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# ON CANONICAL RAYS OF GENERALIZED ANALYTIC FUNCTIONS

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*Dedicated to the memory of Professor Elene Obolashvili*

**Abstract.** In this paper, we investigate the solution space of generalized entire analytic functions in the sense of Vekua and Bers. The main results (Theorem 2.1 and Theorem 2.2), in a slightly modified form can be generalized to a sufficiently wide class of multidimensional elliptic systems as well.

## 1. Introduction

Everywhere below, $\mathbb{R}$ will be the real number axis, $\mathbb{C}_{z=x+iy}$ be the plane of complex numbers, and $\mathbb{A}$ be a class of all possible entire analytic functions (monogeneous functions at every point $z$ of the complex plane $\mathbb{C}$) and of a complex variable $z = x+iy$. One of the current directions of contemporary mathematical analysis is the systematic study of the mentioned class of functions; it originates from the second half of the 19th century in the classical works of Weierstrass, Mittag–Leffleur, Picard and others. One of the most important results of the theory of entire functions is the classical Liouville theorem, according to which every entire function $f(z)$ (without any exceptions) is either constant or unbounded. Moreover, if the rate of growth of the entire function $f(z)$ in the neighborhood of the point at infinity does not exceed the rate of growth of the function $|z|^N$, i.e.

$$f(z) = O\left(|z|^N\right), \quad z \to \infty,$$

for some non-negative integer $N$, then the function $f(z)$ is a polynomial whose order does not exceed $N$.

There are very few non-trivial properties that every member of this class obeys. Therefore, it is quite natural to separate subclasses from it (on the basis of certain criteria) and study these subclasses separately, rather than study the class $\mathbb{A}$ as a whole. As a such criterion, we can use the characteristic growth parameters of the value of the entire function $f(z)$ when the modulus of its argument $|z| \to \infty$ can be used. To obtain these parameters, a simple function $h(z)$ is fixed in some sense, unbounded in the neighborhood of the point at infinitely, and from the whole class of functions $\mathbb{A}$, a subclass is separated, each representative of which grows no faster than this fixed function $h(z)$. From this point of view, the function $h(z) = |z|^N$ is uninteresting by virtue of the classical Liouville theorem. As it turned out, from the mentioned point of view, the function with exponential growth

$$h(z) = \exp\left\{|z|^\delta\right\}$$

is very interesting and important.

Here, the positive number $\delta \in \mathbb{R}$, through which the most important (both theoretically and practically) subclasses are separated from the whole class $\mathbb{A}$ of the entire functions $f(z)$. These subclasses probably first appeared in Hadamard's work [4]. In particular, this work gives the most important classification of all functions according to exponential growth in the neighborhood of the point at infinitely. Hadamar's classification of all analytic functions is the basis of the most important directions of research in complex analysis. A similar classification of generalized analytic functions is given in this work.

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## 2. Main Results

Suppose that $W = W(z)$ is a continuous complex function defined on the whole complex plane. Let us use the classical scheme (see, e.g., [2, 4]) and associate this function with a non-negative real number $R(W)$ or symbol $+$ as follows: if for any positive number $\delta\in \mathbb{R}$ there exists a set $G\subset \mathbb{C}$ such that

$$\sup\left\{|W(z)|\exp\left(-|z|^{\delta}\right) \ : z\in G\right\} = +\infty, \tag{2.1}$$

then

$$R(W) = +\infty. \tag{2.2}$$

In such a case, we will say that the function $W = W(z)$ has an infinite exponential rating in the neighborhood of the point at infinitely (in short, it has an infinite rating). If (2.1) and (2.2) do not hold, then it is obvious that there exists a non-negative number $\delta\in \mathbb{R}$ for the function $W = W(z)$ for which the condition

$$W(z) = O\left(\exp\{|z|^{\delta}\}\right), \quad z\to\infty \tag{2.3}$$

is fulfilled.

For the function $W = W(z)$ with the finite rating, the number $\delta$ satisfying condition (2.3) is not uniquely defined; for every such number $\delta$, condition (2.3) is satisfied by any number greater than this number (and possibly by some numbers less than this one).

Let us denote the set of all such possible numbers by $\Delta_W$ and call a non-negative real number

$$\varrho = \inf \Delta_W$$

an exponential rating of a function in the neighborhood of the point at infinitely (shortly, the rating).

Suppose that $W = W(z)$ is a continuous complex function of the complex variable $z$ defined in the whole complex plane $\mathbb{C}$. We call the sequence of the points of the plane $\mathbb{C}$

$$z_1, z_2, z_3, \dots, z_n, \dots \tag{2.4}$$

the $\alpha$-canonical sequence, if

$$\lim_{n\to\infty} z_n = \infty, \qquad \lim_{n\to\infty} W(z_n) = \infty.$$

The sequence (2.4) is called the $\beta$-canonical sequence, if each of its terms is nonzero, $\lim_{n\to\infty} z_n = \infty$ and for every nonnegative real number $\sigma \geq 0$,

$$\lim_{n\to\infty} \frac{|W(z_n)|}{|z_n|^{\sigma}} = \infty.$$

Everywhere below, $\Gamma_{z_0,\varphi}$ denotes a ray lying on the complex plane $\mathbb{C}$; the origin of this ray is at the point $z_0\in \mathbb{C}$ and the direction is determined by the real number $\varphi\in[0, 2\pi)$

$$\Gamma_{z_0,\varphi} = \{z : z = z_0 + r(\cos\varphi + i\sin\varphi), \ \ r\geq 0\}.$$

We call $\Gamma_{z_0,\varphi}$ an $\alpha$-canonical ($\beta$-canonical) ray of a continuous function $W = W(z)$, if on this ray lies $\alpha$-canonical ($\beta$-canonical) sequence (2.4) of the function $W$, that is

$$\arg(z_n - z_0) = \varphi, \ \ n = 1, 2, 3, \dots .$$

Obviously, every continuous unbounded function has the $\alpha$-canonical sequence (and vice versa); the study of the existence of the $\alpha$-canonical (especially $\beta$-canonical) rays of such functions and their distribution on the plane is undoubtedly an interesting problem. The aim of this paper is not to investigate the mentioned problem; we will not consider all possible classes of continuous functions, but will restrict ourselves to its most important subclass, the class of generalized entire analytic functions.

The following system of elliptic equations, whose complex form is

$$\frac{\partial w}{\partial \overline{z}} + aw + b\overline{w} = 0, \tag{2.5}$$

is called the Carleman–Bers–Vekua equation.

Everywhere below, we mean that the coefficients of this equation satisfy the regularity condition, that is, they belong to the special class $L_{p,2}(\mathbb{C}_z)$, for the number $p > 2$. The function $f \in L_{p,2}(\mathbb{C}_z)$, if the following two integrals satisfy the conditions

$$\iint\limits_{\mathbb{G}} |f(z)|^p \, dz < \infty, \quad \iint\limits_{\mathbb{G}} \left| \frac{1}{z} f(\frac{1}{z}) \right|^p dz < \infty, \quad \mathbb{G} = \{|z| \le 1\}.$$

The solution of equation (2.5) is understood as a continuous generalized solution over the whole complex plane $\mathbb{C}_z$ (see [1,6]).

Let us denote the class of all possible solutions of the system (2.5) by $\mathbb{A}(a,b)$. Let $\mathbb{A}_\varrho(a,b)$ denote all elements of this class whose rating is $\varrho$.

The class of generalized analytic functions $\mathbb{A}(a,b)$ can be represented as a disjunctive union

$$\mathbb{A}(a,b) = \bigcup_{0 \le \varrho \le +\infty} A_\varrho(a,b);$$

furthermore, for every non-negative number $\varrho \in \mathbb{R}$,

$$\text{card } \mathbb{A}_\varrho(a,b) = \text{card } \mathbb{A}_\infty(a,b) = \text{card } \mathbb{C}.$$

**Theorem 2.1.** *Let the function $W \in \mathbb{A}_\varrho(a,b)$, where*

$$0 < \varrho < +\infty.$$

*Let further $z_0 \in \mathbb{C}$ and the complex plane $\mathbb{C}$ be divided into equal angles by a system of following rays*

$$\Gamma_{z_0,\varphi_1}, \Gamma_{z_0,\varphi_2}, \dots, \Gamma_{z_0,\varphi_n}, \tag{2.6}$$

*where the natural number $n \ge 2$. If*

$$2\varrho < n, \tag{2.7}$$

*then in system (2.6) there is (at least one) $\beta$-canonical ray of the function $W = W(z)$.*

*Proof.* We represent the function in the following form:

$$W = \Phi \exp(\Omega), \tag{2.8}$$

where $\Phi$ is the entire analytic function,

$$\Omega(z) = \frac{1}{\pi} \iint\limits_{\mathbb{C}} \left( a(\zeta) + b(\zeta) \frac{\overline{W(\zeta)}}{W(\zeta)} \right) \frac{d_\zeta \mathbb{C}}{\zeta - z}$$

(see [6]). The function $\Omega = \Omega(z)$ is continuous on the whole complex plane and $\lim_{z \to \infty} \Omega(z) = 0$. We say that none of the rays of system (2.6) is a $\beta$-canonical ray of function $W = W(z)$. Then we will be able to find natural numbers $M_1$, $M_2$ and $N$, for which

$$|W(z)| \le M_1 |z|^N, \tag{2.9}$$

where $|z| > M_2, \ z \in \bigcup_{k=1}^n \Gamma_{z_0,\varphi_k}$.

It follows from the classical Phragmen–Lindelöf principle (see, e.g., [5]) that condition (2.9) implies that the function $W = W(z)$ is a generalized polynomial, which is impossible since the rating of this function is $\varrho > 0$. □

Note that if condition (2.7) is not fulfilled, then the conclusion of the theorem is generally not valid. Indeed, consider the entire functions

$$\Phi_1(z) = \sin z, \quad \Phi_2(z) = \cos z$$

and construct the corresponding (see formula (2.8)) generalized entire functions $W_1 = W_1(z)$ and $W_2 = W_2(z)$. It is evident that

$$W_1, \ W_2 \in \mathbb{A}_1(a,b),$$

moreover, the rays $\Gamma_{0,0}$ and $\Gamma_{0,\pi}$ (positive and negative parts of the abscissa axis) are not only $\beta$-canonical, but they are not even $\alpha$-canonical. Quite similarly, the generalized entire function $W_3$

corresponding to the function $\Phi_3 = e^z$ is bounded on the $\Gamma_{0,\pi/2}$, $\Gamma_{0,3\pi/2}$ rays (the upper and lower rays of the ordinate axis).

The classes of generalized analytic functions

$$\mathbb{A}_\varrho(a,b), \quad 0<\varrho<\frac{1}{2}$$

are very interesting. For any $W = W(z)$ functions of these classes, every ray is $\beta$-canonical. Moreover, if the function $W \in A_0(a,b)$ and we additionally require that $W$ have an $\alpha$-canonical sequence (that is, be unbounded), then every ray of the plane $\mathbb{C}$ is an $\alpha$-canonical ray of the function $W = W(z)$. Theorem 2.1 and the results derived from it deal throughout with the generalized entire functions of finite rating (see (2.3)). As we have seen, their canonical rays are sufficiently densely (abundantly) located on the plane, and as is clear from the next theorem, this property is essentially a property of functions with finite rating. A generalized analytic function of infinite rating may have not only $\beta$-canonical, but also an $\alpha$-canonical ray.

**Theorem 2.2.** *For every point $z_0$ of the plane $\mathbb{C}_z$ there exists $W_* \in \mathbb{A}_\infty(a,b)$ such that for every number $\varphi \in \mathbb{R}$,*

$$\lim_{r\to+\infty} W_*\left(z_0 + re^{i\varphi}\right) = 0.$$

*Proof.* Consider the Mittag-Leffler entire function $E(z)$, which is obtained by the analytic extension of the integral

$$\frac{1}{2\pi i}\int_L \frac{\exp(\exp(\zeta))}{\zeta - z} d\zeta$$

(see, e.g., [2,3]).

Let us construct the entire function using the function $E = E(z)$

$$\Phi(z) = E(z)\exp\left(-E(z)\right).$$

This function is such that

$$\lim_{r\to\infty} \Phi\left(re^{i\varphi}\right) = 0$$

for any $\varphi \in \mathbb{R}$. Introducing the entire function

$$\Phi_*(z) = \Phi(z - z_0)$$

and using (2.8), we construct the generalized entire function $W_* = W_*(z)$. Thus the statement of the theorem is well-posed. ☐

The fact that the function $\zeta = |W_*(z)|$ is unbounded is most important and it should be taken into account. The function $W_*(z)$ of the mentioned type "lives" only in the class $\mathbb{A}_\infty(a,b)$; for any number $\varrho \in [0,+\infty)$ in the class of functions $\mathbb{A}_\varrho(a,b)$ there is no function of type $W_*(z)$.

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