits_0^{\xi}d\xi_1\int\limits_{\xi_1}^{l}\exp(-\alpha\xi_1)\psi'(\eta_1-\xi_1)d\eta_1 \\ & +\int\limits_0^{\xi}d\xi_1\int\limits_0^{\xi_1}\exp(-\alpha\eta_1)\varphi'(\xi_1-\eta_1)d\eta_1=h_2(\xi), \quad 0\le\xi\le l, \end{aligned} \tag{13}$$

where

$$h_1(\xi):=f_3(\xi)-f_2(\xi)-f_1(l)+f_2(0)-F_3(l,\xi),$$

$$h_2(\xi) := f_4(\xi) - f_1(\xi) - f_2(l) + f_2(0) - F_4(\xi, l), \quad 0 \le \xi \le l.$$

Based on the structure of system (13) and differentiating this system with respect to the variable ξ, we obtain an equivalent system

$$\begin{aligned} &\int\limits_0^\xi \exp(-\alpha\xi_1)\psi'(\xi-\xi_1)d\xi_1 + \int\limits_\xi^l \exp(-\alpha\xi)\varphi'(\xi_1-\xi)d\xi_1 = h_1'(\xi), \quad 0 \le \xi \le l, \\ &\int\limits_\xi^l \exp(-\alpha\xi)\psi'(\eta_1-\xi)d\eta_1 + \int\limits_0^\xi \exp(-\alpha\eta_1)\varphi'(\xi-\eta_1)d\eta_1 = h_2'(\xi), \quad 0 \le \xi \le l. \end{aligned} \tag{14}$$

Making a suitable replacement of the variables ξ_1 and η_1 in the integrands of system (14), we obtain

$$\begin{aligned} &\int\limits_0^\xi \exp(\alpha\tau)\psi'(\tau)d\tau + \varphi(l-\xi) = h_3(\xi), \quad 0 \le \xi \le l, \\ &\psi(l-\xi) + \int\limits_0^\xi \exp(\alpha\tau)\varphi'(\tau)d\tau = h_4(\xi), \quad 0 \le \xi \le l, \end{aligned} \tag{15}$$

where

$$h_3(\xi) := \exp(\alpha\xi)h_1'(\xi) + f_2'(0), \quad h_4(\xi) := \exp(\alpha\xi)h_2'(\xi) + f_1'(0), \quad 0 \le \xi \le l.$$

Similarly, based on the structure of system (15) and differentiating this system with respect to the variable ξ, we obtain an equivalent system of the equations with respect to the unknown functions φ' and ψ' with a shift

$$\begin{cases} \exp(\alpha\xi)\psi'(\xi) - \varphi'(l-\xi) = h_3'(\xi), \quad 0 \le x \le l, \\ -\psi'(l-\xi) + \exp(\alpha\xi)\varphi'(\xi) = h_4'(\xi), \quad 0 \le \xi \le l. \end{cases} \tag{16}$$

The analysis of system (16) allows us to conclude that:

1) if $\alpha \ne 0$, then system (16) has a unique solution taking the form

$$\begin{cases} \varphi'(\xi) = [\exp(\alpha l) - 1]^{-1}\{h_3'(l-\xi) + \exp[\alpha(l-\xi)]h_4'(\xi)\}, \quad 0 \le \xi \le l, \\ \psi'(\xi) = [\exp(\alpha l) - 1]^{-1}\{h_4'(l-\xi) + \exp[\alpha(l-\xi)]h_3'(\xi)\}, \quad 0 \le \xi \le l. \end{cases} \tag{17}$$

Substituting the expressions for φ' and ψ' from (17) into (11) and returning to the original independent variables x and t, by the formulas (4), we obtain the unique solution to the original problem (1), (3).

2) if $\alpha = 0$, then the corresponding to (16) homogeneous system has an infinite number of solutions, which can be constructed by using the formulas

$$\varphi'(\xi) = \chi(\xi), \quad \psi'(\xi) = \chi(l-\xi), \quad 0 \le \xi \le l,$$

where χ is an arbitrary function from the class $C^1([0,l])$ and $\chi(0) = \chi(l)$. Based on the above-said, the general solution of the corresponding (5)–(7) homogeneous problem can be written in the form

$$\widetilde{u}_0(\xi,\eta) = \begin{cases} \int\limits_0^\eta d\xi_1 \int\limits_{\xi_1}^\eta \exp(-\alpha\xi_1)\chi(l-\eta_1+\xi_1)d\eta_1 \\ \qquad + \int\limits_0^\eta d\eta_1 \int\limits_{\eta_1}^\xi \exp(-\alpha\eta_1)\chi(\xi_1-\eta_1)d\xi_1, \quad \eta \le \xi, \\ \int_0^\xi d\xi_1 \int\limits_{\xi_1}^\eta \exp(-\alpha\xi_1)\chi(l-\eta_1+\xi_1)d\eta_1 \\ \qquad + \int\limits_0^\xi d\xi_1 \int_0^{\xi_1} \exp(-\alpha\eta_1)\chi'(\xi_1-\eta_1)d\eta_1, \quad \eta \ge \xi. \end{cases} \qquad \square$$

At the same time, it is possible to constructively describe the set of the right-hand sides of equation (1) for which the original problem (1), (3) is solvable.

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