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RESEARCH OF A NONLINEAR FOUR-DIMENSIONAL DYNAMIC SYSTEM DESCRIBING THE PROCESS OF INTERACTION BETWEEN GEORGIAN, LAZ, MINGRELIAN, AND SVAN POPULATIONS

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Abstract. This paper examines a mathematical model of interaction between the Georgian, Laz, Mingrelian, and Svan populations, which emerged as a result of the transformation of the Proto–Kartvelian population from the 1st century BC to the present day. The model is described by a four-dimensional nonlinear dynamic system. For certain dependencies of constant coefficients of the differential equation system, the first integrals are found, and the four-dimensional dynamic system is reduced to a two-dimensional one. Using the Poincaré–Bendixson theorem (principle), the theorems on the sign change of the divergence of the vector field in a certain simply connected domain of the first quadrant of the phase plane and the existence of closed integral trajectories within this domain are proven. It is shown that all four populations coexist in a unified geographical space, and none of them undergos complete assimilation.

Indroduction

Mathematical modeling of social processes, compared to modeling in natural sciences, is more original due to the complexity of model justifications [2–4,9–14]. We view mathematical modeling as an innovative approach to describe the area of distribution of the Proto–Kartvelian-speaking population and the process of further transformation of the language, determining the population of people speaking the corresponding language in each time period [5–8].

1. General Mathematical Model. System of Nonlinear Equations

The process of the final transformation of the Proto–Kartvelian ethnolinguistic unity is described by the following four-dimensional nonlinear dynamic system with variable coefficients:

$$\begin{cases} \dfrac{du(t)}{dt} = \alpha_{10}(t)u(t) + \gamma_{15}(t)w(t)u(t) + \gamma_{16}(t)z(t)u(t) + \gamma_{17}(t)v(t)u(t) - \delta_6(t)u(t), \\ \dfrac{dw(t)}{dt} = \alpha_{11}(t)w(t) - \gamma_{18}(t)u(t)w(t) + \gamma_{19}(t)z(t)w(t) + \gamma_{20}(t)v(t)w(t) - \delta_7(t)w(t,) \\ \dfrac{dz(t)}{dt} = \alpha_{12}(t)z(t) - \gamma_{21}(t)u(t)z(t) - \gamma_{22}(t)w(t)z(t) + \gamma_{23}(t)v(t)z(t) - \delta_8(t)z(t), \\ \dfrac{dv(t)}{dt} = \alpha_{13}(t)v(t) - \gamma_{24}(t)u(t)v(t) - \gamma_{25}(t)w(t)v(t) - \gamma_{26}(t)z(t)v(t) - \delta_9(t)v(t), \end{cases} \tag{1.1}$$

where $u(t), w(t), z(t), v(t) \in C^1[t_4, t_5]$ represent the populations of the Georgian, Laz, Mingrelian, and Svan groups at time t, respectively; $t \in [t_4, t_5]$, with $t_4 = 4900$ years (1st century BC) and $t_5 = 7025$ years (AD 2025); $\alpha_i(t) \in C[t_4, t_5]$, $i = \overline{10-13}$, $\alpha_{13}(t) > 0$ are the demographic factors of these populations; $\gamma_j(t) \in C[t_4, t_5]$, $\gamma_j(t) > 0$, $j = \overline{15-26}$ are the coefficients of interaction (assimilation) between them; $\delta_k(t) \in C[t_4, t_5]$, $\delta_k(t) > 0$, $k = \overline{6-9}$ represent the respective losses of these linguistic groups (due to wars, migrations, and other external factors).

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2. First Integrals of the Dynamic System

If the coefficients of the Cauchy problem (1.1) are constant, the system can be rewritten as follows:

$$\begin{cases} \dfrac{du(t)}{udt} = (\alpha_{10} - \delta_6) + \gamma_{15} w(t) + \gamma_{16} z(t) + \gamma_{17} v(t), \\ \dfrac{dw(t)}{wdt} = (\alpha_{11} - \delta_7) - \gamma_{18} u(t) + \gamma_{19} z(t) + \gamma_{20} v(t), \\ \dfrac{dz(t)}{zdt} = (\alpha_{12} - \delta_8) - \gamma_{21} u(t) - \gamma_{22} w(t) + \gamma_{23} v(t), \\ \dfrac{dv(t)}{vdt} = (\alpha_{13} - \delta_9) - \gamma_{24} u(t) - \gamma_{25}(t) w(t) - \gamma_{26} z(t), \\ u(t_4) = u_4,\ w(t_4) = w_4,\ z(t_4) = z_4\ v(t_4) = v_4. \end{cases} \tag{2.1}$$

For the adequacy and non-triviality of the mathematical model (2.1), the following condition must be satisfied:

$$\begin{cases} \alpha_{10} - \delta_6 < 0, \\ \alpha_{13} - \delta_9 > 0. \end{cases} \tag{2.2}$$

Adding the first and fourth equations of system (2.1) and subtracting the sum of the second and third equations, and then adding the first and third equations and subtracting the second equation twice, and imposing the following conditions:

$$\begin{cases} (\alpha_{10} - \delta_6) + (\alpha_{13} - \delta_9) - (\alpha_{11} - \delta_7) - (\alpha_{12} - \delta_8) = 0, \\ \gamma_{18} + \gamma_{21} - \gamma_{24} = 0, \\ \gamma_{15} + \gamma_{22} - \gamma_{25} = 0, \\ \gamma_{16} - \gamma_{19} - \gamma_{26} = 0, \\ \gamma_{17} - \gamma_{20} - \gamma_{23} = 0, \end{cases} \tag{2.3}$$

$$\begin{cases} (\alpha_{10} - \delta_6) + (\alpha_{12} - \delta_8) - 2(\alpha_{11} - \delta_7) = 0, \\ 2\gamma_{18} - \gamma_{21} = 0, \\ \gamma_{15} - \gamma_{22} = 0, \\ \gamma_{16} - 2\gamma_{19} = 0, \\ \gamma_{17} + \gamma_{23} - 2\gamma_{20} = 0, \end{cases} \tag{2.4}$$

we obtain two first integrals of the constant-coefficient dynamic system (2.1):

$$\begin{cases} v(t) = pq\dfrac{w^3(t)}{u^2(t)}, \\ z(t) = q\dfrac{w^2(t)}{u(t)}, \end{cases} \tag{2.5}$$

where $p \equiv \dfrac{u_4 v_4}{w_4 z_4} = \text{const} > 0$, $q \equiv \dfrac{u_4 z_4}{w_4^2} = \text{const} > 0$.

Considering the system of ratios (2.5), the four-dimensional constant-coefficient nonlinear differential equation system (2.1) is reduced to the following two-dimensional dynamic system:

$$\begin{cases} \dfrac{du(t)}{dt} = (\alpha_{10} - \delta_6) u(t) + \gamma_{15} w(t) u(t) + \gamma_{16} q w^2(t) + \gamma_{17} pq \dfrac{w^3(t)}{u(t)}, \\ \dfrac{dw(t)}{dt} = (\alpha_{11} - \delta_7) w(t) - \gamma_{18} u(t) w(t) + \gamma_{19} q \dfrac{w^3(t)}{u(t)} + \gamma_{20} pq \dfrac{w^4(t)}{u^2(t)}, \\ u(t_4) = u_4,\ w(t_4) = w_4. \end{cases} \tag{2.6}$$

We introduce the following notations:

$$\begin{cases} F_1(u(t),w(t)) \equiv (\alpha_{10}-\delta_6)u(t)+\gamma_{15}w(t)u(t)+\gamma_{16}qw^2(t)+\gamma_{17}pq\dfrac{w^3(t)}{u(t)}, \\ F_2(u(t),w(t)) \equiv (\alpha_{11}-\delta_7)w(t)-\gamma_{18}u(t)w(t)+\gamma_{19}q\dfrac{w^3(t)}{u(t)}+\gamma_{20}pq\dfrac{w^4(t)}{u^2(t)}. \end{cases} \tag{2.7}$$

To study the system of nonlinear differential equations (2.6), we consider the divergence of the vector field $\overrightarrow{F}(F_1,F_2)$, which is computed from (2.7):

$$G(w,u) \equiv \operatorname{div}\overrightarrow{F} = \frac{\partial F_1(u,w)}{\partial u}+\frac{\partial F_2(u,w)}{\partial w} = (\alpha_{10}-\delta_6)+(\alpha_{11}-\delta_7)$$
$$+\gamma_{15}w(t)-\gamma_{18}u(t)+3\gamma_{19}q\frac{w^2(t)}{u(t)}+(4\gamma_{20}-\gamma_{17})pq\frac{w^3(t)}{u^2(t)}. \tag{2.8}$$

On the phase plane $(O,w(t),u(t))$, we consider the curve on which the divergence of the vector field is zero:

$$G(w,u)=0. \tag{2.9}$$

3. Theorems on the Coexistence of Four Kartvelian Populations

For equation (2.9), we consider three cases.

First Case. Suppose that the following conditions is satisfied:

$$\begin{cases} (\alpha_{10}-\delta_6)+(\alpha_{11}-\delta_7)=0, \\ 4\gamma_{20}-\gamma_{17}=0, \end{cases} \tag{3.1}$$

which does not contradict conditions (2.2), (2.3) and (2.4). Then, according to (2.8) and (2.9), the divergence of the vector field vanishes on the next ray in the phase plane $(O,w(t),u(t))$:

$$\begin{cases} u=\varphi w, \\ \varphi=\dfrac{\gamma_{15}+\sqrt{\gamma_{15}^2+12\gamma_{18}\gamma_{19}q}}{2\gamma_{18}}=\text{const}>0, \\ w(t)>0, \end{cases} \tag{3.2}$$

since only the first quadrant of the phase plane $(O,w(t),u(t))$ has physical meaning. The following theorem holds.

Theorem 1. *For the problem defined by* (2.6), (2.7), (2.2), (2.3), (2.4) *and* (3.1)*, in some simply connected domain* $D\subset(O,w(t),u(t))$ *of the first quadrant of the phase plane* $(O,w(t),u(t))$ *with physical meaning, there exists a solution in the form of a closed integral trajectory that lies entirely within this domain.*

Proof. We consider the line in the phase plane $(O,w(t),u(t))$, where the divergence of the vector field $\overrightarrow{F}(F_1,F_2)$ is zero. Taking into account (2.8), (2.9) and (3.1), the curve (2.9) takes the form of the ray (3.2). The divergence of the vector field $\overrightarrow{F}(F_1,F_2)$ in the first quadrant of the phase plane $(O,w(t),u(t))$ vanishes on the ray (3.2). Suppose this ray also contains the point $M(w_4,u_4)$, where $w(t)>0$. It is clear that the divergence $G(w(t),u(t))$ of the vector field $\overrightarrow{F}(F_1,F_2)$ changes sign in some simply connected domain $D\subset(O,w(t),u(t))$ that contains the point $M(w_4,u_4)$. According to the Poincaré–Bendixson theorem [1,15], for the problem defined by (2.6), (2.7), (2.2), (2.3), (2.4) and (3.1), there exists a closed integral curve lying entirely within this domain. □

Second Case. Suppose that the following conditions:

$$\begin{cases} (\alpha_{10}-\delta_6)+(\alpha_{11}-\delta_7)=0, \\ \gamma_{17}-4\gamma_{20}>0, \\ \dfrac{\gamma_{18}}{\gamma_{15}}=\dfrac{3\gamma_{19}}{p(\gamma_{17}-4\gamma_{20})}, \end{cases} \tag{3.3}$$

is satisfied, which does not contradict conditions (2.2), (2.3) and (2.4). Then, considering (2.8), from equation (2.9), we obtain

$$G(w,u)=\gamma_{15}\left[w(t)-\frac{\gamma_{18}}{\gamma_{15}}u(t)\right]\left[1-3q\frac{\gamma_{19}}{\gamma_{18}}\frac{w^2(t)}{u^2(t)}\right]=0. \tag{3.4}$$

For equation (3.4), we consider two subcases:

$$w(t)=\frac{\gamma_{18}}{\gamma_{15}}u(t) \tag{3.5}$$

and

$$w(t)=\sqrt{\frac{\gamma_{18}}{3q\gamma_{19}}}u(t). \tag{3.6}$$

The following theorems hold.

Theorem 2. *For the problem defined by* (2.6), (2.7), (2.2), (2.3), (2.4), (3.3) *and* (3.5), *in some simply connected domain* $D\subset(O,w(t),u(t))$ *of the first quadrant of the phase plane* $(O,w(t),u(t))$ *with physical meaning, there exists a solution in the form of a closed integral trajectory that lies entirely within this domain.*

Theorem 3. *For the problem defined by* (2.6), (2.7), (2.2), (2.3), (2.4), (3.3) *and* (3.6), *in some simply connected domain* $D\subset(O,w(t),u(t))$ *of the first quadrant of the phase plane* $(O,w(t),u(t))$ *with physical meaning, there exists a solution in the form of a closed integral trajectory lying entirely within this domain.*

Third Case. Suppose that the following conditions:

$$\begin{cases}(\alpha_{10}-\delta_6)+(\alpha_{11}-\delta_7)=0,\\ 4\gamma_{20}-\gamma_{17}\neq 0,\end{cases} \tag{3.7}$$

is satisfied, which does not contradict the previously established conditions (2.2), (2.3) and (2.4)). Then, taking into account (2.8), from (2.9), we obtain:

$$G(w,u)=\gamma_{15}w(t)-\gamma_{18}u(t)+3q\gamma_{19}\frac{w^2(t)}{u(t)}+(4\gamma_{20}-\gamma_{17})pq\frac{w^3(t)}{u^2(t)}=0. \tag{3.8}$$

In the first quadrant of the phase plane $(O,w(t),u(t))$ of physical significance, we seek a solution to (3.8) in the form

$$w(t)=xu(t), \tag{3.9}$$

where $u>0, w>0, x>0$. Substituting into (3.8), we obtain

$$G(xu,u)=u(t)f(x),$$

where

$$f(x)\equiv(4\gamma_{20}-\gamma_{17})pqx^3+3q\gamma_{19}x^2+\gamma_{15}x-\gamma_{18}.$$

Consider the case

$$4\gamma_{20}-\gamma_{17}>0. \tag{3.10}$$

The following theorem holds.

Theorem 4. *For the problem defined by* (2.6), (2.7), (2.2), (2.3), (2.4), (3.7) *and* (3.10), *in some simply connected domain* $D\subset(O,w(t),u(t))$ *of the first quadrant of the phase plane* $(O,w(t),u(t))$ *of physical significance, there exists a solution in the form of a closed integral trajectory lying entirely within this domain.*

Conclusion

Thus, the dynamic system (1.1) with the given specific constant coefficients and initial conditions, has solutions in the form of closed integral trajectories, indicating that the Georgian, Laz, Mingrelian, and Svan populations do not undergo complete assimilation. Instead, they achieve historical-linguistic coexistence, stable and sustainable development $(u(t)>0, w(t)>0, z(t)>0, v(t)>0)$.

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