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BOUNDEDNESS OF ONE-SIDED OPERATORS IN VARIABLE EXPONENT MORREY SPACES

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Abstract. The paper studies one-sided operators in the framework of variable-exponent Morrey spaces. First, we prove the boundedness of the one-sided Hardy–Littlewood maximal operators and then establish the boundedness theorems of Spanne and Adams type for the right-sided Weyl-type fractional integral operator $W_{\alpha(\cdot)}$ both on finite intervals and on the entire real line $\mathbb{R}$. The theorems are formulated for the right-sided operators, while the corresponding results remain valid for the left-sided Hardy–Littlewood maximal operator and the Riemann–Liouville fractional integral operator $R_{\alpha(\cdot)}$.

1. Preliminaries

Morrey spaces, introduced by C. B. Morrey in 1938, were originally developed in connection with the study of regularity properties of solutions to partial differential equations (PDEs). Such spaces provide a more precise description of local integrability than the classical Lebesgue spaces.

The concept of Morrey spaces with variable exponents was introduced in the paper of A. Almeida, J. Hasanov, and S. Samko [1] (see also V. Kokilashvili and A. Meskhi [9]). These spaces are particularly useful for modelling problems with spatially variable growth and local regularity arising, for instance, in the study of PDEs with nonstandard structure or in the analysis of phenomena occurring in inhomogeneous media (for the case of variable exponent Lebesgue spaces, see, e.g., paper [12] and monographs [5, 11]).

One-sided maximal and fractional operators arise naturally in harmonic analysis and in PDEs with directional structure. They appear in problems involving causality or asymmetry such as boundary value problems, parabolic equations with drift terms, and various classes of evolution equations. Moreover, such operators play a significant role in the theory of weighted inequalities, non-symmetric filtration, and anisotropic settings, where classical symmetric tools are no longer sufficient.

Let $p(\cdot), q(\cdot) : I \to (1, \infty)$ be measurable functions such that $q(\cdot) \le p(\cdot)$. The variable exponent Morrey space $\mathscr{M}^{p(\cdot),q(\cdot)}(I)$ is defined by the norm

$$\|f\|_{\mathscr{M}^{p(\cdot),q(\cdot)}(I)} := \sup_{a\in I,\, R>0} R^{\frac{1}{p_R(a)}-\frac{1}{q_R(a)}} \|f\|_{L^{q(\cdot)}((a,a+R))},$$

where the norm in the variable-exponent Lebesgue space is given in terms of the Luxemburg norm,

$$\|f\|_{L^{q(\cdot)}(E)} := \inf\left\{\lambda > 0 : \int_E \left(\frac{|f(x)|}{\lambda}\right)^{q(x)} dx \le 1\right\}.$$

Here,

$$p_R(a) = \begin{cases} p(a), & R < 1, \\ p(\infty), & R \ge 1, \end{cases} \qquad q_R(a) = \begin{cases} q(a), & R < 1, \\ q(\infty), & R \ge 1. \end{cases}$$

If the interval I is bounded, then the norm has the form

$$\|f\|_{\mathscr{M}^{p(\cdot),q(\cdot)}(I)} := \sup_{a\in I,\, R>0} R^{\frac{1}{p(a)}-\frac{1}{q(a)}} \|f\|_{L^{q(\cdot)}((a,a+R))}.$$

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If $p(\cdot) = q(\cdot)$, then the Morrey space coincides with the variable-exponent Lebesgue space,

$$\mathscr{M}^{p(\cdot),p(\cdot)}(I) = L^{p(\cdot)}(I).$$

The main operators under consideration are the one-sided maximal and fractional integral operators. The one-sided Hardy–Littlewood maximal operators are defined by

$$M^+ f(x) := \sup_{0<h<b-x} \frac{1}{h} \int_x^{x+h} |f(t)|\,dt, \qquad M^- f(x) := \sup_{0<h<x-a} \frac{1}{h} \int_{x-h}^{x} |f(t)|\,dt,$$

where $x \in (a,b) \subset \mathbb{R}$. The one-sided fractional integrals of Weyl and Riemann–Liouville type are given by

$$W_{\alpha(\cdot)} f(x) := \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}}\,dt, \qquad R_{\alpha(\cdot)} f(x) := \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt.$$

For the admissible exponents, we use the following classes. Let $I \subset \mathbb{R}$ be an open interval and $p : I \to \mathbb{R}$ be a measurable function. Then:

- the class $\mathscr{P}$ consists of those p such that
$$1 < p_I^- := \operatorname*{ess\,inf}_{x\in I} p(x) \le p(x) \le \operatorname*{ess\,sup}_{x\in I} p(x) =: p_I^+ < \infty;$$
- $p \in \mathscr{P}^{\log}(I)$ if
$$|p(x) - p(y)| \le \frac{A}{\log\left(\frac{1}{|x-y|}\right)} \quad \text{whenever } 0 \le |x-y| \le \tfrac{1}{2};$$
- the class $\mathscr{P}_-(I)$ consists of those p such that for some constant $c_1 > 0$,
$$p(x) \le p(y) + \frac{c_1}{\ln \frac{1}{x-y}}, \qquad 0 < x - y \le \tfrac{1}{2}, \text{ a.e. } x, y \in I.$$
- the class $\mathscr{P}_+(I)$ is defined by the condition
$$p(x) \le p(y) + \frac{c_2}{\ln \frac{1}{y-x}}, \qquad 0 < y - x \le \tfrac{1}{2}, \text{ a.e. } x, y \in I,$$
for some constant $c_2 > 0$.
- the class $\mathscr{P}_\infty(I)$ requires that there exists $C > 0$ such that
$$|p(x) - p(y)| \le \frac{C}{\log(e + |x|)}, \qquad |y| \ge |x|, \quad x, y \in I.$$

The conditions defining $\mathscr{P}_\infty(I)$ guarantee controlled behaviour of the exponent at infinity and are essential when dealing with unbounded domains.

The classes $\mathscr{P}_-(I)$ and $\mathscr{P}_+(I)$ are much wider than the class of log-Hölder continuous exponents, which can be described as $\mathscr{P}^{\log}(I) := \mathscr{P}_-(I) \cap \mathscr{P}_+(I)$. In particular, any non-decreasing function $p(x)$ belongs to $\mathscr{P}_+(I)$, and any non-increasing function $p(x)$ belongs to $\mathscr{P}_-(I)$ in the case of bounded interval I.

The condition $\mathscr{P}^{\log}(I) \cap \mathscr{P}_\infty(I)$ guarantees the boundedness of the operators in the classical harmonic analysis in variable-exponent Lebesgue spaces. For the details, see, e.g., the monograph by L. Diening, P. Harjulehto, P. Hästö and M. Růžička [5], D. Cruz–Uribe and A. Fiorenza [2], V. Kokilashvili, A. Meskhi, H. Rafeiro and S. Samko [10].

For a bounded domain $\Omega \subset \mathbb{R}^n$, the following statement was proved by L. Diening [4]:

Theorem A. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Define the Hardy–Littlewood maximal operator*

$$(\mathscr{M}_\Omega f)(x) := \sup_{r>0} \frac{1}{r^n} \int_{B(x,r)\cap\Omega} |f(y)|\,dy, \quad x \in \Omega.$$

Then $\mathscr{M}_\Omega$ is bounded on $L^{p(\cdot)}(\Omega)$ if

- $p \in \mathscr{P}$;
- $p \in \mathscr{P}^{\log}(\Omega)$.

A related result was obtained by D. Cruz–Uribe, A. Fiorenza and C. J. Neugebauer [3].

Theorem B. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $p \in \mathscr{P}(\Omega)$. Then the Hardy–Littlewood maximal operator $\mathscr{M}_\Omega$ is bounded on $L^{p(\cdot)}(\Omega)$ if:*

(1) $p \in \mathscr{P}^{\log}(\Omega)$,
(2) $p \in \mathscr{P}_\infty(\Omega)$.

The boundedness of the operators of one-sided harmonic analysis in variable-exponent Lebesgue spaces, under the conditions $\mathscr{P}_+(I) \cap \mathscr{P}_\infty(I)$ or $\mathscr{P}_-(I) \cap \mathscr{P}_\infty(I)$, was established by D. E. Edmunds, V. Kokilashvili and A. Meskhi [6] (see also the monograph by V. Kokilashvili, A. Meskhi, H. Rafeiro and S. Samko [10]).

Theorem C ([6])**.** *Let $I \subset \mathbb{R}$ be a bounded interval. Then:*

- *If $p \in \mathscr{P}_-(I)$, the left-sided maximal operator M^- is bounded,*
$$M^- : L^{p(\cdot)}(I) \to L^{p(\cdot)}(I);$$
- *If $p \in \mathscr{P}_+(I)$, the right-sided maximal operator M^+ is bound,*
$$M^+ : L^{p(\cdot)}(I) \to L^{p(\cdot)}(I).$$

Theorem D ([6])**.** *The following statements hold:*

- *If $p \in \mathscr{P}_+(\mathbb{R}) \cap \mathscr{P}_\infty(\mathbb{R})$, then*
$$M^+ : L^{p(\cdot)}(\mathbb{R}) \to L^{p(\cdot)}(\mathbb{R});$$
- *If $p \in \mathscr{P}_-(\mathbb{R}) \cap \mathscr{P}_\infty(\mathbb{R})$, then*
$$M^- : L^{p(\cdot)}(\mathbb{R}) \to L^{p(\cdot)}(\mathbb{R}).$$

The boundedness of classical operators in variable-exponent Morrey spaces was studied by A. Almeida, J. Hasanov and S. Samko [1], V. Kokilashvili and A. Meskhi [9] for the case of bounded sets, and by V. S. Guliyev and S. G. Samko [7] in the case of unbounded sets (see also [8]).

2. Main Results

In this section, we formulate our main results on the boundedness of one-sided maximal operators and one-sided fractional integral operators of Weyl type $W_{\alpha(\cdot)}$ in variable-exponent Morrey spaces. We distinguish between finite and infinite interval cases, as well as Spanne and Adams type results.

We begin with the boundedness of the right-sided Hardy–Littlewood maximal operator in variable exponent Morrey spaces:

Theorem 2.1. *Let $I = (a, b) \subset \mathbb{R}$ and suppose $p(\cdot), q(\cdot) \in \mathscr{P}(I) \cap \mathscr{P}_+(I)$, provided $q(\cdot) \le p(\cdot)$. Then the operator M^+ is bounded on $\mathscr{M}^{p(\cdot),q(\cdot)}(I)$:*
$$\|M^+ f\|_{\mathscr{M}^{p(\cdot),q(\cdot)}(I)} \le C\|f\|_{\mathscr{M}^{p(\cdot),q(\cdot)}(I)}.$$

Theorem 2.2. *Let $p(\cdot), q(\cdot) \in \mathscr{P}(\mathbb{R}) \cap \mathscr{P}_+(\mathbb{R}) \cap \mathscr{P}_\infty(\mathbb{R})$, provided $q(\cdot) \le p(\cdot)$. Then the operator M^+ is bounded on $\mathscr{M}^{p(\cdot),q(\cdot)}(\mathbb{R})$,*
$$\|M^+ f\|_{\mathscr{M}^{p(\cdot),q(\cdot)}(\mathbb{R})} \le C\|f\|_{\mathscr{M}^{p(\cdot),q(\cdot)}(\mathbb{R})}.$$

Next, we consider the one-sided fractional integral operator of Weyl type $W_{\alpha(\cdot)}$ in variable-exponent Morrey spaces. The following theorem is the Spanne-type Results for the finite interval case.

Theorem 2.3. *Let $I = (a, b) \subset \mathbb{R}$ and assume*
$$p(\cdot), q(\cdot) \in \mathscr{P}(I) \cap \mathscr{P}_+(I), \quad r(\cdot), s(\cdot) \in \mathscr{P}(I),$$
with $r(\cdot) \le p(\cdot)$, $s(\cdot) \le q(\cdot)$, and
$$\frac{1}{p(\cdot)} - \frac{1}{r(\cdot)} = \frac{1}{q(\cdot)} - \frac{1}{s(\cdot)}, \qquad \alpha(x) := \frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{r(x)} - \frac{1}{s(x)} \in (0, 1).$$

Then the Weyl-type one-sided fractional integral

$$W_{\alpha(\cdot)}:\ \mathscr{M}^{p(\cdot),r(\cdot)}(I)\longrightarrow \mathscr{M}^{q(\cdot),s(\cdot)}(I)$$

is bounded.

Now, let us formulate Adams-type results for the Weyl-type $W_{\alpha(\cdot)}$ operator in the finite interval case:

Theorem 2.4. *Let $I=(a,b)\subset\mathbb{R}$ and assume*

$$p(\cdot),q(\cdot)\in\mathscr{P}(I)\cap\mathscr{P}_+(I),\quad r(\cdot),s(\cdot)\in\mathscr{P}(I),$$

with $r(\cdot)\le p(\cdot)$, $s(\cdot)\le q(\cdot)$, and

$$\theta(x):=\frac{p(x)}{r(x)}=\frac{q(x)}{s(x)}>1,\qquad \alpha(x):=\frac{1}{r(x)}-\frac{1}{s(x)}\in(0,1).$$

Then

$$W_{\alpha(\cdot)}:\ \mathscr{M}^{p(\cdot),r(\cdot)}(I)\longrightarrow \mathscr{M}^{q(\cdot),s(\cdot)}(I).$$

To formulate the statement for the unbounded I, we need some definitions. Define the Morrey gauge

$$\omega(x,r)=r^{\frac{1}{p_r(x)}-\frac{1}{r_r(x)}},\qquad \omega_1(x,r):=[\omega(x,r)]^{p_r(x)/q_r(x)},$$

where $u_r(x)=u(x)$ for $0<r\le 1$ and $u_r(x)=u(\infty)$ for $r>1$.

Define the space $\mathscr{M}^{p(\cdot),\omega(\cdot)}(\mathbb{R})$ by the norm

$$\|f\|_{\mathscr{M}^{p(\cdot),\omega(\cdot)}(\mathbb{R})}:=\sup_{x\in\mathbb{R},\,r>0}\frac{1}{\omega(x,r)}\|f\|_{L^{p(\cdot)}(B(x,r))}<\infty,$$

where $B(x,r)=(x-r,x+r)$.

We now turn to the case of the entire real line. In this setting, additional care is required in order to capture the asymptotic behaviour of the variable exponents at infinity. To this end, we employ the Morrey gauges.

Theorem 2.5. *Suppose $p,q,r,s:\mathbb{R}\to(1,\infty)$, provided $1<r(\cdot)\le p(\cdot)$ and $1<s(\cdot)\le q(\cdot)$. Let $\alpha:\mathbb{R}\to(0,1)$ satisfy*

$$\frac{1}{q(x)}=\frac{1}{p(x)}-\alpha(x),\qquad \frac{1}{s(x)}=\frac{1}{r(x)}-\alpha(x),\qquad \theta(x):=\frac{p(x)}{r(x)}=\frac{q(x)}{s(x)}>1.$$

Assume that $p,q,r,s\in\mathscr{P}\cap\mathscr{P}_+(\mathbb{R})\cap\mathscr{P}_\infty(\mathbb{R})$ and

$$\sup_{x\in\mathbb{R}}\alpha(x)\,r(\infty)<1.$$

Then

$$W_{\alpha(\cdot)}:\ \mathscr{M}^{p(\cdot),\omega(\cdot,\cdot)}(\mathbb{R})\longrightarrow \mathscr{M}^{q(\cdot),\omega_1(\cdot,\cdot)}(\mathbb{R})\ +\ \mathscr{M}^{q(\infty),\omega_1(\cdot,\cdot)}(\mathbb{R}).$$

Finally, we note that all the results formulated above for the right-sided maximal and fractional integral operators M^+ and $W_{\alpha(\cdot)}$ also remain valid for their left-sided counterparts M^- and $R_{\alpha(\cdot)}$ under the corresponding assumptions on the exponents. Thus, our results provide a complete picture of the boundedness of one-sided maximal and fractional integral operators in variable-exponent Morrey spaces.

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