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ON THE DIRICHLET PROBLEM IN POLY-HARDY CLASS

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Abstract. In this paper, we study the Dirichlet problem in a poly-Hardy class. Firstly, we state the Dirichlet problem for inhomogeneous polyanalytic equation in the unit disc $\mathbb{D}$. Then we give the properties of the functions in a poly-Hardy class. Lastly, we obtain the solutions of the Dirichlet problem for polyanalytic equations in a poly-Hardy class.

1. Introduction

As we know, a function on an open set $G \subset \mathbb{C}$ is a polyanalytic function of order n on G if it satisfies the generalized Cauchy–Riemann equation

$$\partial_{\bar{z}}^n f(z) = 0, \quad z \in G,$$

where $\partial_{\bar{z}}$ is the Cauchy–Riemann operator. We use the notation $H_n(G)$ for the functions in this particular class [19]. $H_1(G)$ is the set of analytic functions on G. We may consult with Balk [5] for the properties of polyanalytic functions. The relevant boundary value problems are investigated extensively in [2, 11–13, 21, 24, 25].

The BVPs were initiated by Riemann. Later, the problems were modified by Hilbert. Let us note first that $\mathbb{D} = \{z : |z| < 1\} \subset \mathbb{C}$ is the unit circle and $\mathbb{T} = \{t : |t| = 1\}$ is its boundary, oriented counter-clockwise, $V \in H_n(\mathbb{D})$ such that $\partial_{\bar{z}}^k V$, $k = 0, 1, \dots, n-1$ are continuous on the closed unit disc $\mathbb{D} \cup \mathbb{T}$. So, a Hilbert-type BVP states to determine a function $V \in H_n(\mathbb{D})$ satisfying the Hilbert-type boundary conditions

$$\operatorname{Re}\Big\{[a_j(t) + ib_j(t)] \cdot \left[\partial_{\bar{z}}^k V\right]^+(t)\Big\} = c_j(t), \quad t \in \mathbb{T}, \quad j = 0, 1, \dots, n-1$$

with $a_j, b_j, c_j \in H(\mathbb{T}; \mathbb{R})$, Hölder continuous, for $j = 0, 1, \dots, n-1$.

For particular choices of a_j, b_j, we reach to the BVPs known as the Dirichlet, Neumann, Robin and Schwarz problems. These problems are also investigated in different kinds of domains in $\mathbb{C}$ [1, 3, 4, 6–8, 10, 14, 15, 17, 18, 20, 22, 23, 27].

Our aim is to find the solution of Dirichlet problem in the poly-Hardy class. This paper is organized as follows: In Section 2, we state the Dirichlet problem in the unit disc. In Section 3, we need the properties of the functions in the poly-Hardy class. In the last part, we obtain the solution of the Dirichlet problem in the poly-Hardy class.

2. Dirichlet Problem in the Unit Disc

Integral representation formulas play an important role for the boundary value problems in $\mathbb{C}$. To obtain such integral representation formulas, we state the Gauss Theorem in a regular domain D, i.e., bounded domain with a smooth boundary ∂D.

Theorem 2.1 ([7]). *Let $w \in C^1(D; \mathbb{C}) \cap C(\overline{D}; \mathbb{C})$ in a regular domain of the complex plane $\mathbb{C}$, then*

$$\int\limits_D w_{\overline{z}}(z)\,dxdy = \frac{1}{2i} \int\limits_{\partial D} w(z)\,dz$$

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and

$$\int\limits_D w_z(z)dxdy = -\frac{1}{2i}\int\limits_{\partial D} w(z)\overline{dz}.$$

From the Gauss theorem, we can prove the following.

Theorem 2.2 ([7]). *Let $D \subset \mathbb{C}$ be a regular domain and $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$. Then*

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int\limits_D w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}, \tag{2.1}$$

and

$$w(z) = -\frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{\overline{d\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\int\limits_D w_{\zeta}(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}} \tag{2.2}$$

hold in which $\zeta = \xi + i\eta$.

These formulas (2.1) and (2.2) are called the Cauchy–Pompeiu formulas. We may observe that if $w_{\overline{\zeta}}$ is given in D and the values of w along the boundary are known, we can identify a unique function $w(z)$. This representation is an example of the solution of the Dirichlet problem.

Now, we can state the Dirichlet problem for the inhomogeneous polyanalytic equation in the unit disc [8].

Theorem 2.3. *The Dirichlet problem for the inhomogeneous polyanalytic equation in the unit disc*

$$\partial_{\bar{z}}^n w = f \;\text{ in }\; \mathbb{D},\;\; \partial_{\bar{z}}^{\nu} w = \gamma_\nu \;\text{ on }\; \partial\mathbb{D},\;\; 0 \le \nu \le n-1$$

is uniquely solvable for $f \in L_1(\mathbb{D};\mathbb{C})$, $\gamma_\nu \in C(\partial\mathbb{D};\mathbb{C})$ if and only if

$$\sum_{\lambda=\nu}^{n-1}\frac{\bar{z}}{2\pi i}\int\limits_{|\zeta|=1}(-1)^{\lambda-\nu}\frac{\gamma_\lambda(\zeta)}{1-\bar{z}\zeta}\frac{(\overline{\zeta - z})^{\lambda-\nu}}{(\lambda-\nu)!}\,d\zeta$$
$$+\frac{(-1)^{n-\nu}\bar{z}}{\pi}\int\limits_{|\zeta|<1}\frac{f(\zeta)}{1-\bar{z}\zeta}\frac{(\overline{\zeta - z})^{n-1-\nu}}{(n-1-\nu)!}\,d\xi d\eta = 0.$$

The solution then is

$$w(z) = \sum_{\nu=0}^{n-1}\frac{(-1)^\nu}{2\pi i}\int\limits_{|\zeta|=1}\frac{\gamma_\nu(\zeta)}{\nu!}\frac{(\overline{\zeta - z})^{\nu}}{\zeta - z}d\zeta + \frac{(-1)^n}{\pi}\int\limits_{|\zeta|<1}\frac{f(\zeta)}{(n-1)!}\frac{(\overline{\zeta - z})^{n-1}}{\zeta - z}\,d\xi d\eta.$$

3. Functions in Poly–Hardy Class and their Properties over $\mathbb{D}$

To investigate the Dirichlet problem in the poly-Hardy class, we need to know the properties and boundary behaviour of the functions in this class. Firstly we state some definitions barrowed from [28].

Let f be a function on $\mathbb{D}$, and define

$$f_r(\theta) = f\left(re^{i\theta}\right), \quad 0 \le r < 1. \tag{3.1}$$

We may define the Hardy class on $\mathbb{D}$ as

$$H^q(\mathbb{D}) = \left\{f \in H_1(\mathbb{D}) : \|f\|_{1,q} < \infty\right\},$$

where

$$\|f\|_{1,q} = \sup\left\{\|f_r\|_q : 0 \le r < 1\right\} \tag{3.2}$$

with

$$\|f_r\|_q = \left\{ \frac{1}{2\pi} \int_{\mathbb{T}} |f_r(\theta)|^q \, \mathrm{d}\theta \right\}^{\frac{1}{q}}, \quad 0 < q < \infty,$$
$$\|f_r\|_\infty = \sup \{ f_r(\theta) : \theta \in [0, 2\pi] \}. \tag{3.3}$$

It is known that the class $H^q(\mathbb{D})$ is a Banach space under the norm (3.2) if $q > 1$ (see [26]).

If $f \in H_n(\mathbb{D})$, $n > 1$, we define

$$f^j(z) = \partial_{\bar{z}}^j f(z), \ \ z \in \mathbb{D} \ \text{ for } \ j = 0, 1, 2, \ldots, n-1,$$

which also provide us

$$f(z) = f_0(z) + \bar{z} f_1(z) + \cdots + \bar{z}^{n-1} f_{n-1}(z), \ \ z \in \mathbb{D},$$

where $f_j \in H_1(\mathbb{D})$. If $q > 0$ and $n > 1$, the subset of the polyanalytic functions

$$\left\{ f \in H_n(\mathbb{D}) : \left\| f^j \right\|_{1,q} < \infty, \ \ j = 0, 1, \ldots, n-1 \right\}$$

is called the poly-Hardy class of order n on $\mathbb{D}$ and denoted by $H_n^q(\mathbb{D})$. Wang [28] has proved in Theorem 2.1 that

$$H_n^q(\mathbb{D}) = H_1^q(\mathbb{D}) \oplus \bar{z} H_1^q(\mathbb{D}) \oplus \cdots \oplus z^{n-1} H_1^q(\mathbb{D}),$$

where

$$\bar{z}^j H_1^q(\mathbb{D}) = \left\{ \bar{z}^j f(z) \, : \, f \in H_1^q(\mathbb{D}) \right\}$$

for $j = 0, 1, \ldots, n-1$.

To prove this, we need to show that

$$H_n^q(\mathbb{D}) \subset H_1^q(\mathbb{D}) + \bar{z} H_1^q(\mathbb{D}) + \cdots + \bar{z}^{n-1} H_1^q(\mathbb{D})$$

and the converse of this relation. We use a simple computation to observe that

$$f^{\#}(z) = A(\bar{z}) f_{\#}(z), \quad z \in \mathbb{D},$$

with

$$f^{\#}(z) = \begin{pmatrix} f^0(z) \\ f^1(z) \\ \vdots \\ f^{n-1}(z) \end{pmatrix}, \qquad f_{\#}(z) = \begin{pmatrix} f_0(z) \\ f_1(z) \\ \vdots \\ f_{n-1}(z) \end{pmatrix}$$

and

$$A(z) = \begin{pmatrix} 1 & z & z^2 & \cdots & z^{n-1} \\ 0 & 1! & 2z & \cdots & (n-1)z^{n-2} \\ 0 & 0 & 2! & \cdots & (n-1)(n-2)z^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (n-1)! \end{pmatrix}$$

which leads to

$$f_\sharp(z) = A^{-1}(\bar{z}) f^\sharp(z), \ \ z \in \mathbb{D},$$

where $A^{-1}(z)$ is the inverse of $A(z)$ given in [29].

Conversely, if

$$0 = f_0(z) + \bar{z} f_1(z) + \cdots + \bar{z}^{n-1} f_{n-1}(z)$$

with $f_j \in H_1^q(\mathbb{D}) \Rightarrow f_j(z) \equiv 0, z \in \mathbb{D}$ for $j = 0, 1, \ldots, n-1$.

Remark 3.1. If $f \in H_n^q(\mathbb{D})$, then the decomposition of polyanalytic functions is written as

$$f(z) = f_0(z) + (\bar{z} + z) f_1(z) + \cdots + (\bar{z} + z)^{n-1} f_{n-1}(z), \ \ z \in \mathbb{D},$$

with $f_j(z) \in H_1^q(\mathbb{D})$ for $j = 0, 1, \ldots, n-1$.

The last theorem in this section is about the boundary behaviour of the functions.

Theorem 3.1. *If $f \in H_n^q(\mathbb{D})$, $q > 0$, then f has the nontangential limits $f^+(t)$ almost everywhere on $\mathbb{T}$, and*

$$\lim_{r \to 1} \left\| f^+ - f_r \right\|_q = 0,$$

where $f_r, \| \cdot \|_q$ are given in (3.1) *and* (3.3)*, respectively.*

For the proof see Theorem 17.12 in [26].

4. Dirichlet Problem in Poly–Hardy Class

In [9], Begehr, Du and Wang studied a Dirichlet problem for polyharmonic functions in the unit disc by the reflection method. In [16], Begehr and Wang solved this problem by a new approach different from the reflection method. This approach consists of two steps. Firstly, we transform the problem into the equivalent Hilbert boundary value problems and then into several equivalent Dirichlet boundary value problems. The weak decomposition theorem for polyharmonic functions was used in this transformation.

We consider the classical Dirichlet problem for a polyanalytic equation in the poly-Hardy class as:

'*Find a function $V \in H_n(\mathbb{D})$ satisfying the homogeneous Dirichlet type boundary condition*

$$\left[\partial_{\bar{z}}^j V\right]^+ (t) = 0, \ \ t \in \mathbb{T} \ \text{ for } \ j = 0, 1, \ldots, n-1.\text{'} \tag{4.1}$$

For the solution of this problem, we obtain the following results.

Lemma 4.1. *The unique solution of the homogeneous problem* (4.1) *for $n = 1$ is the zero solution.*

Theorem 4.1. *The unique solution of the homogeneous problem* (4.1) *is $V(z) \equiv 0$.*

Proof. By Lemma 1, the case $n = 1$ is obvious. Suppose that the unique solution of the homogeneous problem (4.1) is $V(z) \equiv 0$. The boundary conditions of the homogeneous problem (4.1) for $n = k+1$ can be rewritten as

$$\left[\partial_{\bar{z}}^j \left(\partial_{\bar{z}} V\right)\right]^+ (t) = 0, \ \ t \in \mathbb{T},$$

for $j = 0, 1, \ldots, k-1$, and

$$V^+(t) = 0, \ \ t \in \mathbb{T}. \tag{4.2}$$

It is clear that $V \in H_{k+1}(\mathbb{D})$. So, $\partial_{\bar{z}} V \in H_k(\mathbb{D})$. We know that the unique solution of the homogeneous problem (4.1) is $\partial_{\bar{z}} V \equiv 0$. That is, $V \in H_1(\mathbb{D})$. Using the boundary condition (4.2) and Lemma 1, we obtain $V(z) \equiv 0$. □

Now, we investigate the inhomogeneous first-order equation in the unit disc $\mathbb{D}$.

'Find a function V satisfying the inhomogeneous Dirichlet-type boundary condition

$$\partial_{\bar{z}} V = f, \ \ z \in \mathbb{D},$$
$$V^+(t) = \gamma(t), \ \ t \in \mathbb{T},$$

where $\gamma(t) \in H(\mathbb{T})$, $f \in C^1(\bar{\mathbb{D}})$.'

Let

$$Tf(z) = -\frac{1}{\pi} \int_{\mathbb{D}} \frac{f(\zeta)}{\zeta - z} \, d\xi d\eta$$

with $f \in C^1(\bar{D})$. We know that $\partial_{\bar{z}}(Tf(z)) = f(z)$. By the statement of the problem and the properties of $Tf(z)$ we have $V - Tf \in H_1(\mathbb{D})$. Also, $Tf \in H(\bar{\mathbb{D}})$. Hence, the Dirichlet problem for the inhomogeneous first-order equation becomes:

'Find a function $\varphi \in H_1(\mathbb{D})$ satisfying the Dirichlet-type boundary condition

$$\varphi^+(t) = \gamma(t) - Tf(t), \ \ t \in \partial\mathbb{D}. \tag{4.3}$$

Using Theorem 2.3 for $n = 1$, under the boundary conditions (4.3) subject to the solvability conditions

$$\frac{1}{2\pi i} \int_{\mathbb{T}} \gamma(\zeta) \frac{\bar{z} d\zeta}{1 - \bar{z}\zeta} = \frac{1}{\pi} \int_{\mathbb{D}} f(\zeta) \frac{\bar{z} d\xi d\eta}{1 - \bar{z}\zeta} \, ,$$

the solution is

$$w(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \gamma(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int_{\mathbb{D}} f(\zeta) \frac{d\xi d\eta}{\zeta - z}\,.\text{'},$$

Generalizing these thoughts, the following problem can be solved.

'Find a function $V \in H_4(\mathbb{D})$ satisfying the corresponding Dirichlet type condition

$$\left[\partial_{\bar{z}}^{j} V\right]^{+}(t) = \gamma_j(t), \quad t \in \partial\mathbb{T}, \quad j = 0, 1, 2, 3$$

where $\gamma_j \in H(\partial\mathbb{D})$.'

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