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HIERARCHICAL MODELS FOR MICROPOLAR FLUIDS

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Dedicated to the memory of Professor Elene Obolashvili

Abstract. Applying Ilia Vekua's Dimension Reduction Method, the hierarchical models are constructed for micropolar fluids occupying containers having prismatic shell-like forms with three-dimensional angular edges, in general.

1. Introdaction

In [3, 9], mathematical hierarchical models for shallow fluids occupying non-Lipschitz, in general, prismatic shell-like $3D$ domains Ω (see also the end of Subsection 2.2 below) are constructed within the scheme of small displacements, linearized with respect to the rest state. Based on this, the hierarchical models have been constructed using the Lagrangian coordinates. In contrast, the aim of paper [12] was to construct mathematical hierarchical models for the Newtonian fluid flow without restricting the smallness of displacements with respect to the rest state. In other words, in [7], we constructed mathematical hierarchical models in the Eulerian coordinates for a Newtonian viscous fluid flow in prismatic shell-like domains (containers).

In [4], applying Ilia Vekua's Dimension Reduction Method [12] (see also [3, 6, 8]), the hierarchical models were constructed for micropolar elastic prismatic shells. The aim of the present paper is to construct, using that method, the hierarchical models for micropolar fluids occupying containers in the form of a prismatic shell, generally with three-dimensional angular edges.

2. About Prismatic Shell-like Containers of Fluids

Let $Ox_1x_2x_3$ be an anticlockwise-oriented rectangular Cartesian frame of origin O. We conditionally assume the x_3-axis vertical.

The container Ω is called a prismatic shell-like if it is bounded from above and below by, correspondingly, the surfaces (the so-called face surfaces)

$$x_3 = \overset{(+)}{h}(x_1, x_2) \text{ and } x_3 = \overset{(-)}{h}(x_1, x_2), \quad (x_1,\, x_2) \in \omega,$$

laterally by a cylindrical surface Γ with a generatrix, parallel to the x_3-axis, and its vertical dimension is sufficiently small compared with the other dimensions of the body. $\overline{\omega} := \omega \cup \partial\omega$ is the so-called projection of the prismatic shell onto $x_3 = 0$.

Let the thickness of the prismatic shell be

$$2h(x_1, x_2) := \overset{(+)}{h}(x_1, x_2) - \overset{(-)}{h}(x_1, x_2) \begin{cases} > 0 & \text{for } (x_1, x_2) \in \omega, \\ \geq 0 & \text{for } (x_1, x_2) \in \partial\omega \end{cases}$$

and

$$2\widetilde{h}(x_1, x_2) := \overset{(+)}{h}(x_1, x_2) + \overset{(-)}{h}(x_1, x_2).$$

If the thickness of the prismatic shell-like container vanishes on some subset of $\partial\omega$, it is called cusped one.

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Let us note that the lateral boundary of the standard shell-like container is orthogonal to the "middle surface" of the shell, while the lateral boundary of the prismatic shell-like container is orthogonal to the prismatic shell's-like container projection ω on $x_3 = 0$.

3. Governing Equations of the Newtonian Viscous Micropolar Fluid

Let $t \in T := [0, +\infty[$ be time, $T_+ :=]0, \infty[$, $\overline{\Omega} \times T$ denote the Cartesian product, u_i, and $v_i \in C^2(\Omega \times T_+)$, $i = 1, 2, 3$, be displacements and velocities, respectively, $\underline{\omega}_i$ and $\underline{\underline{\omega}}_i \in C^2(\Omega \times T_+)$, $i = 1, 2, 3$, be microrotations and microrotation velocities, respectively, $e_{ij} \in C^1(\Omega \times T_+)$ be the symmetric strain tensor, $u_{ji} \in C^1(\Omega \times T_+)$ be the asymmetric microstrain (torsion-flexure) tensor, $X_{ji} \in C^1(\Omega \times T_+)$ be the asymmetric force-stress tensor, p be the presure, $\chi_{ji} \in C^1(\Omega \times T_+)$ be the asymmetric couple stress tensor, $\Phi_i \in C(\Omega \times T_+)$ and $\Psi_i \in C(\Omega \times T_+)$ be the fields of volume forces and volume couples, respectively, ρ be the density, $\mathcal{I}$ be the rotational inertia of the medium, λ, μ, $\widetilde{\alpha}$, $\widetilde{\beta}$, ν and ε be the physical constants of the fluid, $\mu > 0$, $3\lambda + 2\mu > 0$, $\widetilde{\alpha} > 0$, $\widetilde{\beta} > 0$, $\nu > 0$, $3\varepsilon + 2\nu > 0$, $\in_{ijk}$ be the Levi–Civita symbol. Here, C^2 and C^1 are the classes of twice and once continuously differentiable functions in the domain under consideration; C is a class of continuous functions on the sets we consider. Throughout the paper, the Einstein rule of summation is used for Latin indices from 1 to 3, and for Greek indices from 1 to 2.

The governing equations of the micropolar theory of fluids (see [5, pp. 422–423]) have the following form:

Motion equations in the linearized case

$$X_{ji,j} + \Phi_i = \rho \dot{v}_i, \quad i=1,2,3, \tag{3.1}$$

$$\chi_{ji,j} + \in_{ijk} X_{jk} + \Psi_i = \mathcal{I} \dot{\underline{\underline{\omega}}}_i, \quad i=1,2,3. \tag{3.2}$$

Kinematic equations

$$u_{ji} = u_{i,j} - \in_{kji} \underline{\omega}_k = e_{ji} + \in_{kji} (\theta_k - \underline{\omega}_k), \quad i,j=1,2,3, \tag{3.3}$$

i.e., $$v_{ji} = v_{i,j} - \in_{kji} \underline{\underline{\omega}}_k = \dot{l}_{ji} + \in_{kji} (\dot{\theta}_k - \underline{\underline{\omega}})$$

and $$\underline{\omega}_{ji} = \underline{\omega}_{i,j}, \quad i,j=1,2,3; \tag{3.4}$$

i.e., $$\underline{\underline{\omega}}_{ji} = \underline{\underline{\omega}}_{ij.}$$

Constitutive equations in the case of a viscous, homogeneous, isotropic, incomprosible micropolar fluid (see [1, p. 6], and also [11, p. 21] and [5, p. 423]),

$$X_{ij} = -p\delta_{ij} + (\mu + \tilde{\alpha}) v_{ij} + (\mu - \tilde{\alpha}) v_{ji} \quad i,j = 1, 2, 3, \tag{3.5}$$

$$\chi_{ij} = \varepsilon \delta_{ij} \underline{\underline{\omega}}_{kk} + (\nu + \tilde{\beta}) \underline{\underline{\omega}}_{ij} + (\nu - \tilde{\beta}) \underline{\underline{\omega}}_{ji} \quad i,j = 1, 2, 3, \tag{3.6}$$

which we consider in the space domain (container), with the projection ω on the plane $x_3 = 0$,

$$\Omega := \{x := (x_1, x_2, x_3) \in R^3 \ : \ (x_1, x_2) \in \omega, \quad \overset{(-)}{h}(x_1, x_2) < x_3 < \overset{(+)}{h}(x_1, x_2)\}$$

occupied by the fluid.

4. Construction of Hierarchical Models

4.1. **Mathematical moments.** Here, we follow Section 10 of [6].[1]

Let $f(x_1, x_2, x_3)$ be a given function on $\overline{\Omega}$ having integrable partial derivatives, let f_r denote its r-th order moment defined as follows:

$$f_r(x_1, x_2) := \int\limits_{\overset{(-)}{h}(x_1, x_2)}^{\overset{(+)}{h}(x_1, x_2)} f(x_1, x_2, x_3) P_r(a x_3 - b) dx_3, \tag{4.1}$$

[1] where I. Vekua's Dimension Reduction Method is reformulated and presented in the unified form. Those formulas for arbitrary functions, independent of physical meaning, allow one to construct easily hierarchical models for any physical model having the thickness or something like that as parameter.

where (see Section 3 of [6])

$$a(x_1,x_2) := \frac{1}{h(x_1,x_2)}, \quad b(x_1,x_2) := \frac{\widetilde{h}(x_1,x_2)}{h(x_1,x_2)},$$
$$2h(x_1,x_2) = \overset{(+)}{h}(x_1,\, x_2) - \overset{(-)}{h}(x_1,\, x_2) > 0,$$
$$2\widetilde{h}(x_1,x_2) = \overset{(+)}{h}(x_1,\, x_2) + \overset{(-)}{h}(x_1,\, x_2) > 0,$$

and

$$P_r(\tau) = \frac{1}{2^r r!}\frac{d^r(\tau^2-1)^r}{d\tau^r}, \quad r = 0,1,\dots,$$

are the r-th order Legendre polynomials with the orthogonality property

$$\int\limits_{-1}^{+1} P_m(\tau)P_n(\tau)d\tau = \frac{2}{2m+1}\delta_{mn}.$$

From here, substituting[2]

$$\tau = ax_3 - b = \frac{2}{\overset{(+)}{h}(x_1,\, x_2) - \overset{(-)}{h}(x_1,\, x_2)}x_3 - \frac{\overset{(+)}{h}(x_1,\, x_2) + \overset{(-)}{h}(x_1,\, x_2)}{\overset{(+)}{h}(x_1,\, x_2) - \overset{(-)}{h}(x_1,\, x_2)},$$

we have

$$\Big(m+\frac{1}{2}\Big)a\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_m(ax_3-b)P_n(ax_3-b)dx_3 = \delta_{mn}.$$

Using the well-known formulas of integration by parts (with respect to x_3) and differentiating with respect to a parameter of integrals depending on the parameters (x_α), taking into account $P_r(1)=1$, $P_r(-1)=(-1)^r$, we deduce

$$\begin{aligned}\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,\alpha}\,dx_3 &= f_{r,\alpha} - \overset{(+)}{f}\overset{(+)}{h}_{,\alpha} + (-1)^r\overset{(-)}{f}\overset{(-)}{h}_{,\alpha}\\ &- \int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r'(ax_3-b)(a_{,\alpha}x_3 - b_{,\alpha})f dx_3, \quad \alpha=1,2,\end{aligned} \tag{4.2}$$

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,3}\,dx_3 = -a\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r'(ax_3-b)f dx_3 + \overset{(+)}{f} - (-1)^r\overset{(-)}{f}, \tag{4.3}$$

where superscript prime means differentiation with respect to the argument $ax_3 - b$, subscripts preceded by a comma mean partial derivatives with respect to the corresponding variables, $\overset{(\pm)}{f} :=$

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[2]

$$x_3 = [\overset{(+)}{h}(x_1,\, x_2) - \overset{(-)}{h}(x_1,\, x_2)]\frac{\tau}{2} + \frac{\overset{(+)}{h}(x_1,\, x_2) + \overset{(-)}{h}(x_1,\, x_2)}{2}.$$

$f[x_1,x_2,\overset{(\pm)}{h}(x_1,x_2)]$. Applying the following relations from the theory of the Legendre polynomials (see [7, pp. 197, 198], [13, p. 27], [3, p. 12], and [5, p. 299 or pp. 335-339], of the second edition)

$$P'_r(\tau)=\sum_{s=0}^{r}(2s+1)\frac{1-(-1)^{r+s}}{2}P_s(\tau),^{3} \tag{4.4}$$

$$\tau P'_r(\tau)=rP_r(\tau)+P'_{r-1}(\tau)=rP_r(\tau)+\sum_{s=0}^{r-1}(2s+1)\frac{1+(-1)^{r+s}}{2}P_s(\tau)^{4} \tag{4.5}$$

and in view of $\frac{a_{,\alpha}}{a}=(\ln a)_{,\alpha}=-\frac{h_{,\alpha}}{h},\quad \frac{a_{,\alpha}}{a}b=\widetilde{h}a_{,\alpha},\quad b_{,\alpha}=(\widetilde{h}a)_{,\alpha}$, it is easily seen that

$$\begin{aligned}P'_r(ax_3-b)(a_{,\alpha}x_3-b_{,\alpha})^{5}&=\frac{a_{,\alpha}}{a}(ax_3-b)P'_r(ax_3-b)+(\frac{a_{,\alpha}}{a}b-b_{,\alpha})P'_r(ax_3-b)\\&=-h_{,\alpha}h^{-1}(ax_3-b)P'_r(ax_3-b)-\widetilde{h}_{,\alpha}h^{-1}P'_r(ax_3-b)=-\overset{r}{a}_{\alpha\underline{r}}P_r(ax_3-b)\end{aligned}$$

$$-\sum_{s=0}^{r-1}\overset{r}{a}_{\alpha s}P_s(ax_3-b)=-\sum_{s=0}^{r}\overset{r}{a}_{\alpha s}P_s(ax_3-b)^{6} \tag{4.6}$$

since, taking into account (4.5) and (4.4), it follows that

$$\begin{aligned}&-h_{,\alpha}h^{-1}(ax_3-b)P'_r(ax_3-b)-\widetilde{h}_{,\alpha}h^{-1}P'_r(ax_3-b)=-rh_{,\alpha}h^{-1}P_r(ax_3-b)\\&-h_{,\alpha}h^{-1}\sum_{s=0}^{r-1}(2s+1)\frac{1+(-1)^{r+s}}{2}P_s(ax_3-b)-\widetilde{h}_{,\alpha}h^{-1}\sum_{s=0}^{r}(2s+1)\frac{1-(-1)^{r+s}}{2}P_s(ax_3-b)\\&=-r\frac{h_{,\alpha}}{h}P_r(ax_3-b)-\sum_{s=0}^{r-1}(2s+1)\left[\frac{h_{,\alpha}+(-1)^{r+s}h_{,\alpha}}{2h}+\frac{\widetilde{h}_{,\alpha}-(-1)^{r+s}\widetilde{h}_{,\alpha}}{2h}\right]P_s(ax_3-b)\\&=-r\frac{h_{,\alpha}}{h}P_r(ax_3-b)-\sum_{s=0}^{r-1}\frac{(2s+1)}{2h}\Bigg(\frac{\overset{(+)}{h}_{,\alpha}-\overset{(-)}{h}_{,\alpha}+\overset{(+)}{h}_{,\alpha}(-1)^{r+s}-\overset{(-)}{h}_{,\alpha}(-1)^{r+s}}{2}\\&\qquad+\frac{\overset{(+)}{h}_{,\alpha}+\overset{(-)}{h}_{,\alpha}-\overset{(+)}{h}_{,\alpha}(-1)^{r+s}-\overset{(-)}{h}_{,\alpha}(-1)^{r+s}}{2}\Bigg)P_s(ax_3-b)\\&=-r\frac{h_{,\alpha}}{h}P_r(ax_3-b)-\sum_{s=0}^{r-1}(2s+1)\frac{\overset{(+)}{h}_{,\alpha}-(-1)^{r+s}\overset{(-)}{h}_{,\alpha}}{2h}P_s(ax_3-b)=-\sum_{s=0}^{r}\overset{s}{a}_{\alpha s}P_s(ax_3-b),\end{aligned}$$

because of

$$h_{,\alpha}=\frac{\overset{(+)}{h}_{,\alpha}-\overset{(-)}{h}_{,\alpha}}{2},\quad \widetilde{h}_{,\alpha}=\frac{\overset{(+)}{h}_{,\alpha}+\overset{(-)}{h}_{,\alpha}}{2}.$$

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[3] on the top of the symbol $\sum$, both $r-1$ and r are true, since the last term equals zero.

[4] on the top of the symbol $\sum$, both $r-2$ and $r-1$ are true, since the last term equals zero.

[5] Clearly,

$$a_{,\alpha}x_3-b_{,\alpha}=\frac{a_{,\alpha}}{a}ax_3-b_{,\alpha}=\frac{a_{,\alpha}}{a}ax_3-\frac{a_{,\alpha}}{a}b+\frac{a_{,\alpha}}{a}b-b_{,\alpha}=\frac{a_{,\alpha}}{a}(ax_3-b)-\widetilde{h}_{,\alpha}\frac{1}{h},$$

because of

$$\frac{a_{,\alpha}}{a}b-b_{,\alpha}=a_{,\alpha}\widetilde{h}-(\widetilde{h}a)_{,\alpha}=\widetilde{h}a_{,\alpha}-\widetilde{h}a_{,\alpha}-\widetilde{h}_{,\alpha}a=-\widetilde{h}_{,\alpha}a.$$

[6]

$$\overset{r}{a}_{\alpha s}:=(2s+1)\frac{\overset{(+)}{h}_{,\alpha}-(-1)^{r+s}\overset{(-)}{h}_{,\alpha}}{2h},\quad s\neq r,\quad \overset{r}{a}_{\alpha r}=r\frac{h_{,\alpha}}{h}.$$

4.2. **N-th order approximation.** In order to construct the governing equations of the N-th approximation of hierarchical models, using Vekua's Dimension Reduction Method, after multiplying (3.1)–(3.6) by $P_r(ax_3-b)$ and then integrating within the limits $\overset{(-)}{h}$, $\overset{(+)}{h}$ with respect to the thickness variable x_3, we rewrite them in terms of mathematical moments for $r=0,1,\dots$. Substituting the obtained kinematic relations into the obtained constitutive relations and the result into the obtained motion (equilibrium) equations, we get an infinite system with respect to the mathematical moments. Further, assuming all moments of greater than N-th order are equal to zero and retaining the first $3N+3$ equations of the infinite system with respect to the mathematical moments, we will have the governing system of the N-th approximation of the hierarchical models, meanwhile we use formulas (4.2) and (4.3) (with (4.6) and (4.4), respectively).

By the corresponding calculations, we take the prescribed values as the values of tractions and couple stress vectors on the face surfaces, while for displacements and microrotations we use their approximate values calculated from their Fourier–Legendre expansions (namely, their $N+1$ partial sum) on the face surfaces corresponding to the N-th approximation.

4.3. $N=0$ **approximation.** The governing equations of the $N=0$ approximation can be deduced from the governing equations of the N-th approximation by taking $N=0$.

On the other hand, the governing equations of the $N=0$ approximation of hierarchical models can be directly obtained by integrating the $3D$ governing equations (3.1)–(3.6) within the limits $\overset{(-)}{h}$, $\overset{(+)}{h}$ with respect to the thickness variable x_3.

From (3.1) and (3.2) we get

$$X_{\beta i0,\beta}+X_i^0=\rho\dot{v}_{i0},\qquad i=\alpha,3,\ \ \alpha=1,2, \tag{4.7}$$

and

$$\chi_{\beta i0,\beta}+\in_{ijk}X_{jk0}+\chi_i^0=\mathcal{I}\dot{\underline{\underline{\omega}}}_{i0},\ \ i=\alpha,3,\ \ \alpha=1,2, \tag{4.8}$$

respectively, in ω, where

$$X_i^0:=Q_{\overset{(+)}{\nu}i}\cdot\sqrt{\Big(\overset{(+)}{h}_{,1}\Big)^2+\Big(\overset{(+)}{h}_{,2}\Big)^2+1}+Q_{\overset{(-)}{\nu}i}\cdot\sqrt{\Big(\overset{(-)}{h}_{,1}\Big)^2+\Big(\overset{(-)}{h}_{,2}\Big)^2+1}+\Phi_{i0},$$

$$\chi_j^0:=\Theta_{\overset{(+)}{\nu}i}\cdot\sqrt{\Big(\overset{(+)}{h}_{,1}\Big)^2+\Big(\overset{(+)}{h}_{,2}\Big)^2+1}+\Theta_{\overset{(-)}{\nu}i}\cdot\sqrt{\Big(\overset{(-)}{h}_{,1}\Big)^2+\Big(\overset{(-)}{h}_{,2}\Big)^2+1}+\Psi_{i0},$$

$Q_{\overset{(\pm)}{\nu}}$ and $\Theta_{\overset{(\pm)}{\nu}}$ are the tractions and couple stress vectors prescribed on the face surfaces (in what follows, the superscripts $(+)$ and $(-)$ mean the values on upper and lower face surfaces, correspondingly).

From (3.3) and (3.4), taking into account (4.2) and (4.3), for $r=0$, we get

$$v_{\beta i0}=v_{i0,\beta}-\overset{(+)}{v}_i\overset{(+)}{h}_{,\beta}+\overset{(-)}{v}_i\overset{(-)}{h}_{,\beta}-\in_{k\beta i}\underline{\underline{\omega}}_{k0},\quad \beta=1,2;\ \ i=1,2,3, \tag{4.9}$$

$$v_{3i0}=\overset{(+)}{v}_i-\overset{(-)}{v}_i-\in_{k3i}\underline{\underline{\omega}}_{k0}=\overset{(+)}{v}_i-\overset{(-)}{v}_i-\in_{\gamma 3i}\underline{\underline{\omega}}_{\gamma 0},\qquad i=1,2,3, \tag{4.10}$$

and

$$\underline{\underline{\omega}}_{\beta i0}=\underline{\underline{\omega}}_{i0,\beta}-\overset{(+)}{\underline{\underline{\omega}}}_i\overset{(+)}{h}_{,\beta}+\overset{(-)}{\underline{\underline{\omega}}}_i\overset{(-)}{h}_{,\beta},\qquad \beta=1,2,\quad i=1,2,3; \tag{4.11}$$

$$\underline{\underline{\omega}}_{3i0}=\overset{(+)}{\underline{\underline{\omega}}}_i-\overset{(-)}{\underline{\underline{\omega}}}_i,\qquad i=1,2,3, \tag{4.12}$$

respectively. Under the indices 0 we mean integrated values of the corresponding quantities which are called zero order mathematical moments (see (4.1)).

In the $N=0$ approximation, we assume approximately that

$$v_i(x_1,x_2,x_3,t)=\frac{1}{2}\tilde{v}_{i0}(x_1,x_2,t)=\frac{1}{2}\frac{v_{i0}(x_1,x_2,t)}{h(x_1,x_2)}\,, \tag{4.13}$$

$$\underline{\underline{\omega}}_i(x_1,x_2,x_3,t)=\frac{1}{2}\eta_{i0}(x_1,x_2,t)=\frac{1}{2}\frac{\underline{\underline{\omega}}_{i0}(x_1,x_2,t)}{h(x_1,x_2)}\,. \tag{4.14}$$

Evidently, from (4.9), by virtue of (4.13), (4.14),

$$v_{\beta i0}=(h\tilde{v}_{i0}),_\beta-\tilde{v}_{i0}h,_\beta-\in_{k\beta i}h\eta_{k0}=h\tilde{v}_{i0,\beta}-\in_{k\beta i}h\eta_{k0}, \tag{4.15}$$

from (4.10), in view of (4.13), (4.14),

$$v_{3i0}=-\in_{\gamma 3i}h\eta_{\gamma 0}, \tag{4.16}$$

whence

$$v_{330}=0,\qquad v_{320}=\in_{123}\underline{\underline{\omega}}_{10}=\eta_{10},\qquad v_{310}=-\in_{231}\underline{\underline{\omega}}_{20}=-\eta_{20}. \tag{4.17}$$

From (4.11), by virtue of (4.14),

$$\underline{\underline{\omega}}_{\beta i0}=(h\eta_{i0}),_\beta-\eta_{i0}h,_\beta=h\eta_{i0,\beta}\,,\qquad \beta=1,2,\quad i=1,2,3. \tag{4.18}$$

From (4.12), in view of (4.14),

$$\underline{\underline{\omega}}_{3i0}=0,\qquad i=1,2,3. \tag{4.19}$$

From (3.5) and (3.6), we obtain

$$X_{ij0}=-p_0\delta_{ij}+(\mu+\tilde{\alpha})v_{ij0}+(\mu-\tilde{\alpha})v_{ji0},\qquad i,j=1,2,3, \tag{4.20}$$

and

$$\chi_{ij0}=\varepsilon\delta_{ij}\underline{\underline{\omega}}_{kk0}+(\nu+\tilde{\beta})\underline{\underline{\omega}}_{ij0}+(\nu-\tilde{\beta})\underline{\underline{\omega}}_{ji0},\qquad i,j=1,2,3, \tag{4.21}$$

respectively.

From (4.17) and (4.15), it follows that

$$v_{kk0}=v_{\gamma\gamma 0}=h\tilde{v}_{\gamma 0,\gamma}. \tag{4.22}$$

From (4.19) and (4.18), we obtain

$$\underline{\underline{\omega}}_{kk0}=\underline{\underline{\omega}}_{\gamma\gamma 0}=h\eta_{\gamma 0,\gamma}. \tag{4.23}$$

Since

$$\in_{k\alpha\beta}\eta_{k0}=\in_{3\alpha\beta}\eta_{30}+\in_{2\alpha\beta}\eta_{20}+\in_{1\alpha\beta}\eta_{10}, \tag{4.24}$$

we have

$$\in_{312}\eta_{30}=\eta_{30},\quad \in_{321}\eta_{30}=-\eta_{30}.$$

From (4.20), taking into account (4.22), (4.15), (4.24), we find that

$$\begin{aligned}X_{\beta\alpha 0}&=-p_0\delta_{\beta\alpha}+(\mu+\tilde{\alpha})h(\tilde{v}_{\alpha 0,\beta}-\in_{k\beta\alpha}\eta_{k0})+(\mu-\tilde{\alpha})h(\tilde{v}_{\beta 0,\alpha}-\in_{k\alpha\beta}\eta_{k0})\\&=-p_0\delta_{\beta\alpha}+(\mu+\tilde{\alpha})h(\tilde{v}_{\alpha 0,\beta}-\in_{3\beta\alpha}\eta_{30})+(\mu-\tilde{\alpha})h(\tilde{v}_{\beta 0,\alpha}-\in_{3\alpha\beta}\eta_{30})\\&=-p_0\delta_{\beta\alpha}+(\mu+\tilde{\alpha})h\tilde{v}_{\alpha 0,\beta}+(\mu-\tilde{\alpha})h\tilde{v}_{\beta 0,\alpha}+2\tilde{\alpha}h\in_{3\alpha\beta}\eta_{30},\ \alpha,\beta=1,2.\end{aligned} \tag{4.25}$$

From (4.20), taking into account (4.15), (4.16), we find that

$$\begin{aligned}X_{3\beta 0}&=(\mu+\tilde{\alpha})h(-\in_{\gamma 3\beta}\eta_{\gamma 0})+(\mu-\tilde{\alpha})h(v_{30,\beta}-\in_{k\beta 3}\eta_{k0})\\&=-(\mu+\tilde{\alpha})h\in_{\gamma 3\beta}\eta_{\gamma 0}+(\mu-\tilde{\alpha})h(v_{30,\beta}-\in_{\gamma\beta 3}\eta_{\gamma 0})\\&=(\mu-\tilde{\alpha})hv_{30,\beta}+2\tilde{\alpha}h\in_{\gamma\beta 3}\eta_{\gamma 0}.\end{aligned} \tag{4.26}$$

From (4.20), by virtue of (4.22), (4.16), we get

$$X_{330}=-p_0.$$

From (4.20), in view of (4.16), (4.15), we obtain

$$\begin{aligned}X_{\beta 30}&=(\mu+\tilde{\alpha})h(\tilde{v}_{30,\beta}-\in_{\gamma\beta 3}\eta_{\gamma 0})+(\mu-\tilde{\alpha})h(-\in_{\gamma 3\beta}\eta_{\gamma 0})\\&=(\mu+\tilde{\alpha})h\tilde{v}_{30,\beta}-2\tilde{\alpha}h\in_{\gamma\beta 3}\eta_{\gamma 0}.\end{aligned} \tag{4.27}$$

From (4.21), taking into account (4.23), (4.18), we have

$$\chi_{\beta\alpha 0} = \varepsilon h\eta_{\gamma 0,\gamma}\delta_{\beta\alpha} + (\nu+\tilde{\beta})h\eta_{\alpha 0,\beta} + (\nu-\tilde{\beta})h\eta_{\beta 0,\alpha}, \quad \alpha,\beta = 1,2. \tag{4.28}$$

From (4.21), by virtue of (4.7), (4.18), we get

$$\chi_{3\beta 0} = (\nu-\tilde{\beta})h\eta_{30,\beta}.$$

From (4.21), in view of (4.23), (4.19), we obtain

$$\chi_{330} = \varepsilon h\eta_{\gamma 0,\gamma}.$$

From (4.21), according to (4.18), (4.19), we have

$$\chi_{\beta 30} = (\nu+\tilde{\beta})h\eta_{30,\beta}. \tag{4.29}$$

Substituting (4.25) into (4.7), we arrive at the equation

$$\begin{aligned}&-p_{0,\beta}\delta_{\beta\alpha} + (\mu+\tilde{\alpha})(h\tilde{v}_{\alpha 0,\beta})_{,\beta} + (\mu-\tilde{\alpha})(h\tilde{v}_{\beta 0,\alpha})_{,\beta}\\ &+2\tilde{\alpha} \in_{3\alpha\beta} (h\eta_{30})_{,\beta} + X^0_\alpha = \rho h\dot{\tilde{v}}_{\alpha 0}, \quad \alpha = 1,2.\end{aligned}$$

Hence,

$$\begin{aligned}&-p_{0,\alpha} + (\mu+\tilde{\alpha})(h\tilde{v}_{\alpha 0,\beta})_{,\beta} + (\mu-\tilde{\alpha})(h\tilde{v}_{\beta 0,\alpha})_{,\beta}\\ &+2\tilde{\alpha} \in_{\alpha\beta 3} (h\eta_{30})_{,\beta} + X^0_\alpha = \rho h\dot{\tilde{v}}_{\alpha 0}, \quad \alpha = 1,2.\end{aligned} \tag{4.30}$$

Substituting (4.27) into (4.7), we obtain

$$(\mu+\tilde{\alpha})(h\tilde{v}_{30,\beta})_{,\beta} - 2\tilde{\alpha} \in_{\gamma\beta 3} (h\eta_{\gamma 0})_{,\beta} + X^0_3 = \rho h\dot{\tilde{v}}_{30}.$$

Therefore

$$(\mu+\tilde{\alpha})(h\tilde{v}_{30,\beta})_{,\beta} - 2\tilde{\alpha}[(h\eta_{10})_{,2} - (h\eta_{20})_{,1}] + X^0_3 = \rho h\dot{\tilde{v}}_{30}. \tag{4.31}$$

Substituting (4.28) into (4.8), for $i = \alpha = 1,2$, we get

$$\begin{aligned}&\varepsilon(h\eta_{\gamma 0,\gamma})_{,\alpha} + (\nu+\tilde{\beta})(h\eta_{\alpha 0,\beta})_{,\beta} + (\nu-\tilde{\beta})(h\eta_{\beta 0,\alpha})_{,\beta}\\ &+ \in_{\alpha jk} X_{jk0} + \chi^0_\alpha = \mathcal{I}h\dot{\eta}_{\alpha 0}, \quad \alpha = 1,2.\end{aligned} \tag{4.32}$$

Substituting (4.29) into (4.8), for $i = 3$, we obtain

$$(\nu+\tilde{\beta})(h\eta_{30,\beta})_{,\beta} + \in_{3jk} X_{jk0} + \chi^0_3 = \mathcal{I}h\dot{\eta}_{30}. \tag{4.33}$$

Since by virtue of (4.25)–(4.27),

$$\begin{aligned}\in_{1jk} X_{jk0} &= \in_{123} X_{230} + \in_{132} X_{320} = X_{230} - X_{320} = 2\tilde{\alpha}h\tilde{v}_{30,2} - 4\tilde{\alpha}h\eta_{10},\\ \in_{2jk} X_{jk0} &= \in_{213} X_{130} + \in_{231} X_{310} = -X_{130} + X_{310} = -2\tilde{\alpha}h\tilde{v}_{30,1} - 4\tilde{\alpha}h\eta_{20},\\ \in_{3jk} X_{jk0} &= \in_{312} X_{120} + \in_{321} X_{210} = X_{120} - X_{210}\\ &= (\mu+\tilde{\alpha})h\tilde{v}_{20,1} + (\mu-\tilde{\alpha})h\tilde{v}_{10,2} + 2\tilde{\alpha}h \in_{321} \eta_{30}\\ &- (\mu+\tilde{\alpha})h\tilde{v}_{10,2} - (\mu-\tilde{\alpha})h\tilde{v}_{20,1} - 2\tilde{\alpha}h \in_{312} \eta_{30}\\ &= 2\tilde{\alpha}h(\tilde{v}_{20,1} - \tilde{v}_{10,2}) - 4\tilde{\alpha}h\eta_{30},\end{aligned}$$

from (4.32), (4.33), we get

$$\varepsilon(h\eta_{\gamma 0,\gamma})_{,1} + (\nu+\tilde{\beta})(h\eta_{10,\beta})_{,\beta} + (\nu-\tilde{\beta})(h\eta_{\beta 0,1})_{,\beta} + 2\tilde{\alpha}h\tilde{v}_{30,2} - 4\tilde{\alpha}h\eta_{10} + \chi^0_1 = \mathcal{I}h\dot{\eta}_{10}, \tag{4.34}$$

$$\varepsilon(h\eta_{\gamma 0,\gamma})_{,2} + (\nu+\tilde{\beta})(h\eta_{20,\beta})_{,\beta} + (\nu-\tilde{\beta})(h\eta_{\beta 0,2})_{,\beta} - 2\tilde{\alpha}h\tilde{v}_{30,1} - 4\tilde{\alpha}h\eta_{20} + \chi^0_2 = \mathcal{I}h\dot{\eta}_{20}, \tag{4.35}$$

$$(\nu+\tilde{\beta})(h\eta_{30,\beta})_{,\beta} + 2\tilde{\alpha}h(\tilde{v}_{20,1} - \tilde{v}_{10,2}) - 4\tilde{\alpha}h\eta_{30} + \chi^0_3 = \mathcal{I}h\dot{\eta}_{30}. \tag{4.36}$$

5. Analysis of the Constructed System

Dirichlet Problem. Find a solution $u_{30} \in C^2(\omega)\cap C(\overline{\omega})$ of equation (5.1) in ω, satisfying the BC,

$$u(x_1,x_2)=\varphi(x_1,x_2),\ \ (x_1,x_2)\in\partial\omega,$$

where φ_i is the given continuous on $\partial\omega$ function.

Keldysh Problem. Find a bounded solution $u\in C^2(\omega)\cap C(\overline{\omega}\backslash\omega_0)$ of equation (5.1) in ω, satisfying the BC,

$$u(x_1,x_2)=\varphi(x_1,x_2),\ \ (x_1,x_2)\in\partial\omega\backslash\omega_0,\ \ i=\overline{1,6},$$

where φ is the given continuous on $(\partial\omega)\setminus\omega_0$ function.

The following theorem is true [2] (compare with [10], where $m_1=0$).

Theorem. *If the coefficients a_α, $\alpha=1,2$, and c of the equation*

$$x_2^{m_\alpha}u_{,\alpha\alpha}+a_\alpha(x_1,x_2)u_{,\alpha}+c(x_1,x_2)u=0,\ \ c\le 0,\ \ m_\alpha=\text{const}\ge 0,\ \ \alpha=1,2, \tag{5.1}$$

are analytic in $\overline{\omega}$, bounded both by a sufficiently smooth arc $(\partial\omega\setminus\omega_0)$, lying in the half-plane $x_2\ge 0$, and by a segment ω_0 of the x_1-axis, then

(i) *if either $m_2<1$, or $m_2\ge 1$, $a_2(x_1,x_2)<x_2^{m_2-1}$ in $\bar I_\delta$ for some $\delta=\text{const}>0$, where*

$$I_\delta:=\{(x_1,x_2)\in\omega\,:\,0<x_2<\delta\},$$

the Dirichlet problem is well-posed.

(ii) *If $m_2\ge 1$, $a_2(x_1,x_2)\ge x_2^{m_2-1}$ in I_δ and $a_1(x_1,x_2)=O(x_2^{m_1})$, $x_2\to 0_+$ (O is the Landau symbol), the Keldysh problem is well-posed.*

Let us consider a prismatic shell-like container with a cusped edge $\omega_0\subseteq\partial\omega$, where the thickness $2h(x_1,x_2)$ vanishes:

$$\omega_0:=\{(x_1,x_2)\in\partial\omega:2h(x_1,x_2)=0\}.$$

Evidently, ω_0 is a closed set.

Let $v_{20,1}=v_{10,2}$, then equation (4.36) will be separated from the system (4.30), (4.31), (4.34)–(4.36).

Since, if on a part of the boundary of the container projection ω $h=0$ equation (4.36) degenerates, this may lead to peculiarities in posing boundary conditions for the weighted mathematical moment of microrotation velocity η_{30} for the well-posedness of the problems under consideration. In order to study this point, for the sake of simplicity, we assume

$$h(x_2)=h_0x_2^{\varkappa},\quad x_2\in[0,L],\quad h_0,\varkappa=\text{const}>0,\ \ L=\text{const},$$

then from (4.36), for η_{10}, we obtain

$$(\nu+\tilde\beta)h_0(x_2^{\varkappa}\eta_{30,11}+(x_2^{\varkappa}\eta_{30,2})_2)-4\alpha h_0x_2^{\varkappa}\eta_{30}=0,$$

i.e.,

$$x_2^{\varkappa}\eta_{30,11}+x_2^{\varkappa}\eta_{30,22}+\varkappa x_2^{\varkappa-1}\eta_{30,2}-\frac{4\alpha}{\nu+\tilde\beta}=0,$$

provided the applicates of the traction and couple stress vectors prescribed on face surfaces, and volume forces are equal to zero.

So, in terms of the above Theorem, $m_1=\varkappa$, $m_2=\varkappa$, $a_2(x_1,x_2)=\varkappa x_2^{\varkappa-1}\ge x_2^{\varkappa-1}$ for $\varkappa\ge 1$. Therefore, the Dirichlet problem for $\varkappa<1$ and the Keldysh problem for $\varkappa\ge 1$ are well-posed. In other words, for $\varkappa<1$, the microrotation velocity η_{30} should be prescribed at the cusped edge, while for $\varkappa\ge 1$ $\eta_{30,}$ it cannot be prescribed at the cusped edge, and this boundary condition should be replaced by its boundedness near the cusped edge.

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