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ON THE PROXIMITY OF SEQUENCES OF DISTRIBUTIONS OF SUMS OF INDEPENDENT RANDOM VARIABLES (IN A SERIES SCHEME)

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Abstract. Many problems in probability theory and mathematical statistics are reduced to the summation of independent random variables. In turn, the latter is closely connected with the study of sequences of distribution functions for sums of independent random variables. It is well known that the distribution function of the sum of independent random variables is a convolution (composition) of the distribution functions of the summands. It is natural to consider sequences of distribution functions $\{F_n, n \geq 1\}$ and $\{G_n, n \geq 1\}$ to be close if $F_n - G_n \to 0$ properly or weakly. In the proposed work, using modifications of the Rotar numerical characteristic introduced by the author of the paper, we study problems related to the proximity of sequences of convolutions of distributions $\{F_n\}$ and $\{G_n\}$ generated by the corresponding sums of independent random variables forming "a triangular array".

1. Numerical Characteristics of Sequences of Convolutions

The results of the theory of weak convergence related to the validity of the Central Limit Theorem for sequences of independent random variables are given in the monographs [2, 12, 14] ([2, Chapter 6, §1, §2, §3, p. 14–29]; [14, Chapter III, §1, §2, p. 427–437]; [12, Chapter 2, §8, p. 163–181]). In this connection, the numerical Lindeberg and Rotar characteristics, given in these books, play an essential role. In [7, 10], general generalized variants of the numerical Lindeberg and Rotar characteristics are proposed, which are used in proving the results of works [7, 9–11]. In [16] and [1], Ch. Stein created a "powerful method" of proof in limit theorems on the convergence of the distribution of sums of independent and weakly dependent random variables to the normal distribution law. This method is based on one characteristic property of the normal distribution, discovered by Ch. Stein, which is defined in terms of distribution functions. The main results of works [3, 4, 6, 8] are established by direct application of the Stein method. In [4] the Stein method is modified using the characterization property of the normal distribution law in terms of characteristic functions (the Stein-Tikhomirov method). This method is used to prove the results of the works [13, 15]. In [5], the problems on the validity of the strong law of large numbers for sequences of sums of independent random variables are investigated.

Let in a probability space $(\Omega, \mathcal{F}, P)$, for each $n \geq 1$, be given two sequences of random variables (r. v.'s)

$$X_{n1}, X_{n2}, \ldots, X_{nn}, \tag{1.1}$$

$$Y_{n1}, Y_{n2}, \ldots, Y_{nn} \tag{1.2}$$

with the corresponding distribution functions (d. f.'s)

$$F_{nj}(x) = P\left(X_{nj} < x\right), \quad G_{nj}(x) = P\left(Y_{nj} < x\right), \ \ j = 1, 2, \ldots, n.$$

In most cases, it is said that the sequences of independent r. v.'s (1.1) and (1.2) forming "triangular arrays" are given (see [2, Chapter 8, §3, p. 157], [14, Chapter III, §5, p. 463], [12, Chapter 4, §16, p. 266]).

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Recall that the convolution (or composition) of d. f's $F(x)$ and $G(x)$ is the distribution function

$$(F*G)(x)=\int_{-\infty}^{\infty}F(x-u)dG(u)=F*G=\int_{-\infty}^{\infty}G(x-u)dF(u)=(G*F)(x)=G*F.$$

Then by virtue of the last equalities,

$$\overline{F}_n(x)=P\left(X_{n1}+\ldots+X_{nn}<x\right)=\left(\prod_{j=1}^{n}(*)F_{nj}\right)(x)$$
$$=\left(F_{n1}*\ldots*F_{nn}\right)(x)=F_{n1}*\ldots*F_{nn}=\overline{F}_n,$$
$$\overline{G}_n(x)=P\left(Y_{n1}+\ldots+Y_{nn}<x\right)=\left(\prod_{j=1}^{n}(*)G_{nj}\right)(x)$$
$$=\left(G_{n1}*\ldots*G_{nn}\right)(x)=G_{n1}*\ldots*G_{nn}=\overline{G}_n.$$

Distributions $\overline{F}_n$ and $\overline{G}_n$ are called n-fold convolutions of sequences $\{F_{nj},1\le j\le n\}$ and $\{G_{nj},1\le j\le n\}$, respectively. The terms of these sequences are called components of n-fold convolutions $\overline{F}_n$ and $\overline{G}_n$, respectively.

Next, we put

$$S_n^X=X_{n1}+\cdots+X_{nn},\quad S_n^Y=Y_{n1}+\cdots+Y_{nn}$$

and suppose that r. v.'s included in triangular arrays (1.1) and (1.2) satisfy the conditions

$$EX_{nj}=EY_{nj}=0,\ \ \sigma_{nj}^2=EX_{nj}^2=\int_{-\infty}^{\infty}x^2dF_{nj}(x)=EY_{nj}^2$$
$$=\int_{-\infty}^{\infty}x^2dG_{nj}(x)<\infty,\ \ j=1,2,\ldots,n. \tag{1.3}$$

Under conditions (1.3), without any loss of generality, we may assume that

$$DS_n^X=DS_n^Y=\sum_{j=1}^{n}\sigma_{nj}^2=1.$$

The last equalities implies that

$$\int_{-\infty}^{\infty}x^2d\overline{F}_n(x)=\int_{-\infty}^{\infty}x^2d\overline{G}_n(x)=1.$$

Following the work by V. I. Rotar [12, Chapter 4, §16.5, p. 271], we introduce the numerical characteristic

$$R_n(\varepsilon)=\sum_{j=1}^{n}\int_{|x|>\varepsilon}|x|\left|F_{nj}(x)-G_{nj}(x)\right|dx,\ \ \varepsilon>0.$$

The limit relation

$$R_n(\varepsilon)\to 0,\quad n\to\infty \tag{1.4}$$

for any $\varepsilon>0$ is said to be the Rotar condition. In [10], the authors proved that the Rotar numerical characteristic can be written in a more general form. To do this, they introduced a class $\mathbf{B}$ of bounded nonnegative functions $b(x)$ on the real line $\mathbb{R}$ such that

$$\lim_{x\to 0}b(x)=0,\quad m_b(\delta)=\inf_{|x|>\delta}b(x)>0$$

for all $\delta>0$. Now, set

$$R_n^b=\sum_{j=1}^{n}\int_{-\infty}^{\infty}|x|\left|F_{nj}(x)-G_{nj}(x)\right|b(x)dx, \tag{1.5}$$

where the function $b(x)$ belongs to the class $\mathbf{B}$.

In particular, when

$$b(x) = \begin{cases} 1 & \text{at } |x| > \varepsilon, \\ 0 & \text{at } |x| \le \varepsilon, \end{cases}$$

we have $R_n^b = R_n(\varepsilon)$.

Proposition A. *The following three conditions are equivalent:*

a) *the Rotar condition* (1.4) *is satisfied;*

b) $\lim\limits_{n\to\infty} R_n^b = 0$ *for some* $b(\cdot) \in \mathbf{B}$;

c) $\lim\limits_{n\to\infty} R_n^b = 0$ *for all* $b(\cdot) \in \mathbf{B}$.

We give one modification of the numerical Rotar characteristic $R_n(\varepsilon)$. To do this, we put $b(x) = \min(|x|, 1)$ in formula (1.5). It is easy to check that $b(x) \in \mathbf{B}$. Then

$$D_n = \sum_{j=1}^{n} \int_{-\infty}^{\infty} |x|\,|F_{nj}(x) - G_{nj}(x)| \min(|x|, 1)\,dx = \sum_{j=1}^{n} \int_{|x|\le 1} x^2\,|F_{nj}(x) - G_{nj}(x)|\,dx$$
$$+ \sum_{j=1}^{n} \int_{|x|>1} |x|\,|F_{nj}(x) - G_{nj}(x)|\,dx = M_n + L_n.$$

The limit relation

$$D_n \to 0 \quad (M_n \to 0,\, L_n \to 0)\,, \quad n \to \infty \tag{1.6}$$

is said to be the condition (D). Similarly, the convergence

$$\sup_{b\in\mathbf{B}} R_n^b \to 0, \quad n \to \infty \tag{1.7}$$

is said to be the condition (R_b).

Theorem 1.1. *All three conditions* (1.7), (1.4), *and* (1.6) *are mutually equivalent.*

Proof. By virtue of Proposition A, conditions (1.7) and (1.4) are mutually equivalent. Therefore, it remains to prove the equivalence of conditions (1.4) and (1.6).

1) Let condition (1.4) be satisfied. Then

$$L_n = R_n(1) = \sum_{j=1}^{n} \int_{|x|>1} |x|\,|F_{nj}(x) - G_{nj}(x)|\,dx \to 0 \tag{1.8}$$

as $n \to \infty$. Further, for $0 < \varepsilon < 1$

$$M_n = \sum_{j=1}^{n} \int_{|x|\le 1} x^2\,|F_{nj}(x) - G_{nj}(x)|\,dx \le \varepsilon \Bigg[\sum_{j=1}^{n} \int_{|x|\le 1} |x|\,(1 - F_{nj}(x))\,dx$$
$$+ \sum_{j=1}^{n} \int_{|x|\le 1} |x|\,(1 - G_{nj}(x))\,dx \Bigg] + \sum_{j=1}^{n} \int_{\varepsilon<|x|\le 1} |x|\,|F_{nj}(x) - G_{nj}(x)|\,dx$$
$$\le 2\varepsilon \sum_{j=1}^{n} \sigma_{nj}^2 + \sum_{j=1}^{n} \int_{|x|>\varepsilon} |x|\,|F_{nj}(x) - G_{nj}(x)|\,dx \le 2\varepsilon + R_n(\varepsilon).$$

Therefore, we have the estimate

$$M_n \le 2\varepsilon + R_n(\varepsilon) \tag{1.9}$$

for any $0 < \varepsilon < 1$.

It should be noted that in proving estimate (1.9), we have used the equality

$$\int_{-\infty}^{0} |x|\, F(x)dx + \int_{0}^{\infty} x\,(1 - F(x))\, dx = \frac{1}{2} \int_{-\infty}^{\infty} x^2 dF(x),$$

which is valid for any d. f. $F(x)$.

Now estimate (1.9) implies

$$\limsup_{n\to\infty} M_n \le 2\varepsilon$$

for any $\varepsilon > 0$. Therefore

$$M_n \to 0, \quad n \to \infty, \tag{1.10}$$

and it follows from (1.8) and (1.10) that condition (1.6) is valid.

Now, let condition (1.6) be satisfied. Then

$$R_n(\varepsilon) = \sum_{j=1}^{n} \int_{|x|>\varepsilon} |x|\, |F_{nj}(x) - G_{nj}(x)|\, dx = \sum_{j=1}^{n} \int_{\varepsilon<|x|\le 1} |x|\, |F_{nj}(x) - G_{nj}(x)|\, dx$$

$$+ \sum_{j=1}^{n} \int_{|x|>1} |x|\, |F_{nj}(x) - G_{nj}(x)|\, dx \le \frac{1}{\varepsilon} \sum_{j=1}^{n} \int_{\varepsilon<|x|\le 1} x^2\, |F_{nj}(x) - G_{nj}(x)|\, dx + L_n$$

$$\le \frac{M_n}{\varepsilon} + L_n \to 0, \quad n \to \infty$$

for any $0 < \varepsilon < 1$. Therefore, the validity of the implication (1.6) $\to$ (1.4) is proved. Earlier we proved the implication (1.4) $\to$ (1.6). Thus Theorem 1.1 is proved. □

2. Definition of the Proximity of Sequences of Probability Distributions

Consider two sequences of probability distributions $\{F_n, n \ge 1\}$ and $\{G_n, n \ge 1\}$. Suppose that one needs to establish a weak convergence of $\{F_n, n \ge 1\}$ to a d. f. F, i.e., $F_n \Rightarrow F$. To achieve this, one often proceeds as follows: a sequence of probability distributions $\{G_n, n \ge 1\}$ is considered, for which the convergence $G_n \to F$ is almost obvious, and then the convergence $F_n - G_n \to 0$ is proved in a certain sense. The last problem (on the convergence $F_n - G_n \to 0$) is more general than the problem of the weak convergence of probability distributions, since $\{F_n, n \ge 1\}$ and $\{G_n, n \ge 1\}$ may be close even if the sequences $\{F_n\}$ and $\{G_n\}$ have no limit.

It is well-known that any d. f. $F(x)$ generates a probability measure $F(A)$ (A is a Borel set on $\mathbb{R}$) by the formula

$$F(A) = \int_{\mathbb{R}} I_A dF = \int_{A} 1dF = P(X \in A),$$

where $I_A(\cdot)$ is the indicator of the event A, X is an r. v. with the d. f. $F(x)$.

Recall that a set A is called F-continuous if $F\delta(A)) = 0$, where $\delta(A)$ is the topological boundary of A.

Let $F_n(\cdot)$ be a probability measure generated by the d. f. $F_n(x)$. Then the weak convergence $F_n \Rightarrow F$ is equivalent to the convergence

$$F_n(A) \to F(A) \tag{2.1}$$

for any F-continuous set A.

It is no longer easy to determine the convergence $F_n - G_n \to 0$ $(n \to \infty)$ by the conditions similar to condition (2.1), since it is not clear with respect to which distribution the set A should be declared a set of continuity. The most universal condition is based on the general form of the weak convergence of distributions.

Definition 2.1. The difference of distributions $F_n - G_n \to 0$ properly or weakly if

$$\int_{-\infty}^{\infty} u(x)d(F_n(x) - G_n(x)) \to 0$$

for any continuous bounded function $u(x)$.

This definition of weak convergence $F_n - G_n \to 0$ is borrowed from [12, Chapter 4, §8.5, p.152], and it generalizes the concept of weak (or proper) convergence of distributions $F_n \Rightarrow F$. The last is obtained from $F_n - G_n \to 0$, when

$$G_1 = G_2 = \cdots = G_n = F(x) = F,$$

i.e., when the sequence $\{G_n, n \geq 1\} = \{F, \ldots, F\}$ is stationary. It is easy to verify the existence of such a sequence by setting

$$G_1 = (F * E_0)(x) = \int_{-\infty}^{\infty} F(x-u)dE_0(u) = F(x),$$

where $E_0(x)$ is the unit distribution degenerate at zero.

Next, for $n \geq 2$, we put

$$G_2 = F * E_0 * E_0 = \cdots = F * E_{n-1} * E_0 = F * E_n = G_n = F,$$

where $E_n = \underbrace{E_0 * \cdots * E_0}_{n} = E_0$.

Therefore, from the above, taking into account Definition 2.1, we can come to the following conclusion: if the sequence of convolutions $\{G_n, n \geq 1\}$ consists of the same distribution $F(x)$ (it means that $G_n = \{F, \ldots, F\}$), then the proper or weak convergence of the difference $F_n - G_n \to 0$ turns into a weak convergence of the sequence of d. f.'s $\{F_n, n \geq 1\}$ to the d. f. $F(x)$:

$$F_n \Rightarrow F, \quad n \to \infty. \tag{2.2}$$

But the function $F(x)$ in the limit relation (2.2) may not be a probability d.f. (see [12, Chapter 3, §8.3, Examples 6 and 7, p.150]). To exclude such situations, the following definition is introduced.

Definition 2.2. A sequence of random variables $\{X_n, n \geq 1\}$ is called stochastically bounded, and the corresponding sequence of distribution functions $\{X_n, n \geq 1\}$ is called dense if the relation

$$\lim_{N\to\infty} P(|X_n| > N) = \lim_{N\to\infty} [1 - F_n(N) + F_n(-N)]$$

holds uniformly in n (see [12, Chapter 3, §8.3, Examples 6 and 7, p.150–151]).

3. Limit Distributions for Convolutions of Distributions

The asymptotic closeness of the convolutions of distributions $\overline{F}_n = \prod_{j=1}^{n}(*)F_{nj}$ and $\overline{G}_n = \prod_{j=1}^{n}(*)G_{nj}$ can be studied independently of the sequences of random variables (1.1) and (1.2) (directly), as convolutions of the distributions $\{F_{nj}, 1 \leq j \leq n\}$ and $\{G_{nj}, 1 \leq j \leq n\}$. First of all, we introduce the necessary notations for the characteristic functions (ch. f.'s) of random variables included in the "triangular arrays" (1.1) and (1.2). Let

$$f_{nj}(t) = \int_{-\infty}^{\infty} e^{itx}dF_{nj}(x), \quad g_{nj}(t) = \int_{-\infty}^{\infty} e^{itx}dG_{nj}(x),$$

where the index $j = 1, 2, \dots, n$. Then the characteristic functions of the sums S_n^X and S_n^Y of sequences of r. v.s (1.1) and (1.2)

$$\overline{f}_n(t) = Ee^{itS_n^X} = \int_{-\infty}^{\infty} e^{itx} d\overline{F}_n(x) = \prod_{j=1}^{n} f_{nj}(t),$$

$$\overline{g}_n(t) = Ee^{itS_n^Y} = \int_{-\infty}^{\infty} e^{itx} d\overline{G}_n(x) = \prod_{j=1}^{n} g_{nj}(t).$$

In [12, Chapter 4, §15.7, p.265], the following generalized version of the continuity (convergence) theorem is given.

Theorem C. *Let one of the sequences of convolutions* $\left\{\overline{F}_n, n \geq 1\right\}$ *and* $\left\{\overline{G}_n, n \geq 1\right\}$ *be dense. Suppose that for each* t*,*

$$\overline{f}_n(t) - \overline{g}_n(t) \to 0, \;\; n \to \infty. \tag{3.1}$$

Then $\overline{F}_n - \overline{G}_n \to 0$ *properly, and both sequences are dense. If the sequence* $\left\{\overline{G}_n, n \geq 1\right\}$ *consists of a single distribution* F $(G = \{F, \dots, F\})$*, then the convergence* $\overline{F}_n - \overline{G}_n \to 0$ *turns properly into a limit theorem*

$$\overline{F}_n(x) \to F(x) \;\; \left(\overline{F}_n \Rightarrow F\right), \;\; n \to \infty,$$

at each point of continuity of the function $F(x)$*.*

Now, we are able to prove the following

Theorem 3.1. *Let r. v.'s of the "triangular arrays"* (1.1) *and* (1.2) *be satisfied conditions* (1.3) *and* (1.6)*. Then for any continuous and bounded function* $u(x)$*,*

$$\int_{-\infty}^{\infty} u(x) d\left(\overline{F}_n(x) - \overline{G}_n(x)\right) \to 0, \;\; n \to \infty,$$

i.e., $\overline{F}_n - \overline{G}_n \to 0$ *properly as* $n \to \infty$*.*

Proof. By virtue of condition (1.3),

$$DS_n^X = DS_n^Y = E\left(S_n^X\right)^2 = E\left(S_n^Y\right)^2 = \int_{-\infty}^{\infty} x^2 d\overline{F}_n(x) = \int_{-\infty}^{\infty} x^2 d\overline{G}_n(x) = \sum_{j=1}^{n} \sigma_{nj}^2 = 1.$$

Using the Chebyshev inequality, from the last equalities, we obtain

$$\int_{|x|>N} d\overline{F}_n(x) = 1 - F_n(N) + F_n(-N) \leq \frac{1}{N^2} \cdot \sum_{j=1}^{n} \sigma_{nj}^2 = \frac{1}{N^2},$$

$$\int_{|x|>N} d\overline{G}_n(x) = 1 - G_n(N) + G_n(-N) \leq \frac{1}{N^2} \cdot \sum_{j=1}^{n} \sigma_{nj}^2 = \frac{1}{N^2}.$$

Therefore

$$\begin{aligned} &\lim_{N\to\infty} \int_{|x|>N} d\overline{F}_n(x) = 0, \\ &\lim_{N\to\infty} \int_{|x|>N} d\overline{G}_n(x) = 0 \end{aligned} \tag{3.2}$$

uniformly in n. These equalities (3.2) imply that the sequences of the sums $\left\{S_n^X, n \geq 1\right\}$ and $\left\{S_n^Y, n \geq 1\right\}$ are stochastically bounded, and the corresponding sequences of d. f.'s $\left\{\overline{F}_n, n \geq 1\right\}$ and $\left\{\overline{G}_n, n \geq 1\right\}$

are dense. Taking into account these considerations, we can come to the following conclusion: to prove Theorem 3.1, it suffices to check the validity of condition (3.1) of Theorem 3. First, we prove some auxiliary statements. □

Let the numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be such that

$$\max_j \left(|a_j, |, |b_j|\right) \leq 1.$$

Using the method of induction for $n \geq 2$, one can verify the validity of the equality

$$\left|\prod_{j=1}^{n} a_j - \prod_{j=1}^{n} b_j\right| = \left|\sum_{k=1}^{n} \prod_{j=1}^{k-1} a_j (a_k - b_k) \prod_{j=k+1}^{n} b_j\right|$$

in which

$$\prod_{j=1}^{0} a_j = \prod_{j=n+1}^{n} b_j = 1$$

is considered. Now, using formula (22) and Lemma 3 given in the book by A. A. Borovkov (Chapter 7, §6, p. 143) we obtain

$$\left|\prod_{j=1}^{n} a_j - \prod_{j=1}^{n} b_j\right| \leq \sum_{j=1}^{n} |a_j - b_j|.$$

Thus, if $f_{nj}(t)$ and $g_{nj}(t)$ are characteristic functions, then from the last inequality we obtain

$$\left|\prod_{j=1}^{n} f_{nj}(t) - \prod_{j=1}^{n} g_{nj}(t)\right| \leq \sum_{j=1}^{n} |f_{nj}(t) - g_{nj}(t)| \tag{3.3}$$

for any $t \in \mathbb{R}$.

Lemma 3.1. *If condition* (1.3) *is satisfied, then the estimate*

$$\sum_{j=1}^{n} |f_{nj}(t) - g_{nj}(t)| \leq 4 \max\left(t^2, |t|^3\right) D_n$$

holds for any $t \in \mathbb{R}$.

Proof. By virtue of condition (1.3), it can be seen that

$$\sum_{j=1}^{n} |f_{nj}(t) - g_{nj}(t)| = \sum_{j=1}^{n} \left| \int_{-\infty}^{\infty} e^{itx} d\left(F_{nj} - G_{nj}\right) \right|$$

$$= \sum_{j=1}^{n} \left| \int_{-\infty}^{\infty} \left(e^{itx} - 1 - itx - \frac{(itx)^2}{2}\right) d\left(F_{nj}(x) - G_{nj}(x)\right) \right|.$$

After integrating by parts here and using the limit relations

$$\lim_{x\to\infty} x\left(1 - F_{nj}(x) + F_{nj}(-x)\right) = \lim_{x\to\infty} x\left(1 - G_{nj}(x) + G_{nj}(-x)\right) = 0,$$

that are valid for $j = 1, \ldots, n$, we obtain

$$\sum_{j=1}^{n} |f_{nj}(t) - g_{nj}(t)| = \sum_{j=1}^{n} \left| t \int_{-\infty}^{\infty} \left(e^{itx} - 1 - itx\right) \left(F_{nj}(x) - G_{nj}(x)\right) dx \right|$$

$$\leq \frac{|t|^3}{2} \sum_{j=1}^{n} \int_{|x|\leq 1} x^2 \left|F_{nj}(x) - G_{nj}(x)\right| dx + 2t^2 \sum_{j=1}^{n} \int_{|x|>1} |x| \left|F_{nj}(x) - G_{nj}(x)\right| dx$$

$$= \frac{|t|^3}{2} M_n + 2t^2 L_n \leq 2\left(t^2 + |t|^3\right)\left(M_n + L_n\right) \leq 4 \max\left(t^2, |t|^3\right) D_n. \tag{3.4}$$

The chain of inequalities (3.4) proves the validity of Lemma 1. □

Now, the proof of Theorem 3.1 follows easily from Lemma 2 and from equation (3.3). From equation (3.3), we have

$$\left|\overline{f}_n(t) - \overline{g}_n(t)\right| = \left|\prod_{j=1}^{n} f_{nj}(t) - \prod_{j=1}^{n} g_{nj}(t)\right| \leq \sum_{j=1}^{n} \left|f_{nj}(t) - g_{nj}(t)\right|. \tag{3.5}$$

Therefore, by Lemma 1 (estimate (3.5)),

$$\left|\overline{f}_n(t) - \overline{g}_n(t)\right| \leq 4 \max\left(t^2, |t|^3\right) D_n \to 0, \tag{3.6}$$

as $n \to \infty$ for any $t \in \mathbb{R}$.

Thus, by virtue of relations (3.2) and (3.6), all the conditions of Theorem 3 are satisfied, and by the conclusion of this theorem,

$$\overline{F}_n - \overline{G}_n \to 0$$

properly, as $n \to \infty$. Theorem 3.1 is proved.

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