Transactions of A. Razmadze
Mathematical Institute
Vol. **179** (2025), issue 2, 221–230

THE THIRD AND FOURTH BOUNDARY VALUE PROBLEMS OF CONSOLIDATION THEORY WITH DOUBLE POROSITY

LEVAN GIORGASHVILI AND SHOTA ZAZASHVILI

Abstract. In this paper, we consider the boundary value problems of statics of a three-dimensional version of the Aifantis equations of the theory of consolidation with double porosity for a half-space, when either the limiting values of the tangential components of a stress vector and the normal component of a displacement vector are given on the boundary, or the limiting values of the normal components of a stress vector and the tangential components of a displacement vector are given on the boundary. A new approach is developed which is based on the explicit solutions of the Dirichlet and Neumann problems for the Laplace equation for a half-place. The solutions of the boundary value problems are constructed explicitly in quadratures.

1. Introduction

The theory of consolidation with double porosity was proposed by E. C. Aifantis. The physical and mathematical foundations of the theory were considered in papers [4, 8, 12]. Namely, in [12], the detailed physical interpretations of the phenomenological coefficients appearing in the model are given and several particular boundary value problems (BVP) are solved.

In [4], the uniqueness of solution for some standard boundary value problems is proved and the variational principles for the equations of double porosity are considered. In [8], for the Aifantis equations, the finite element formulation is employed to obtain numerical results. The basic bibliographical review concerning the theory of porous media can be found in R. De Boer's paper [5].

The Dirichlet and Neumann type BVPs of the theory of consolidation with double porosity for a half-space are solved explicitly in [3] by using the potential method and the theory of integral equations.

For a wider overview of the subject area of applications we refer to [1, 2, 6, 7, 9, 11].

In this paper, we consider the so-called third and fourth BVPs of the theory of consolidation with double porosity for a half-space.

In the case of the third problem, the limiting values of the tangential components of a stress vector and the normal components of a displacement vector are given on the boundary along with the Neumann conditions for the pleasure functions.

In the case of the forth problem, the limiting values of the normal components of a stress vector and the tangential components of a displacement vector are given on the boundary along with the Dirichlet conditions for the pleasure functions.

We offer an approach for solving the mentioned boundary value problems based on the Poison type formulas for the Dirichlet and Neumann boundary value problems for the Laplace equation in the case of a half-space. The solutions are represented explicitly in quadratures.

2. Basic Differential Equations and Formulation of Boundary Value Problems

Let Ω^- be a half-space, $\Omega^- := \{x : x \in \mathbb{R}^3,\ x_3 > 0\}$, whose boundary $\partial\Omega^-$ is a plane $\partial\Omega^- = \{x : x \in \mathbb{R}^3,\ x_3 = 0\}$. By $n = (0,0,1)^\top$ we denote the unit normal vector to $\partial\Omega^-$. The symbol $(\cdot)^\top$ denotes transposition operation.

2020 *Mathematics Subject Classification.* 74G20, 74G10, 74G30.

Key words and phrases. Porous media; Double porosity; Consolidation; Fundamental solution.

The homogeneous system of partial differential equations of statics of the theory of consolidation with double porosity reads as [4,12]:

$$\mu\Delta u(x)x+(\lambda+\mu)\operatorname{grad}\operatorname{div}u(x)-\operatorname{grad}(\beta_1p_1(x)+\beta_2p_2(x))=0, \tag{2.1}$$

$$(m_1\Delta-\varkappa)\,p_1(x)+\varkappa\,p_2(x)=0, \tag{2.2}$$

$$\varkappa\,p_1(x)+(m_2\Delta-\varkappa)\,p_2(x)=0, \tag{2.3}$$

where $u=(u_1,u_2,u_3)^\top$ is the displacement vector, p_1 is the fluid pressure within the primary pores and p_2 is the fluid pressure within the secondary pores, $m_j=k_j/\mu^*$, $j=1,2$, k_1 and k_2 are the permeabilities of the primary and secondary systems of pores; μ^* denotes the viscosity of the pore fluid, constant $\varkappa$ measures the transfer of fluid from the secondary pores to the primary pores, λ and μ are the Lame constants, β_1 and β_2 are measures of the change of porosities due to an applied volumetric strain. The quantities λ, μ, $\varkappa$, β_j, k_j, $j=1,2$, and μ^* are positive constants; Δ is the tree-dimensional Laplace operator.

Definition 1. A vector-function $U=(u_1,\,u_2,\,u_3,\,p_1,\,p_2)^\top$ is said to be regular in a domain Ω^-, if $U\in C^2(\Omega^-)\cap C^1(\overline{\Omega^-})$ and at infinity satisfies the following conditions:

$$\begin{aligned}
&U_j(x)=O(|x|^{-1}), && \frac{\partial U_j}{\partial x_i}=O(|x|^{-2}),\ \ x_3>0,\ \ |x|\to\infty,\\
&U_j(x)=o(1), && \frac{\partial U_j}{\partial x_i}=O(|x|^{-1}),\ \ x_3=0,\ \ |x|\to\infty,\\
&j=1,2,\dots,5,\ \ i=1,2,3;\ \ |x|^2=x_1^2+x_2^2+x_3^2.
\end{aligned}$$

Here, we use the notation $U_j=u_j$, $j=1,2,3$, $U_4=p_1$, $U_5=p_2$.

For the system of equations (2.1)–(2.3) we consider the following boundary value problems.

Problem (III$^-$)**.** Find a regular vector-function $U=(u_1,\,u_2,\,u_3,\,p_1,\,p_2)^\top$ satisfying the system of differential equations (2.1)–(2.3) in Ω^- and the boundary conditions

$$\{P(\partial,n)U(z)\}^- - n(z)\,\{n(z)\cdot P(\partial,n)U(z)\}^-=F(z),\quad \{n(z)\cdot u(z)\}^-=f_3(z), \tag{2.4}$$

$$\left\{\frac{\partial p_1(z)}{\partial n(z)}\right\}^-=f_4(z),\ \ \left\{\frac{\partial p_2(z)}{\partial n(z)}\right\}^-=f_5(z),\ \ z\in\partial\,\Omega^-. \tag{2.5}$$

We assume that the vector-function $F=(f_1,f_2,0)^\top$ and the functions f_j, $j=3,4,5$, are given on the boundary $\partial\,\Omega^-$; moreover, $f_j\in C^{\,0,\alpha}(\partial\,\Omega^-)$, $j=1,2,4,5$, and $f_3\in C^{\,1,\alpha}(\partial\,\Omega^-)$ satisfy at infiniy the following decay conditions:

$$|f_j(z)|<\frac{A}{1+|z|^2},\ \ j=1,2,4,5,\qquad |f_3(z)|<\frac{A}{1+|z|},\ \ z\in\partial\,\Omega^-,\ \ A=\text{const}>0.$$

Problem (IV$^-$)**.** Find a regular vector-function $U=(u_1,\,u_2,\,u_3,\,p_1,\,p_2)^\top$ satisfying the system of differential equations (2.1)–(2.3) in Ω^- and the boundary conditions

$$\{u(z)\}^- - n(z)\{n(z)\cdot u(z)\}^-=F(z),\quad \{n(z)\cdot P(\partial,n)U(z)\}^-=f_3(z), \tag{2.6}$$

$$\{p_1(z)\}^-=f_4(z),\ \ \{p_2(z)\}^-=f_5(z),\ \ z\in\partial\,\Omega^-, \tag{2.7}$$

where $P(\partial,n)U$ is the stress vector of the form [12]

$$P(\partial,n)U=2\mu\frac{\partial u}{\partial n}+\lambda\,n\operatorname{div}u+\mu[n\times\operatorname{rot}u]-n\,(\beta_1p_1+\beta_2p_2),$$

$$\frac{\partial}{\partial n}=\sum_{j=1}^{3}n_j\frac{\partial}{\partial x_j};$$

$n=(n_1,n_2,n_3)^\top$ is a unit vector; the vector-function $F=(f_1,f_2,0)^\top$ and the functions f_j, $j=3,4,5$, are given on the boundary $\partial\,\Omega^-$; moreover, $f_j\in C^{\,1,\alpha}(\partial\,\Omega^-)$, $j=1,2,4,5$, $f_3\in C^{\,0,\alpha}(\partial\,\Omega^-)$,

$0 < \alpha < 1$, satisfy at infinity the decay conditions

$$|f_j(z)| < \frac{A}{1+|z|}, \quad j=1,2,4,5, \quad |f_3(z)| < \frac{A}{1+|z|^2}, \quad z \in \partial\,\Omega^-, \quad A = \text{const} > 0.$$

Here and in what follows, the central dot denotes the real scalar product $a \cdot b = \sum_{k=1}^{3} a_k b_k$ for $a, b \in \mathbb{R}^3$, and the symbol $[a \times b]$ denotes a cross product of two vectors in $\mathbb{R}^3$. The symbol $\{\cdot\}^-$ denotes the limiting value on the boundary $\partial\Omega^-$ from Ω^-,

$$\{w(z)\}^- = \lim_{\Omega^- \ni x \to z \in \partial\,\Omega^-} w(x), \quad \left\{\frac{\partial w(z)}{\partial x_j}\right\}^- = \lim_{\Omega^- \ni x \to z \in \partial\,\Omega^-} \frac{\partial w(x)}{\partial x_j}, \quad j=1,2,3.$$

First of all, we will construct the fundamental matrix for the system of equations (2.2) and (2.3).

Introduce the following operator:

$$L(\partial) = \begin{bmatrix} m_1\Delta - \varkappa & \varkappa \\ \varkappa & m_2\Delta - \varkappa \end{bmatrix}_{2\times 2}.$$

Denote by

$$\Gamma(x) = [\Gamma_{kj}(x)]_{2\times 2}$$

the matrix of fundamental solutions of the operator $L(\partial)$,

$$L(\partial)\,\Gamma(x) = \delta(x)\, I_2. \tag{2.8}$$

Here, $\delta(\,\cdot\,)$ is the Dirac delta distribution and I_2 is the second order unit matrix.

Let $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1}_{\xi\to x}$ denote the generalized direct and inverse Fourier transforms. Then for an arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and a tempered distribution g, we have

$$\mathcal{F}_{x\to\xi}[\partial^{\,\alpha} g(x)] = (-i\,\xi)^{\alpha}\mathcal{F}_{x\to\xi}[g(x)], \qquad \mathcal{F}^{-1}_{\xi\to x}[\xi^{\,\alpha}\widehat{g}(\xi)\,] = (i\,\partial)^{\alpha}\mathcal{F}^{-1}_{\xi\to x}[\widehat{g}(\xi)\,], \tag{2.9}$$

where $\widehat{g}(\xi) = \mathcal{F}_{x\to\xi}[g(x)]$, $|\alpha| = \alpha_1+\alpha_2+\alpha_3$, $\xi^{\,\alpha} = \xi_1^{\alpha_1}\,\xi_2^{\alpha_2}\,\xi_3^{\alpha_3}$, and $\partial^{\,\alpha} = \partial_1^{\alpha_1}\,\partial_2^{\alpha_2}\,\partial_3^{\alpha_3}$ with $\partial_j = \frac{\partial}{\partial x_j}$. Perform the Fourier transforms of equation (2.8) using the first formula in (2.9) and the equality $\mathcal{F}[\delta(\cdot)] = 1$ to get

$$L(-i\,\xi)\,\widehat{\Gamma}(\xi) = I_2, \tag{2.10}$$

where $\widehat{\Gamma}(\xi)$ is the Fourier transform of fundamental matrix $\Gamma(x)$.

If $\det L(-i\,\xi) \neq 0$, than from equation (2.10), we obtain

$$\widehat{\Gamma}(\xi) = L^{-1}(-i\,\xi)$$

$$= \frac{1}{m_1 m_2}\begin{bmatrix} -\dfrac{m_2}{|\xi|^2+\lambda_0^2} + \dfrac{\varkappa}{\lambda_0^2}\left(\dfrac{1}{|\xi|^2+\lambda_0^2} - \dfrac{1}{|\xi|^2}\right) & \dfrac{\varkappa}{\lambda_0^2}\left(\dfrac{1}{|\xi|^2+\lambda_0^2} - \dfrac{1}{|\xi|^2}\right) \\ \dfrac{\varkappa}{\lambda_0^2}\left(\dfrac{1}{|\xi|^2+\lambda_0^2} - \dfrac{1}{|\xi|^2}\right) & -\dfrac{m_1}{|\xi|^2+\lambda_0^2} + \dfrac{\varkappa}{\lambda_0^2}\left(\dfrac{1}{|\xi|^2+\lambda_0^2} - \dfrac{1}{|\xi|^2}\right) \end{bmatrix}_{2\times 2}, \tag{2.11}$$

where

$$\lambda_0^2 = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\varkappa > 0.$$

Using the relations

$$\mathcal{F}^{-1}_{\xi\to x}\left[\frac{1}{|\xi|^2}\right] = \frac{1}{4\pi|x|}, \qquad \mathcal{F}^{-1}_{\xi\to x}\left[\frac{1}{|\xi|^2+\lambda_0^2}\right] = \frac{e^{-\lambda_0|x|}}{4\pi|x|},$$

from (2.11), we get the following representation of the fundamental matrix:

$$\Gamma(x) = \mathcal{F}^{-1}_{\xi\to x}\left[\widehat{\Gamma}(\xi)\right]$$

$$= \frac{1}{4\pi m_1 m_2}\begin{bmatrix} -m_2\dfrac{e^{-\lambda_0|x|}}{|x|} + \dfrac{\varkappa}{\lambda_0^2}\dfrac{e^{-\lambda_0|x|}-1}{|x|} & \dfrac{\varkappa}{\lambda_0^2}\dfrac{e^{-\lambda_0|x|}-1}{|x|} \\ \dfrac{\varkappa}{\lambda_0^2}\dfrac{e^{-\lambda_0|x|}-1}{|x|} & -m_1\dfrac{e^{-\lambda_0|x|}}{|x|} + \dfrac{\varkappa}{\lambda_0^2}\dfrac{e^{-\lambda_0|x|}-1}{|x|} \end{bmatrix}_{2\times 2}.$$

3. Solution of Problem (III)$^-$.

Let's start solving problem (III)$^-$ by defining a vector $p=(p_1,p_2)^\top$ that satisfies equations (2.2) and (2.3) in Ω^- and the Neumann type boundary condition (2.5) on $\partial\Omega^-$. We look for the vector $p=(p_1,p_2)^\top$ in the form of a single layer potential

$$p(x)=\int\limits_{\partial\Omega^-}\Gamma(x-y)h(y)dy_1dy_2, \tag{3.1}$$

where $h=(h_1,h_2)^\top$ is a two-dimensional unknown density vector.

Using the jump relations for the single layer potential [10] and taking into consideration the boundary conditions (2.5) for the unknown density vector function $h=(h_1,h_2)^\top$, we get the Fredholm integral equation of the second kind

$$\frac{1}{2m_1m_2}\begin{bmatrix} m_2 & 0\\ 0 & m_1\end{bmatrix}h(z)+\int\limits_{\partial\Omega^-}\Big[\frac{\partial}{\partial x_3}\Gamma(x-y)\Big]_{x=z}h(y)\,dy_1\,dy_2=\begin{bmatrix} f_4(z)\\ f_5(z)\end{bmatrix}, \tag{3.2}$$

$$z=(z_1,z_2,0)\in\partial\Omega^-.$$

Since

$$\Big[\frac{\partial}{\partial x_3}\Gamma(x-y)\Big]_{x=z}=0,$$

from (3.2), we conclude

$$h(z)=2\begin{bmatrix} m_1f_4(z)\\ m_2f_5(z)\end{bmatrix}.$$

Therefore (3.1) takes the form

$$p(x)=2\int\limits_{\partial\Omega^-}\Gamma(x-y)\begin{bmatrix} m_1f_4(y)\\ m_2f_5(y)\end{bmatrix}\,dy_1\,dy_2,\quad x\in\Omega^-. \tag{3.3}$$

Thus, to determine the solution $U=(u_1,\,u_2,\,u_3,\,p_1,\,p_2)^\top$ of problem (III)$^-$, it remains to find the displacement vector $u=(u_1,u_2,u_3)^\top$.

Substitution $p_1(x)$ and $p_2(x)$ into equation (2.1) leads to the non-homogeneous differential equation with respect to $u(x)$,

$$\mu\,\Delta u(x)+(\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,u(x)=\mathrm{grad}(\beta_1p_1(x)+\beta_2p_2(x)). \tag{3.4}$$

Denote by $u^{(0)}(x)$ a particular solution of equation (3.4). Then a general solution of equation (3.4) can be represented in the form $u(x)=v(x)+u^{(0)}(x)$, where $v(x)$ is a general solution of the homogeneous equation

$$\mu\,\Delta v(x)+(\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,v(x)=0, \tag{3.5}$$

and $u^{(0)}(x)$ is the particular solution of equation (3.4). We can construct $u^{(0)}(x)$ explicitly [10] as

$$u^{(0)}(x)=\int\limits_{\Omega^-}\Gamma^{(1)}(x-y)\,\mathrm{grad}\,(\beta_1p_1(y)+\beta_2p_2(y))\,dy,\quad x\in\Omega^-, \tag{3.6}$$

where

$$\Gamma^{(1)}(x-y)=\Big[\Gamma^{(1)}_{lj}(x-y)\Big]_{3\times3},\qquad \Gamma^{(1)}_{lj}(x-y)=\lambda'\,\frac{\delta_{lj}}{|x-y|}-\mu'\,\frac{\partial^2|x-y|}{\partial x_l\partial x_j}\,,$$

$$\lambda'=\frac{\lambda+3\mu}{4\pi\mu(\lambda+2\mu)}\,,\qquad \mu'=\frac{\lambda+\mu}{4\pi\mu(\lambda+2\mu)}\,.$$

The vector $\mathrm{grad}\,(\beta_1p_1(x)+\beta_2p_2(x))$ is continues in Ω^- along with its first order derivatives and satisfy at infinity the following decay conditions:

$$\mathrm{grad}\,(\beta_1p_1(x)+\beta_2p_2(x))=O(|x|^{-2-\alpha}),\quad \alpha>0.$$

The boundary conditions (2.4) we can transform as follows:

$$\{u_3(z)\}^- = f_3(z), \quad \left\{\frac{\partial u_j(z)}{\partial x_3}\right\}^- = \frac{1}{\mu} f_j(z) - \frac{\partial f_3(z)}{\partial z_j}, \;\; j=1,2, \;\; z \in \partial\,\Omega^-, \tag{3.7}$$

The boundary conditions (3.7) for the vector $v(x)$ take the following form:

$$\{v_3(z)\}^- = f_3(z) - \left\{u_3^{(0)}(z)\right\}^-, \quad \left\{\frac{\partial\, v_j(z)}{\partial x_3}\right\}^- = \frac{1}{\mu} f_j(z) - \frac{\partial\, f_3(z)}{\partial z_j} - \left\{\frac{\partial\, u_j^{(0)}(z)}{\partial x_3}\right\}^-,$$
$$j=1,2, \;\; z \in \partial\,\Omega^-.$$

From these conditions we find that

$$\{[\operatorname{rot} v(z)]_j\}^- = \varphi_j(z), \;\; j=1,2, \tag{3.8}$$

$$\left\{\frac{\partial}{\partial x_3}[\operatorname{rot} v(z)]_3\right\}^- = \varphi_3(z), \;\; z \in \partial\,\Omega^-, \tag{3.9}$$

where

$$\varphi_1(z) = -\frac{1}{\mu} f_2(z) + 2\frac{\partial f_3(z)}{\partial z_2} - \left\{\frac{\partial\, u_3^{(0)}(z)}{\partial z_2}\right\}^- + \left\{\frac{\partial\, u_2^{(0)}(z)}{\partial x_3}\right\}^-,$$

$$\varphi_2(z) = \frac{1}{\mu} f_1(z) - 2\frac{\partial f_3(z)}{\partial z_1} + \left\{\frac{\partial\, u_3^{(0)}(z)}{\partial z_1}\right\}^- - \left\{\frac{\partial\, u_1^{(0)}(z)}{\partial x_3}\right\}^-,$$

$$\varphi_3(z) = \frac{1}{\mu}\left(\frac{\partial f_2(z)}{\partial z_1} - \frac{\partial f_1(z)}{\partial z_2}\right) + \frac{\partial}{\partial z_2}\left\{\frac{\partial\, u_1^{(0)}(z)}{\partial x_3}\right\}^- - \frac{\partial}{\partial z_1}\left\{\frac{\partial\, u_2^{(0)}(z)}{\partial x_3}\right\}^-.$$

From equation (3.5), we get

$$\Delta \operatorname{rot} v(x) = 0, \;\; x \in \Omega^-.$$

Taking into consideration the boundary conditions (3.8) and (3.9), we obtain the Dirichlet problem for the functions $[\operatorname{rot} v(x)]_j$, $j=1,2$, and the Neumann problem for $[\operatorname{rot} v(x)]_3$.

The solutions of the Dirichlet and Neumann problems can be written explicitly as [7]

$$[\operatorname{rot} v(x)]_j = -\frac{1}{2\pi}\int\limits_{\partial\,\Omega^-} \frac{\partial}{\partial x_3}\frac{1}{|x-y|}\varphi_j(y)\,dy_1\,dy_2, \;\; j=1,2, \tag{3.10}$$

$$[\operatorname{rot} v(x)]_3 = -\frac{1}{2\pi}\int\limits_{\partial\,\Omega^-} \frac{1}{|x-y|}\varphi_3(y)\,dy_1\,dy_2. \tag{3.11}$$

From equalities (3.10) and (3.11), for the vector $x \times \operatorname{rot} v(x)$, we have

$$\begin{aligned}[x \times \operatorname{rot} v(x)]_j =& \frac{1}{2\pi}\int\limits_{\partial\,\Omega^-} \frac{x_1\delta_{2j} - x_2\delta_{1j}}{|x-y|}\,\varphi_3(y)\,dy_1\,dy_2 \\ &+ \frac{1}{2\pi}\int\limits_{\partial\,\Omega^-} x_3\,\frac{\partial}{\partial x_3}\frac{\delta_{1j}\varphi_2(y) - \delta_{2j}\varphi_1(y)}{|x-y|}\,dy_1\,dy_2, \;\; j=1,2,\end{aligned} \tag{3.12}$$

$$[x \times \operatorname{rot} v(x)]_3 = \frac{1}{2\pi}\int\limits_{\partial\,\Omega^-} \frac{\partial}{\partial x_3}\frac{x_2\varphi_1(y) - x_1\varphi_2(y)}{|x-y|}\,dy_1\,dy_2. \tag{3.13}$$

Using the identity $\operatorname{grad}\operatorname{div} v = \Delta v + \operatorname{rot}\operatorname{rot} v$, from (3.5), we get

$$(\lambda + 2\mu)\Delta v(x) + (\lambda+\mu)\operatorname{rot}\operatorname{rot} v(x) = 0. \tag{3.14}$$

Since $\Delta \operatorname{rot} v = 0$, the equality $\Delta\,[x \times \operatorname{rot} v] = 2 \operatorname{rot}\operatorname{rot} v$ holds, and from equation (3.14), we deduce

$$\Delta w(x) = 0, \;\; x \in \Omega^-,$$

where

$$w(x)=2(\lambda+2\mu)\,v(x)+(\lambda+\mu)[x\times \operatorname{rot} v(x)]. \tag{3.15}$$

Using the boundary conditions (3.7), (3.8) and (3.9), for the vector $w(x)$, we obtain the following boundary conditions:

$$\{w_3(z)\}^-=\varphi_4(z),\quad z\in\partial\,\Omega^-,$$
$$\left\{\frac{\partial}{\partial x_3}w_1(z)\right\}^-=\varphi_5(z),\quad \left\{\frac{\partial}{\partial x_3}w_2(z)\right\}^-=\varphi_6(z),\quad z\in\partial\,\Omega^-,$$

where

$$\varphi_4(z)=2(\lambda+2\mu)\left(f_3(z)-\left\{u_3^{(0)}(z)\right\}^-\right)+(\lambda+\mu)\left(z_1\varphi_2(z)-z_2\varphi_1(z)\right),$$
$$\varphi_5(z)=2(\lambda+2\mu)\left(\frac{1}{\mu}f_1(z)-\frac{\partial f_3(z)}{\partial z_1}-\left\{\frac{\partial\, u_1^{(0)}(z)}{\partial x_3}\right\}^-\right)+(\lambda+\mu)\left(z_2\varphi_3(z)-\varphi_2(z)\right),$$
$$\varphi_6(z)=2(\lambda+2\mu)\left(\frac{1}{\mu}f_2(z)-\frac{\partial f_3(z)}{\partial z_2}-\left\{\frac{\partial\, u_2^{(0)}(z)}{\partial x_3}\right\}^-\right)$$
$$+(\lambda+\mu)\left(\varphi_1(z)-z_1\varphi_3(z)\right),\quad z\in\partial\,\Omega^-.$$

As we see, for the components of the vector $w(x)$ we have the Dirichlet and Neumann problems in the domain Ω^-. The solutions to these problems can be written explicitly:

$$\begin{aligned}
w_3(x)&=-\frac{1}{2\pi}\int\limits_{\partial\,\Omega^-}\frac{\partial}{\partial x_3}\frac{1}{|x-y|}\varphi_4(y)\,dy_1\,dy_2,\\
w_1(x)&=-\frac{1}{2\pi}\int\limits_{\partial\,\Omega^-}\frac{1}{|x-y|}\varphi_5(y)\,dy_1\,dy_2,\\
w_2(x)&=-\frac{1}{2\pi}\int\limits_{\partial\,\Omega^-}\frac{1}{|x-y|}\varphi_6(y)\,dy_1\,dy_2,\quad x\in\Omega^-.
\end{aligned}\tag{3.16}$$

From equation (3.15), we get

$$v(x)=\frac{1}{2(\lambda+2\mu)}w(x)-\frac{\lambda+\mu}{2(\lambda+2\mu)}\left[x\times\operatorname{rot} v(x)\right]. \tag{3.17}$$

Let a function ν satisfy the following conditions: $\nu\in C^{1,\alpha}(\partial\,\Omega^-)$ and, at infinity, $\nu(x)=O(|x|^{-\alpha})$, $\frac{\nu(x)}{\partial x_j}=O(|x|^{-1-\alpha})$, $\alpha>0$, $j=1,2$. Then, for $x\in\Omega^-$, the following identities hold:

$$\int\limits_{\partial\,\Omega^-}\frac{1}{|x-y|}\frac{\partial\nu(y)}{\partial y_l}\,dy_1\,dy_2=\int\limits_{\partial\,\Omega^-}\frac{\partial}{\partial x_l}\frac{1}{|x-y|}\,\nu(y)\,dy_1\,dy_2, \tag{3.18}$$

$$\int\limits_{\partial\,\Omega^-}(x_l-y_l)\frac{\partial}{\partial x_3}\frac{1}{|x-y|}\frac{\partial\nu(y)}{\partial y_j}\,dy_1\,dy_2=\int\limits_{\partial\,\Omega^-}\frac{\partial^3|x-y|}{\partial x_l\,\partial x_j\,\partial x_3}\nu(y)\,dy_1\,dy_2,\quad l,j=1,2, \tag{3.19}$$

$$\lim_{\Omega^-\ni x\to z\in\partial\,\Omega^-}\frac{1}{2\pi}\int\limits_{\partial\,\Omega^-}\frac{\partial}{\partial x_3}\frac{1}{|x-y|}\,\nu\,(y)\,dy_1\,dy_2=-\,\nu\,(z),\quad z\in\partial\,\Omega^-. \tag{3.20}$$

With the help of formulas (3.12), (3.13), (3.16), (3.17), (3.18) and (3.19), for the vector v, we get the following final expression:

$$v(x)=\frac{1}{2\pi}\int\limits_{\partial\,\Omega^-}K(x,y)f(y)\,dy_1\,dy_2+\frac{1}{2\pi}\int\limits_{\partial\,\Omega^-}M(x,y)\widetilde{u}^{(0)}(y)\,dy_1\,dy_2,\quad x\in\Omega^-, \tag{3.21}$$

where

$$K(x,y)=\left[K_{lj}(x,y)\right]_{3\times3},\qquad M(x,y)=\left[M_{lj}(x,y)\right]_{3\times3},$$

$$K_{lj}(x,y) = -\frac{\delta_{lj}}{\mu|x-y|} + a\frac{\partial^2|x-y|}{\partial x_l \partial x_j}, \quad K_{3j}(x,y) = a\,\frac{\partial^2|x-y|}{\partial x_3 \partial x_j},$$
$$K_{l3}(x,y) = (1-4a\mu)\frac{\partial}{\partial x_l}\frac{1}{|x-y|} + 2a\mu\frac{\partial^3|x-y|}{\partial x_l \partial x_3^2},$$
$$K_{33}(x,y) = -(1+4a\mu)\frac{\partial}{\partial x_3}\frac{1}{|x-y|} + 2a\mu\frac{\partial^3|x-y|}{\partial x_3^3},$$
$$M_{lj}(x,y) = -\mu K_{lj}(x,y), \quad M_{3j}(x,y) = -\mu K_{3j}(x,y), \ \ l,j=1,2,$$
$$M_{l3}(x,y) = \frac{1}{2}\frac{\partial}{\partial x_l}\frac{1}{|x-y|} - \frac{1}{2}K_{l3}(x,y), \ \ l=1,2,3,$$

$$f = (f_1,f_2,f_3)^\top, \quad a = \frac{\lambda+\mu}{2\mu(\lambda+2\mu)}; \quad \widetilde{u}^{(0)} = \left(\left\{\frac{\partial\, u_1^{(0)}}{\partial x_3}\right\}^-, \left\{\frac{\partial\, u_2^{(0)}}{\partial x_3}\right\}^-, \left\{u_3^{(0)}\right\}^-\right)^\top.$$

Using relations (3.3), (3.6), (3.20) and (3.21), one can prove that the vector $U = (u,p)^\top$, where $u = v + u^{(0)}$ and $p = (p_1,p_2)^\top$, is a regular solution of problem $(\mathrm{III})^-$.

4. Solution of Problem $(\mathrm{IV})^-$.

We can solve problem $(\mathrm{IV})^-$ in a quite similar way. We start with finding an explicit form for the vector $p = (p_1,p_2)^\top$, satisfying differential equations (2.2) and (2.3) in Ω^- and the Dirichlet boundary conditions (2.7). We look for the vector $p = (p_1,p_2)^\top$ in the form of a double layer potential

$$p(x) = \int\limits_{\partial\,\Omega^-} \frac{\partial}{\partial x_3}\Gamma(x-y)\,g(y)\,dy_1\,dy_2, \quad x\in\Omega^-, \tag{4.1}$$

where $g = (g_1,g_2)^\top$ is a two-dimensional unknown vector.

Passing to the limit $\Omega^- \ni x \to z \in \partial\,\Omega^-$, for the unknown vector g, we obtain the following Fredholm integral equation of the second kind:

$$-\frac{1}{2m_1m_2}\begin{bmatrix} m_2 & 0\\ 0 & m_1\end{bmatrix} g(z) + \int\limits_{\partial\,\Omega^-}\left[\frac{\partial}{\partial x_3}\Gamma(x-y)\right]_{x=z} g(y)\,dy_1\,dy_2 = \begin{bmatrix} f_4(z)\\ f_5(z)\end{bmatrix}, \ \ z\in\partial\,\Omega^-. \tag{4.2}$$

Taking into account that

$$\left[\frac{\partial}{\partial x_3}\Gamma(x-y)\right]_{x_3=0} = 0 \ \text{ for } y\in\partial\Omega^-,$$

from (4.2), we find that

$$g(z) = -2\begin{bmatrix} m_1 f_4(z)\\ m_2 f_5(z)\end{bmatrix}$$

and (4.1) implies

$$p(x) = -2\int\limits_{\partial\,\Omega^-}\frac{\partial}{\partial x_3}\Gamma(x-y)\begin{bmatrix} m_1 f_4(y)\\ m_2 f_5(y)\end{bmatrix} dy_1\,dy_2.$$

So, it remains to find the displacement vector $u = (u_1,u_2,u_3)^\top$. Remaind that the vector u satisfies the non-homogeneous equation (3.4). We already know that a particular solution $u^{(0)}(x)$ to equation (3.4) is given by formula (3.6). Then a general solution of equation (3.4) can be represented in the form $u(x) = v(x) + u^{(0)}(x)$, where $v(x)$ is a general solution of the homogeneous equation (3.5). To find the vector $v(x)$, we proceed as follows.

From equation (3.5), we have

$$\Delta\,\mathrm{div}\,v(x) = 0, \quad x\in\Omega^-. \tag{4.3}$$

From the boundary conditions (2.6), we deduce:

$$\{v_1(z)\}^- = f_1(z) - \left\{u_1^{(0)}(z)\right\}^-, \quad \{v_2(z)\}^- = f_2(z) - \left\{u_2^{(0)}(z)\right\}^-, \quad z \in \partial\,\Omega^-, \tag{4.4}$$

$$\left\{\frac{\partial\, v_3(z)}{\partial x_3}\right\}^- = \psi_1(z), \quad z \in \partial\,\Omega^-, \tag{4.5}$$

where

$$\psi_1(z) = \frac{1}{\lambda+2\mu}\left(f_3(z) + \beta_1 f_4(z) + \beta_2 f_5(z)\right) - \frac{\lambda}{\lambda+2\mu}\left(\frac{\partial f_1(z)}{\partial z_1} + \frac{\partial f_2(z)}{\partial z_2}\right) - \left\{\frac{\partial u_3^{(0)}(z)}{\partial x_3}\right\}^-.$$

From these conditions, we get

$$\{\operatorname{div} v(z)\}^- = \psi_2(z), \quad z \in \partial\,\Omega^-, \tag{4.6}$$

where

$$\psi_2(z) = \frac{1}{\lambda+2\mu}\left(f_3(z) + \beta_1 f_4(z) + \beta_2 f_5(z)\right) + \frac{2\mu}{\lambda+2\mu}\left(\frac{\partial f_1(z)}{\partial z_1} + \frac{\partial f_2(z)}{\partial z_2}\right) - \left\{\operatorname{div} u^{(0)}(z)\right\}^-.$$

The Dirichlet problem (4.3), (4.6) with respect to $\operatorname{div} v(x)$ has the following solution:

$$\operatorname{div} v(x) = -\frac{1}{2\pi}\int\limits_{\partial\,\Omega^-} \frac{\partial}{\partial x_3}\frac{1}{|x-y|}\,\psi_2(y)\,dy_1\,dy_2, \quad j=1,2, \quad x \in \Omega^-. \tag{4.7}$$

By the relation $\Delta(x\operatorname{div} v(x)) = 2\operatorname{grad}\operatorname{div} v(x)$, from equation (3.5), we obtain

$$\Delta w(x) = 0, \quad x \in \Omega^-, \tag{4.8}$$

where

$$w(x) = 2\mu v(x) + (\lambda+\mu)[x\operatorname{div} v(x)]. \tag{4.9}$$

For the components of the vector w, from (4.8), we have

$$\Delta w_j(x) = 0, \quad j=1,2,3, \quad x \in \Omega^-. \tag{4.10}$$

On the other hand, bearing in mind the boundary conditions (4.4) and (4.6), we find

$$\{w_j(z)\}^- = \phi_j(z), \quad j=1,2, \quad z \in \partial\,\Omega^-, \tag{4.11}$$

where

$$\phi_j(z) = 2\mu f_j(z) + (\lambda+\mu)\, z_j\, \psi_2(z) - 2\mu\left\{u_j^{(0)}(z)\right\}^-, \quad j=1,2, \quad z \in \partial\,\Omega^-.$$

The Dirichlet problems (4.10), (4.11) with respect to the components $w_j(x)$, $j=1,2$, have the following solutions:

$$w_j(x) = -\frac{1}{2\pi}\int\limits_{\partial\,\Omega^-} \frac{\partial}{\partial x_3}\frac{1}{|x-y|}\,\phi_j(y)\,dy_1\,dy_2, \quad j=1,2, \quad x \in \Omega^-. \tag{4.12}$$

Using relations (4.5) and (4.6), for $w_3(x)$, we get the following Neumann boundary condition:

$$\left\{\frac{\partial w_3(z)}{\partial x_3}\right\}^- = \phi_3(z), \quad z \in \partial\,\Omega^-,$$

where

$$\phi_3(z) = 2\mu\psi_1(z) + (\lambda+\mu)\,\psi_2(z), \quad z \in \partial\,\Omega^-.$$

Therefore, for w_3, we have the following explicit form:

$$w_3(x) = -\frac{1}{2\pi}\int\limits_{\partial\,\Omega^-} \frac{1}{|x-y|}\,\phi_3(y)\,dy_1\,dy_2, \quad x \in \Omega^-. \tag{4.13}$$

From (4.9), we have

$$v(x) = \frac{1}{2\mu}w(x) - \frac{\lambda+\mu}{2\mu}\,x\operatorname{div} v(x), \quad x \in \Omega^-. \tag{4.14}$$

Using relations (4.7), (4.12), (4.13), (3.18) and (3.19), from (4.14), we get

$$v(x)=\frac{1}{2\pi}\int\limits_{\partial\,\Omega^-}\widetilde{K}(x,y)f(y)\,dy_1\,dy_2+\frac{1}{2\pi}\int\limits_{\partial\,\Omega^-}\widetilde{M}(x,y)\widetilde{u}^{(0)}(y)\,dy_1\,dy_2,\ \ x\in\Omega^-,$$

where

$$\widetilde{K}(x,y)=\Big[\widetilde{K}_{lj}(x,y)\Big]_{3\times 5},\quad \widetilde{M}(x,y)=\Big[\widetilde{M}_{lj}(x,y)\Big]_{3\times 3},$$

$$\widetilde{K}_{lj}(x,y)=\frac{\partial}{\partial x_3}\left(-\frac{\delta_{lj}}{|x-y|}+2\mu\,a\,\frac{\partial^2|x-y|}{\partial x_l\partial x_j}\right),\quad \widetilde{K}_{l3}(x,y)=a\,\frac{\partial^2|x-y|}{\partial x_l\partial x_3},$$

$$\widetilde{K}_{l4}(x,y)=a\,\beta_1\frac{\partial^2|x-y|}{\partial x_l\partial x_3},\quad \widetilde{K}_{l5}(x,y)=a\,\beta_2\frac{\partial^2|x-y|}{\partial x_l\partial x_3},$$

$$\widetilde{K}_{3j}(x,y)=\frac{\partial}{\partial x_j}\left(-\frac{1}{|x-y|}+2a\mu\frac{\partial^2|x-y|}{\partial x_3^2}\right),$$

$$\widetilde{K}_{33}(x,y)=-\frac{1}{\mu}\,\frac{1}{|x-y|}+a\,\frac{\partial^2|x-y|}{\partial x_3^2},$$

$$\widetilde{K}_{34}(x,y)=\beta_1\widetilde{K}_{33}(x,y),\quad \widetilde{K}_{35}(x,y)=\beta_2\widetilde{K}_{33}(x,y),$$

$$\widetilde{M}_{lj}(x,y)=\frac{\partial}{\partial x_3}\left(\delta_{lj}\frac{1}{|x-y|}-\frac{\lambda+\mu}{2\mu}\,\frac{\partial^2|x-y|}{\partial x_l\partial x_j}\right),$$

$$\widetilde{M}_{13}(x,y)=-\frac{\lambda+\mu}{2\mu}\,\frac{\partial^2|x-y|}{\partial x_l\partial x_3},$$

$$\widetilde{M}_{3j}(x,y)=\frac{\lambda+\mu}{2\mu}\,\frac{\partial}{\partial x_j}\left(\frac{2}{|x-y|}-\frac{\partial^2|x-y|}{\partial x_3^2}\right),\quad l,j=1,2,$$

$$\widetilde{M}_{33}(x,y)=\frac{\lambda+2\mu}{\mu}\,\frac{1}{|x-y|}-\frac{\lambda+\mu}{2\mu}\frac{\partial^2|x-y|}{\partial x_3^2},$$

$$f=(f_1,f_2,f_3,f_4,f_5)^\top,\quad \widetilde{u}^{(0)}=\left(\left\{u_1^{(0)}\right\}^-,\left\{u_2^{(0)}\right\}^-,\left\{\frac{\partial u_3^{(0)}}{\partial x_3}\right\}^-\right).$$

As in the previous case, one can prove that the vector $U=(u,p)^\top$, where $u=v+u^{(0)}$, $p=(p_1,p_2)^\top$, is a regular solution of problem $(IV)^-$.

References

1. J. R. Barber, The solution of elasticity problems for the half-space by the method of Green and Collins. *Appl. Sci. Res.* **40** (1983), no. 2, 135–157.
2. J. R. Barber, *Contact Mechanics.* Solid Mechanics and its Applications, 250. Springer, Cham, 2018.
3. M. Basheleishvili, L. Bitsadze, Explicit solutions of the BVPs of the theory of consolidation with double porosity for the half-space. *Bull. TICMI* **14** (2010), 9–15.
4. D. E. Beskos, E. C. Aifantis, On the theory of consolidation with double porosity-II. *Int. J. Engng. Sci.* **24** (1986), no. 11, 1697–1716.
5. R. De Boer, *Theory of Porous Media.* Highlights in historical development and current state. Springer-Verlag, Berlin, 2000.
6. L. Giorgashvili, D. Metreveli, Problems of statics of two-component elastic mixtures for a half-space. *Proc. A. Razmadze Math. Inst.* **167** (2015), 43–61.
7. L. Giorgashvili, S. Zazashvili, R. Meladze, About an explicit solution method for boundary value problem of elastostatics for half-space. *Georgian Int. J. Sci. Technol.* **6** (2014), no. 3, 215–224.
8. M. Y. Khaled, D. E. Beskos, E. C. Aifantis, On the theory of consolidation with double porosity-III, A finite element formulation. *Internat. J. Numer. Anal. Methods Geomech.* **8** (1984), no. 2, 101–123.
9. R. Kumar, T. Chadha, Plane problem in micropolar thermoelastic half-space with strech. *Indian J. pure appl. Math.* **176** (1986), 827–842.
10. V. D. Kupradze, T. G. Gegelia, M. O. Basheleishvili, T. V. Burchuladze, *Three Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity.* (Russian) Nauka, Moscow, 1976; English translation: North Holland Series in Applied Mathematics and Mechanics **25**, North Holland Publishing Company, Amsterdam, New York, Oxford, 1979.

11. H. Sherief, H. Saleh, A half-space problem in the theory of generalized thermoelastic diffusion. *Int. J. of Solids and Structures* **42** (2005), no. 15, 4484–4493.
12. R. K. Wilson, E. C. Aifantis, On the theory of consolidation with double porosity-I. *Int. J. Engng. Sci.* **20** (1982), no. 9, 1009–1035.

(Received 20.02.2024)

Department of Mathematics, Georgian Technical University, 77 Kostava Str., Tbilisi 0160, Georgia
Email address: lgiorgashvili@gmail.com
Email address: s.zazashvili@gtu.ge