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PERIODIC PROBLEM WITH RESPECT TO A SPATIAL VARIABLE FOR A SEMILINEAR WAVE EQUATION

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Dedicated to the memory of Professor Elene Obolashvili

Abstract. For a one-dimensional semilinear wave equation, a periodic problem with respect to a spatial variable is studied. Depending on the structure of the nonlinear term included in the equation, the questions of the existence, uniqueness and absence of a solution to the problem are investigated.

1. Statement of the Problem

In the plane of independent variables x and t in the domain $D_T := \{(x,t) \in \mathbb{R}^2 : \ 0 < x < l, 0 < t < T\}$, $\mathbb{R} := (-\infty, +\infty)$ we consider the problem of determining the solution $u(x,t)$ of the semilinear wave equation

$$\square\, u + f(u) = F(x,t), \quad (x,t) \in D_T, \tag{1.1}$$

satisfying the initial conditions

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \ \ 0 \le x \le l, \tag{1.2}$$

with respect to the variable t and the periodic conditions

$$u(0,t) = u(l,t), \quad u_x(0,t) = u_x(l,t), \ \ 0 \le t \le T, \tag{1.3}$$

with respect to the variable x, where f, F, φ and ψ are the given and u is the unknown real functions, $\square := \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$.

Everywhere below, when considering the classical solution $u \in C^2(\overline{D}_T)$ of the problem (1.1)–(1.3), we will assume that the following conditions of smoothness and consistency of data f, F, φ and ψ of problem (1.1)–(1.3):

$$\begin{gathered} f \in C^1(\mathbb{R}), \quad F \in C^1(\overline{D}_T), \quad \varphi \in C^2([0,l]), \quad \psi \in C^1([0,l]), \\ F(0,0) - f[\varphi(0)] + \varphi''(0) = F(l,0) - f[\varphi(l)] + \varphi''(l), \\ \varphi(0) = \varphi(l), \quad \varphi'(0) = \varphi'(l), \quad \psi(0) = \psi(l), \quad \psi'(0) = \psi'(l), \end{gathered} \tag{1.4}$$

are satisfied at the points $(0,0)$ and $(l,0)$.

For some classes of hyperbolic equations, periodic problems have been the subject of research by many authors (see, e.g., [2,3,7–11,13,14,16] and references therein).

2. A Priori Estimate of the Solution to Problem (1.1)–(1.3)

Let us consider the condition imposed on the nonlinear function f

$$(Gf)(s) := \int\limits_0^s f(s_1)ds_1 \ge -M_1 s^2 - M_2 \quad \forall s \in \mathbb{R}, \tag{2.1}$$

where $M_i := \text{const} \ge 0$, $i = 1, 2$.

The following lemma holds:

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Lemma 2.1. *Let the condition* (2.1) *be satisfied. Then for any solution* $u \in C^2(\overline{D}_T)$ *of problem* (1.1)–(1.3) *the a priori estimate*

$$\|u\|_{C(\overline{D}_T)} \leq c_1\|\varphi\|_{C^1([0,l])} + c_2\|\psi\|_{C([0,l])} + c_3\|f\|_{C([-\|\varphi\|_{C(\omega_0)},\|\varphi\|_{C(\omega_0)}])} + c_4\|F\|_{C(\overline{D}_T)} + c_5 \tag{2.2}$$

with positive constants $c_i = c_i(l,T)$, $i = 1,2,3,4$, *independent of the functions* u, F, φ *and* ψ *is valid at the same time* $c_5 \geq 0$.

Proof. Multiplying both sides of equality (1.1) by $2u_t$, and integrating over the domain D_τ, $0 < \tau \leq T$, we obtain

$$\int\limits_{D_\tau} (u_t^2)_t dxdt - 2\int\limits_{D_\tau} u_{xx}u_t dxdt + 2\int\limits_{D_\tau} \big[(Gf)(u)\big]_t dxdt = 2\int\limits_{D_\tau} Fu_t dxdt. \tag{2.3}$$

Set $\omega_\tau : t = \tau$, $0 \leq x \leq l$, $0 \leq \tau \leq T$; $\Gamma := \Gamma_1 \cup \omega_0 \cup \Gamma_2$, where $\Gamma_1 : x = 0$, $0 \leq t \leq T$; $\Gamma_2 : x = l$, $0 \leq t \leq T$. Let $\nu := (\nu_x, \nu_t)$ be a unit vector of the outward normal to ∂D_τ. It is easy to see that

$$\begin{gathered} \nu_x|_{\omega_\tau} = 0, \quad 0 \leq \tau \leq T, \quad \nu_x|_{\Gamma_1} = -1, \quad \nu_x|_{\Gamma_2} = 1, \\ \nu_t|_{\Gamma_1\cup\Gamma_2} = 0, \quad \nu_t|_{\omega_0} = -1, \quad \nu_t|_{\omega_\tau} = 1, \quad 0 < \tau \leq T. \end{gathered} \tag{2.4}$$

Using integration by parts, taking into account (1.2), (1.3) and (2.4), we have

$$\begin{gathered} \int\limits_{D_\tau} (u_t^2)_t dxdt + 2\int\limits_{D_\tau} \big[(Gf)(u)\big]_t dxdt = \int\limits_{\partial D_\tau} u_t^2\nu_t ds + 2\int\limits_{\partial D_\tau} (Gf)(u)\nu_t ds \\ = \int\limits_{\omega_\tau} u_t^2 dx - \int\limits_{\omega_0} \psi^2 dx + 2\int\limits_{\omega_\tau} (Gf)(u)dx - 2\int\limits_{\omega_0} (Gf)(\varphi)dx, \\ -2\int\limits_{D_\tau} u_{xx}u_t dxdt = 2\int\limits_{D_\tau} [u_xu_{tx} - (u_xu_t)_x]dxdt = \int\limits_{D_\tau} (u_x^2)_t dxdt \\ -2\int\limits_{\partial D_\tau} u_xu_t\nu_x ds = \int\limits_{\partial D_\tau} u_x^2\nu_t ds = \int\limits_{\omega_\tau} u_x^2 dx - \int\limits_{\omega_0} \varphi'^2 dx, \end{gathered} \tag{2.5}$$

where $\Gamma_{i,\tau} := \Gamma_i \cap \{t \leq \tau\}$, $i = 1,2$.

In view of (2.5), equality (2.3) can be rewritten as

$$\begin{gathered} w(\tau) := \int\limits_{\omega_\tau} (u_x^2 + u_t^2)dx = \int\limits_{\omega_0} (\varphi'^2 + \psi^2)dx \\ +2\int\limits_{\omega_0} (Gf)(\varphi)dx - 2\int\limits_{\omega_\tau} (Gf)(u)dx + 2\int\limits_{D_\tau} Fu_t dxdt. \end{gathered} \tag{2.6}$$

Taking into account (2.1), from (2.6), it follows that

$$w(\tau) \leq \int\limits_{\omega_0} (\varphi'^2 + \psi^2)dx + 2\int\limits_{\omega_0} (Gf)(\varphi)dx + 2M_1\int\limits_{\omega_\tau} u^2 dx + 2M_2 l + 2\int\limits_{D_\tau} Fu_t dxdt. \tag{2.7}$$

Further, in view of (1.2)

$$u(x,\tau) = \varphi(x) + \int\limits_0^\tau u_t(x,t)dt,$$

then

$$|u(x,\tau)|^2 \leq 2\varphi^2(x) + 2\bigg(\int\limits_0^\tau u_t(x,t)dt\bigg)^2 \leq 2\varphi^2(x) + 2\tau\int\limits_0^\tau u_t^2(x,t)dt.$$

Integrating the obtained inequality with respect to the variable x and taking into account (2.6), we obtain

$$\int\limits_{\omega_\tau} u^2dx \le 2\|\varphi\|^2_{L_2(\omega_0)} + 2T\int\limits_0^\tau \Big[\int\limits_{\omega_t} u_t^2dx\Big]dt \le 2l\|\varphi\|^2_{C(\omega_0)} + 2T\int\limits_0^\tau w(t)dt. \tag{2.8}$$

For $(x,t)\in\overline{D}_T$, integrating the obvious inequality

$$|u(x,t)|^2 = \Big|u(\xi,t) + \int\limits_\xi^x u_x(x_1,t)dx_1\Big|^2 \le 2|u(\xi,t)|^2 + 2l\int\limits_0^l u_x^2(x,t)dx$$

with respect to the variable $\xi\in[0,l]$, similarly to how inequality (2.8) was obtained, we will have

$$|u(x,t)|^2 \le \frac{2}{l}\int\limits_0^l |u(\xi,t)|^2d\xi + 2lw(t) = \frac{2}{l}\int\limits_{\omega_t} u^2dx + 2lw(t). \tag{2.9}$$

From (2.8) and (2.9), it follows that

$$|u(x,t)|^2 \le 4\|\varphi\|^2_{C(\omega_0)} + \frac{4T}{l}\int\limits_0^t w(\sigma)d\sigma + 2lw(t), \quad (x,t)\in\overline{D}_T. \tag{2.10}$$

Taking into account that

$$2\Big|\int\limits_{D_\tau} Fu_tdxdt\Big| \le \int\limits_{D_\tau} F^2dxdt + \int\limits_{D_\tau} u_t^2dxdt \le lT\|F\|^2_{C(\overline{D}_T)}$$

$$+\int\limits_0^\tau\Big[\int\limits_{\omega_t} u_t^2dx\Big]dt \le lT\|F\|^2_{C(\overline{D}_T)} + \int\limits_0^\tau w(t)dt,$$

$$\int\limits_{\omega_0}(\varphi'^2+\psi^2)dx \le l\big(\|\varphi'\|^2_{C(\omega_0)} + \|\psi\|^2_{C(\omega_0)}\big) \le l\Big(\|\varphi\|^2_{C^1(\omega_0)} + \|\psi\|^2_{C(\omega_0)}\Big),$$

$$2\int\limits_{\omega_0}(Gf)(\varphi)dx = 2\int\limits_{\omega_0}\Big[\int\limits_0^{\varphi(x)} f(s_1)ds_1\Big]dx \le 2l\|(G|f|)(|\varphi|)\|_{C(\omega_0)}$$

$$= 2l\Big\|\int\limits_0^{|\varphi|}|f(s_1)|ds_1\Big\|_{C(\omega_0)} \le 2l\|\varphi\|_{C(\omega_0)}\|f\|_{C([-\|\varphi\|_{C(\omega_0)},\|\varphi\|_{C(\omega_0)}])}$$

$$\le l\Big(\|\varphi\|^2_{C^1(\omega_0)} + \|f\|^2_{C([-\|\varphi\|_{C(\omega_0)},\|\varphi\|_{C(\omega_0)}])}\Big)$$

from (2.7), in view of (2.8), we obtain

$$w(\tau) \le \alpha_1\int\limits_0^\tau w(t)dt + \alpha_2, \tag{2.11}$$

where

$$\alpha_1 := 1+4M_1T, \quad \alpha_2 := 2l(1+2M_1)\|\varphi\|^2_{C^1(\omega_0)} + l\|\psi\|^2_{C(\omega_0)}$$

$$+l\|f\|^2_{C([-\|\varphi\|_{C(\omega_0)},\|\varphi\|_{C(\omega_0)}])} + lT\|F\|^2_{C(\overline{D}_T)} + 2M_2l.$$

Applying Gronwall's lemma to inequality (2.11), we obtain

$$w(\tau) \le \alpha_2\exp\big(\alpha_1T\big), \qquad 0<\tau\le T. \tag{2.12}$$

From (2.10) and (2.12), it follows that

$$|u(x,t)|^2 \le 4\|\varphi\|^2_{C^1(\omega_0)} + 2\alpha_2\left(\frac{2T^2}{l} + l\right)\exp\left(\alpha_1 T\right), \quad (x,t)\in \overline{D}_T,$$

whence, taking into account the obvious inequality $\left(\sum_{i=1}^n a_i^2\right)^{\frac{1}{2}} \le \sum_{i=1}^n |a_i|$, we get the a priori estimate (2.2), in which

$$c_1 := 2\big(1 + c_0\sqrt{1+2M_1}\big), \ \ c_2 = c_3 := \sqrt{2}c_0, \ \ c_4 = \sqrt{2T}c_0, \ \ c_5 = 2\sqrt{M_2}c_0, \tag{2.13}$$

where

$$c_0 := \sqrt{2T^2 + l^2}\exp\frac{\alpha_1 T}{2}.$$

Lemma 2.1 is proven. □

Remark 2.1. In particular, for $f = 0$, assuming $M_1 = M_2 = 0$ in inequality (2.1) and taking into account (2.13), the uniqueness of the solution to problem (1.1)–(1.3) follows from the a priori estimate (2.2).

3. Existence of a Solution to Problem (1.1)–(1.3)

Before moving on to the equivalent reduction of problem (1.1)–(1.3) to a nonlinear integral equation, we present the solution of the following mixed problem for the corresponding linear equation (1.1) i.e., for $f = 0$, retaining the previous notation u for the unknown function: in the domain D_T find the solution $u \in C^2(\overline{D}_T)$ of equation (1.1) by the boundary

$$u(0,t) = \mu_1(t), \quad u(l,t) = \mu_2(t), \ \ 0 \le t \le l, \tag{3.1}$$

and by the initial conditions (1.2), where the functions F, φ, ψ, μ_1 and μ_2 satisfy the smoothness and consistency conditions, similar to (1.4).

For simplicity of presentation, let us consider the case where the domain D_T is a square, i.e., $T = l$. In order to solve this problem in quadratures let us divide the domain D_T, which is a square with vertices at the points $A(0,0)$, $B(0,l)$, $C(l,l)$ and $D(l,0)$, into four rectangular triangles $\Delta_1 := \Delta AOD$, $\Delta_2 := \Delta AOB$, $\Delta_3 := \Delta DOC$ and $\Delta_4 := \Delta BOC$, where the point $O(\frac{l}{2},\frac{l}{2})$ is the center of the square D_l [6].

By virtue of d'Alembert's formula (see, e.g., [1]), the solution of problem (1.1), (1.2) is given by the following equality:

$$u(x,t) = \frac{1}{2}\big[\varphi(x-t) + \varphi(x+t)\big]$$
$$+\frac{1}{2}\int\limits_{x-t}^{x+t}\psi(\tau)d\tau + \frac{1}{2}\int\limits_{\Omega^1_{x,t}} F d\xi d\tau, \quad (x,t)\in\Delta_1, \tag{3.2}$$

where $\Omega^1_{x,t}$ is a triangle with vertices at the points (x,t), $(x-t,0)$ and $(x+t,0)$.

As is known, for any twice continuously differentiable function v and any characteristic rectangle $PP_1P_3P_2$ from the domain of its definition for equation (1.1), the identity of the characteristic rectangle

$$v(P) = v(P_1) + v(P_2) - v(P_3) + \frac{1}{2}\int\limits_{PP_1P_3P_2} \Box\, v(\xi,\tau)d\xi d\tau, \tag{3.3}$$

is valid [1], where P and P_3, as well as P_1 and P_2 are opposite vertices of this rectangle, and the ordinate of the point P is greater than the ordinates of the other points.

Let now $(x,t)\in\Delta_2$. Then, applying equality (3.3) for the characteristic rectangle with vertices at the points $P(x,t)$, $P_1(0,t-x)$, $P_2(t,x)$ and $P_3(t-x,0)$, and also formula (3.2) for the point

$P_2(t,x) \in \Delta_1$, taking into account (1.1) and (3.1), we obtain

$$u(x,t) = \mu_1(t-x) + \frac{1}{2}\big[\varphi(t+x) - \varphi(t-x)\big]$$
$$+\frac{1}{2}\int\limits_{t-x}^{t+x} \psi(\tau)d\tau + \frac{1}{2}\int\limits_{\Omega^2_{x,t}} F(\xi,\tau)d\xi d\tau, \quad (x,t) \in \Delta_2. \tag{3.4}$$

Here, $\Omega^2_{x,t}$ is a quadrangle $P\widetilde{P}_2P_3P_1$, where $\widetilde{P}_2 := \widetilde{P}_2(t+x,0)$.

Similarly, we have

$$u(x,t) = \mu_2(x+t-l) + \frac{1}{2}\big[\varphi(x-t) - \varphi(2l-x-t)\big]$$
$$+\frac{1}{2}\int\limits_{x-t}^{2l-x-t} \psi(\tau)d\tau + \frac{1}{2}\int\limits_{\Omega^3_{x,t}} F(\xi,\tau)d\xi d\tau, \quad (x,t) \in \Delta_3 \tag{3.5}$$

and

$$u(x,t) = \mu_1(t-x) + \mu_2(x+t-l) - \frac{1}{2}\big[\varphi(t-x) + \varphi(2l-t-x)\big]$$
$$+\frac{1}{2}\int\limits_{t-x}^{2l-t-x} \psi(\tau)d\tau + \frac{1}{2}\int\limits_{\Omega^4_{x,t}} F(\xi,\tau)d\xi d\tau, \quad (x,t) \in \Delta_4. \tag{3.6}$$

Here, $\Omega^3_{x,t}$ is a quadrilateral with vertices $P^3(x,t)$, $P^3_1(l,x+t-l)$, $P^3_2(x-t,0)$ and $P^3_3(2l-x-t,0)$, and $\Omega^4_{x,t}$ is a pentagon with vertices $P^4(x,t)$, $P^4_1(0,t-x)$, $P^4_2(t-x,0)$, $P^4_3(2l-x-t,0)$ and $P^4_4(l,x+t-l)$.

Thus the unique classical solution $u \in C^2(\overline{D}_l)$ of problem (1.1), (1.2), (3.1) for $f = 0$ is given by formulas (3.2), (3.4)–(3.6), which will be applied to obtain a solution to the periodic problem (1.1)–(1.3) for $f = 0$. For this purpose, we apply the Monge–Ampere theory presented in [5]. Along the family of characteristics $x + t =$ const for the first-order derivatives of the uknown solution u, the relations

$$du_t = u_{tx}dx + u_{tt}dt = (u_{tt} - u_{tx})dt$$

and

$$du_x = u_{xx}dx + u_{xt}dt = (u_{xt} - u_{xx})dt = (u_{xt} + F - u_{tt})dt = (u_{xt} - u_{tt})dt + Fdt$$

are valid. Whence we have

$$d(u_x + u_t) = Fdt,$$

the integration of which from the point $(t,0)$ to the point $(0,t)$ along the corresponding characteristic gives

$$(u_x + u_t)(0,t) - (u_x + u_t)(t,0) = \int\limits_0^t F(t-\tau,\tau)d\tau. \tag{3.7}$$

Similar reasoning applied to the second family of characteristics $x - t =$ const, by integrating the corresponding relation $d(u_t - u_x) = Fdt$ along the corresponding characteristic from the point $(t,0)$ to the point $(l, l-t)$ leads to the equality

$$(u_t - u_x)(l, l-t) - (u_t - u_x)(t,0) = \int\limits_0^{l-t} F(t+\tau,\tau)d\tau. \tag{3.8}$$

Taking into account equalities (3.7) and (3.8), as well as the requirement of the first periodicity condition (1.3), for $\mu := \mu_1 = \mu_2$, we obtain

$$\mu(t) = \frac{1}{2}\bigg\{\varphi(t) + \varphi(l-t) + \int\limits_0^t [\psi(\tau) + \psi(l-\tau)]d\tau$$

$$+ \int\limits_0^t d\tau_1 \int\limits_0^{\tau_1} [F(\tau_1 - \tau, \tau) + F(l - \tau_1 + \tau, \tau)]d\tau\bigg\}, \quad 0 \le t \le l. \tag{3.9}$$

Substituting the obtained expression for the function μ according to formula (3.9) into equalities (3.4)–(3.6), we have

$$u(x,t) = \frac{1}{2}\bigg\{\varphi(t+x) + \varphi(l-t+x) + \int\limits_0^{t+x} \psi(\tau)d\tau + \int\limits_0^{t-x} \psi(l-\tau)d\tau$$

$$+ \int\limits_0^{t-x} d\tau_1 \int\limits_0^{\tau_1} [F(\tau_1 - \tau, \tau) + F(l - \tau_1 + \tau, \tau)]d\tau + \int\limits_{\Omega^2_{x,t}} F(\xi,\tau)d\xi d\tau\bigg\}, \quad (x,t) \in \Delta_2, \tag{3.10}$$

$$u(x,t) = \frac{1}{2}\bigg\{\varphi(x-t) + \varphi(x+t-l) + \int\limits_0^{x+t-l} \big[\psi(\tau) + \psi(l-\tau)\big]d\tau + \int\limits_{x-t}^{2l-x-t} \psi(\tau)d\tau$$

$$+ \int\limits_0^{x+t-l} d\tau_1 \int\limits_0^{\tau_1} \big[F(\tau_1 - \tau, \tau) + F(l - \tau_1 + \tau, \tau)\big]d\tau$$

$$+ \int\limits_{\Omega^3_{x,t}} F(\xi,\tau)d\xi d\tau\bigg\}, \quad (x,t) \in \Delta_3, \tag{3.11}$$

$$u(x,t) = \frac{1}{2}\bigg\{\varphi(x-t+l) + \varphi(x+t-l) + \int\limits_0^{x+t-l} \psi(\tau)d\tau + \int\limits_0^{2l-t-x} \psi(\tau)d\tau$$

$$+ \int\limits_0^{t-x} \psi(l-\tau)d\tau + \int\limits_0^{x+t-l} \psi(l-\tau)d\tau + \int\limits_0^{t-x} d\tau_1 \int\limits_0^{\tau_1} \big[F(\tau_1 - \tau, \tau) + F(l - \tau_1 + \tau, \tau)\big]d\tau$$

$$+ \int\limits_0^{x+t-l} d\tau_1 \int\limits_0^{\tau_1} \big[F(\tau_1 - \tau, \tau) + F(l - \tau_1 + \tau, \tau)\big]d\tau$$

$$+ \int\limits_{\Omega^4_{x,t}} F(\xi,\tau)d\xi d\tau\bigg\}, \quad (x,t) \in \Delta_4. \tag{3.12}$$

Remark 3.1. From the above reasoning it follows that the classical solution $u \in C^2(\overline{D}_l)$ of problem (1.1)–(1.3) for $f = 0$ is represented in the form

$$u = A_1(\varphi, \psi) + A_2 F, \tag{3.13}$$

where the operators A_i, $i = 1, 2$, act based on the formulas: (3.2) for $(x,t) \in \Delta_1$; (3.10) for $(x,t) \in \Delta_2$; (3.11) for $(x,t) \in \Delta_3$; (3.12) for $(x,t) \in \Delta_4$.

Remark 3.2. From the structure of the operator A_2 it follows that this operator acts continuously from the space $C(\overline{D}_l)$ to the space $C^1(\overline{D}_l)$. Now, taking into account that the embedding of the space $C^1(\overline{D}_l)$ into the space $C(\overline{D}_l)$ is compact [4], we obtain that the operator

$$A_2 : \ C(\overline{D}_l) \to C(\overline{D}_l)$$

is compact.

Remark 3.3. Taking into account equality (3.13), it is easy to see that if $u \in C^2(\overline{D}_l)$ is a classical solution of the nonlinear problem (1.1)–(1.3), then it satisfies the following nonlinear integral equation

$$u = Au := A_1(\varphi, \psi) + A_2[F - f(u)]. \tag{3.14}$$

In this case, any solution of the integral equation (3.14) of class C will belong to the space $C^2(\overline{D}_l)$ and satisfy problem (1.1)–(1.3), if the smoothness and second-order consistency conditions (1.4) are satisfied for the data of this problem.

Remark 3.4. Note that by virtue of Remark 3.2, the operator

$$A:\ C(\overline{D}_l) \to C(\overline{D}_l)$$

from (3.14) is continuous and compact.

Further, for $\lambda \in [0,1]$, let $u = u_\lambda$ be a continuous solution of the nonlinear integral equation

$$u = \lambda Au.$$

It is easy to see that u_λ is a classical solution of the nonlinear problem (1.1)–(1.3), when instead of φ, ψ, F and f we take $\lambda\varphi, \lambda\psi, \lambda F$ and λf, respectively. Therefore, since $\lambda \in [0,1]$, from the a priori estimate (2.2) follows the inequality

$$\|u_\lambda\|_{C(\overline{D}_T)} \leq c_1\|\varphi\|_{C^1([0,l])} + c_2\|\psi\|_{C([0,l])} + c_3\|f\|_{C([-\|\varphi\|_{C(\omega_0)}, \|\varphi\|_{C(\omega_0)}])} + c_4\|F\|_{C(\overline{D}_T)} + c_5.$$

From here, by virtue of Remark 3.4 of the Leray–Schauder theorem [15], it follows that there exists a continuous solution to equation (3.14), which, by virtue of Remark 3.3, is also a classical solution to the original problem (1.1)–(1.3).

Thus the following theorem is true.

Theorem 3.1. *Let conditions* (1.4) *and* (2.1) *be satisfied. Then there exists at least one classical solution* $u \in C^2(\overline{D}_l)$ *of problem* (1.1)–(1.3).

Remark 3.5. Applying the linearization method and using the reasoning given in the proof of a priori estimate (2.2), we easily obtain the uniqueness of the solution to problem (1.1)–(1.3).

4. Cases of Violation of the Solvability of Problem (1.1)–(1.3)

Below, using the method of test functions [12], we will show that violation of condition (2.1) may, generally speaking, lead to the absence of a solution to problem (1.1)–(1.3).

Indeed, let it be

$$f(s) \geq \lambda|s|^p, \quad \lambda > 0, \quad p > 1, \quad s \in \mathbb{R}. \tag{4.1}$$

It is easy to see that if inequality (4.1) is satisfied, then condition (2.1) is violated.

Multiplying both sides of equation (1.1) by a test function $\chi \in C^2(\overline{D}_T)$ such that

$$\chi\big|_{D_T} > 0, \qquad \chi,\ \chi_t,\ \chi_x\big|_{\partial D_T} = 0, \tag{4.2}$$

after integration by parts, we get

$$\int\limits_{D_T} u\,\Box\,\chi dxdt + \int\limits_{D_T} f(u)\chi dxdt = \int\limits_{D_T} F\chi dxdt, \tag{4.3}$$

where $u \in C^2(\overline{D}_T)$ is the classical solution to problem (1.1)–(1.3).

By virtue of (4.1)–(4.3), we have

$$\lambda\int\limits_{D_T} |u|^p\chi dxdt \leq \int\limits_{D_T} |u\,\Box\,\chi| dxdt + \int\limits_{D_T} F\chi dxdt. \tag{4.4}$$

If in Young's inequality with parameter $\varepsilon > 0$

$$ab \leq \frac{\varepsilon}{p}a^p + \frac{1}{p'\varepsilon^{p'-1}}b^{p'}; \quad a, b \geq 0,\ \ \frac{1}{p} + \frac{1}{p'} = 1,\ \ p > 1$$

and letting $a=|u|\chi^{\frac{1}{p}}$, $b=\frac{|\Box\chi|}{\chi^{\frac{1}{p}}}$, then taking into account that $\frac{p'}{p}=p'-1$, we obtain

$$|u\,\Box\chi|=|u|\chi^{\frac{1}{p}}\frac{|\Box\chi|}{\chi^{\frac{1}{p}}}\le\frac{\varepsilon}{p}|u|^p\chi+\frac{1}{p'\varepsilon^{p'-1}}\frac{|\Box\chi|^{p'}}{\chi^{p'-1}}. \tag{4.5}$$

By virtue of (4.4) and (4.5), we have

$$\Big(\lambda-\frac{\varepsilon}{p}\Big)\int\limits_{D_T}|u|^p\chi dxdt\le\frac{1}{p'\varepsilon^{p'-1}}\int\limits_{D_T}\frac{|\Box\chi|^{p'}}{\chi^{p'-1}}dxdt+\int\limits_{D_T}F\chi dxdt,$$

whence for $\varepsilon<\lambda p$, we get

$$\int\limits_{D_T}|u|^p\chi dxdt\le\frac{p}{(\lambda p-\varepsilon)p'\varepsilon^{p'-1}}\int\limits_{D_T}\frac{|\Box\chi|^{p'}}{\chi^{p'-1}}dxdt+\frac{p}{\lambda p-\varepsilon}\int\limits_{D_T}F\chi dxdt. \tag{4.6}$$

Taking into account that $p'=\frac{p}{p-1}$, $p=\frac{p'}{p'-1}$ and $\min\limits_{0<\varepsilon<\lambda p}\frac{p}{(\lambda p-\varepsilon)p'\varepsilon^{p'-1}}=\frac{1}{\lambda^{p'}}$, which is achieved when $\varepsilon=\lambda$, it follows from (4.6) that

$$\int\limits_{D_T}|u|^p\chi dxdt\le\frac{1}{\lambda^{p'}}\int\limits_{D_T}\frac{|\Box\chi|^{p'}}{\chi^{p'-1}}dxdt+\frac{p'}{\lambda}\int\limits_{D_T}F\chi dxdt. \tag{4.7}$$

Below we will assume that, along with (4.2), the condition

$$\kappa:=\int\limits_{D_T}\frac{|\Box\chi|^{p'}}{|\chi|^{p'-1}}dxdt<+\infty \tag{4.8}$$

is satisfied. A simple check shows that as a function χ satisfying conditions (4.2) and (4.8) we can take, for example, the function

$$\chi(x,t)=[xt(l-x)(T-t)]^n,\quad (x,t)\in D_T,$$

with a sufficiently large natural n.

Assuming

$$F=-\mu F_0,\quad F_0\ge 0,\quad F_0\not\equiv 0,\quad \mu=\text{const}>0, \tag{4.9}$$

and taking into account (4.8), we rewrite inequality (4.7) in the form

$$\int\limits_{D_T}|u|^p\chi dxdt\le\frac{\kappa}{\lambda^{p'}}-\frac{\mu p'}{\lambda}\int\limits_{D_T}F_0\chi dxdt. \tag{4.10}$$

Due to the requirements imposed on the functions χ and F_0, we have

$$0\le\int\limits_{D_T}|u|^p\chi dxdt,\qquad \int\limits_{D_T}F_0\chi dxdt>0. \tag{4.11}$$

Therefore, under the assumption that problem (1.1)–(1.3) has a classical solution u and

$$\mu>\mu_0:=\frac{\kappa}{\lambda^{p'-1}p'}\bigg(\int\limits_{D_T}F_0\chi dxdt\bigg)^{-1}, \tag{4.12}$$

we arrive at a contradiction, since by virtue of (4.11), the left-hand side of (4.10) is non-negative, and the right-hand side is negative.

Thus the following theorem is true.

Theorem 4.1. *Let conditions* (4.1), (4.9) *and* (4.12) *be satisfied. Then problem* (1.1)–(1.3) *has no classical solution.*

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