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DEFERRED STATISTICAL CONVERGENCE AND DEFERRED STRONG *P*-CESÀRO SUMMABILITY ON TIME SCALES

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Abstract. The concepts of deferred statistical convergence and deferred strong p-Cesàro summability of Δ-measurable real valued functions on an arbitrary time scale are introduced via a pair of increasing functions in the interval $[1, \infty)$. The relationships between the sets of deferred statistical convergence and the set of deferred strong p-Cesàro summability on time scales are investigated under certain conditions. Many inclusion theorems relating these concepts are established. The obtained results are expected to contribute to viewing deferred statistical convergence as a type of deferred summability method in the time scale framework.

1. Introduction

The idea of statistical convergence was studied by Zygmund [20] in 1935. Later statistical convergence of number sequences was formally introduced by Fast [9] and Steinhaus [16] independently in 1951. Agnew [1] defined deferred Cesàro mean in 1932. In connection with the present paper, some works on statistical convergence can be found in [7, 10, 11], and there are many interesting works on statistical convergence and statistical summability in various directions.

A time scale is an arbitrary non-empty closed subset of the real numbers. It is denoted by the symbol $\mathbb{T}$. The time scale calculus was introduced by Stefan Hilger in his PhD thesis supervised by Bernd Aulbach in 1988 [13, 14]. It allowed to unify discrete and continuous analysis. There are many applications of time scales in dynamic equations [4]. The notion of statistical convergence on time scales was first studied in [15] and [17] independently. Turan and Duman introduced the notion of lacunary statistical convergence on time scales in [19]. Seyyidoglu and Tan [15] put forwarded the notions of Δ-convergence and Δ-Cauchy sequences using Δ-density and investigated their relations. Several other studies on time scale calculus have been presented in [2, 3, 5, 6, 12, 18] and many others.

We first discuss some important terms and notions on time scales [12]:

For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is given by

$$\sigma(t) = \inf \{s \in \mathbb{T} : s > t\},$$

the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is given by

$$\rho(t) = \sup \{s \in \mathbb{T} : s < t\},$$

and the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is given by

$$\mu(t) = \sigma(t) - t.$$

Here, we put $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$, if $\mathbb{T}$ has a maximum t) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$, if $\mathbb{T}$ has a minimum t), where $\emptyset$ is the empty set.

A closed interval, open interval and semi-closed (or semi-open) interval on a time scale $\mathbb{T}$ are given by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$, $(a, b)_{\mathbb{T}} = \{t \in \mathbb{T} : a < t < b\}$ and $[a, b)_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t < b\}$, respectively.

Next, let S be the collection all left closed and right open intervals of the form $[a, b)_{\mathbb{T}}$. Then the set function $m : S \to [0, \infty)$ defined by $m([a, b)) = b - a$ is a countably additive measure. An outer

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measure $m^* : P(\mathbb{T}) \to [0,\infty)$ generated by m is defined as follows:

$$m^*(A) = \inf\left\{\sum_{n=1}^{\infty} m(A_n) : (A_n) \text{ is a sequence of } S \text{ with } A \subset \bigcup_{n=1}^{\infty} A_n\right\}.$$

If there exists no sequence (A_n) of S such that $A \subset \bigcup_{n=1}^{\infty} A_n$, then we consider $m^*(A) = \infty$. Now, the family $M(m^*)$ of all m^*-measurable (or Δ-measurable) subsets of $\mathbb{T}$ is considered, i.e.,

$$M(m^*) = \{E \subset \mathbb{T} \ : \ m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \text{ for all } A \subset \mathbb{T}\}.$$

The collection $M(m^*)$ of all m^*-measurable subsets of $\mathbb{T}$ is a σ-algebra and the restriction of m^* to $M(m^*)$ is a countable additive measure on $M(m^*)$, which is denoted by μ_Δ. This measure μ_Δ, which is the Carathéodory extension of the set function m associated with the family S, is called the Lebesgue Δ-measure on $\mathbb{T}$ [12].

We say that a function $f : \mathbb{T} \to \mathbb{R}$ is Δ-measurable if the set $f^{-1}(A)$ is Δ-measurable for every open subset A of $\mathbb{R}$.

Theorem 1.1 ([12]). *For each $a \in \mathbb{T} - \{\max \mathbb{T}\}$, the singleton point set $\{a\}$ is Δ-measurable, and its Δ-measure is given by*

$$\mu_\Delta(a) = \sigma(a) - a.$$

Theorem 1.2 ([12]). *If $a,\ b \in \mathbb{T}$ and $a \le b$, then*

$$\mu_\Delta([a,b)) = b - a, \text{ and } \mu_\Delta((a,b)) = b - \sigma(a).$$

If $a, b \in \mathbb{T} - \{\max \mathbb{T}\}$ and $a \le b$, then

$$\mu_\Delta((a,b]) = \sigma(b) - \sigma(a) \text{ and } \mu_\Delta([a,b]) = \sigma(b) - a.$$

Definition 1.1 ([17]). Let Ω be a Δ-measurable subset of $\mathbb{T}$. Then for $t \in \mathbb{T}$, we define the set $\Omega(t)$ by

$$\Omega(t) = \{s \in [t_o, t]_{\mathbb{T}} : s \in \Omega\}.$$

The density of the set Ω on $\mathbb{T}$, denoted by $\delta_{\mathbb{T}}(\Omega)$, is defined as

$$\delta_{\mathbb{T}}(\Omega) = \lim_{t\to\infty} \frac{\mu_\Delta(\Omega(t))}{\mu_\Delta([t_0, t]_{\mathbb{T}})},$$

provided the above limit exists.

Here, if $\mathbb{T} = \mathbb{N}$, then the concept reduces to asymptotic density (or natural density) and if $\mathbb{T} = [0,\infty)$, then the concept implies approximate density. In this paper, we shall mainly use the Lebesgue Δ-measure μ_Δ introduced by Guseinov in [12]. Here, $\mathbb{T}$ is a time scale satisfying $\inf \mathbb{T} = t_0 > 0$ and $\sup \mathbb{T} = \infty$.

Definition 1.2 ([17]). Let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. Then f is said to be statistically convergent on $\mathbb{T}$ to a real number L if for every $\varepsilon > 0$,

$$\lim_{t\to\infty} \frac{\mu_\Delta(\{s \in [t_0, t]_{\mathbb{T}} : |f(s) - L| \ge \varepsilon\})}{\mu_\Delta([t_0, t]_{\mathbb{T}})} = 0.$$

Definition 1.3 ([17]). Let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function and $0 < p < \infty$. Then f is said to be strongly p-Cesàro summable on time scale $\mathbb{T}$ if there exists $L \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{1}{\mu_\Delta([t_0, t]_{\mathbb{T}})} \int_{[t_0,t]_{\mathbb{T}}} |f(s) - L|^p \Delta s = 0.$$

Let $\theta = (k_r)$ be an increasing sequence of non-negative numbers with $k_0 = 0$ and $\sigma(k_r) - \sigma(k_{r-1}) \to \infty$ as $r \to \infty$, where $\sigma : \mathbb{R} \to \mathbb{T}$ is the forward jump operator defined as $\sigma(s) = \inf\{t \in \mathbb{T} : t > s\}$. Then θ is called a lacunary sequence with respect to $\mathbb{T}$ [17]. Using this definition of lacunary sequence on $\mathbb{T}$, the following notions have been defined.

Definition 1.4 ([19])**.** Let $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$. A Δ-measurable function $f : \mathbb{T} \to \mathbb{R}$ is said to be lacunary statistical convergent to a number L if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{\mu_\Delta(\{s \in (k_{r-1}, k_r]_\mathbb{T} : |f(s) - L| \geq \varepsilon\})}{\mu_\Delta((k_{r-1}, k_r]_\mathbb{T})} = 0.$$

Definition 1.5 ([19])**.** Let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function and $\theta = (k_r)$ be a lacunary sequence. Then f is said to be strongly lacunary Cesàro summable on time scale $\mathbb{T}$ if there exists $L \in \mathbb{R}$ such that

$$\lim_{r\to\infty} \frac{1}{\mu_\Delta((k_{r-1}, k_r]_\mathbb{T})} \int\limits_{(k_{r-1},k_r]_\mathbb{T}} |f(s) - L|\Delta s = 0.$$

2. Deferred Statistical Convergence and Deferred Strongly p-Cesàro Summability on Time Scales

In this section, we define two notions, namely, deferred OPF statistical convergence and strongly OPF deferred k-Cesàro summability on time scales and discuss some results.

Following [8], we consider a pair of functions (p, q), where $p, q : [1, \infty) \to [1, \infty)$ such that $p(u) < q(u)$ and $\lim_{u\to\infty} q(u) = \infty$. The set of all such ordered pairs of functions will be denoted by OPF, i.e.,

$$OPF = \left\{(p, q) : p, q \text{ are increasing functions, } p(u) < q(u) \text{ and} \lim_{u\to\infty} q(u) = \infty\right\}.$$

We shall also use $I_{p,q}(u) = (p(u), q(u)]_\mathbb{T}$.

Definition 2.1. Let $(p, q) \in OPF$ be an arbitrary pair of functions and $A \subset \mathbb{T}$. If the limit

$$\lim_{u\to\infty} \frac{\mu_\Delta\left(A \cap I_{p,q}\left(u\right)\right)}{\mu_\Delta\left(I_{p,q}\left(u\right)\right)}$$

or equivalently, the limit

$$\lim_{u\to\infty} \frac{\mu_\Delta\left(\{t \in (p(u), q(u)]_\mathbb{T} : t \in A\}\right)}{\mu_\Delta\left((p(u), q(u)]_\mathbb{T}\right)}$$

exists, then it is called deferred OPF density of the set A, denoted by $\delta_\mathbb{T}(A)$.

Let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function and $L \in \mathbb{R}$. For any $\varepsilon > 0$, we denote the sets as

$$A^f_\varepsilon = \{\ t \in \mathbb{T} : |f(t) - L| \geq \varepsilon\},$$

and

$$B^f_\varepsilon = \{\ t \in \mathbb{T} : |f(t) - L| < \varepsilon\}.$$

Then, clearly, $A^f_\varepsilon \cup B^f_\varepsilon = \mathbb{T}$ and $A^f_\varepsilon \cap B^f_\varepsilon = \emptyset$.

Definition 2.2. Let $(p, q) \in OPF$ be an arbitrary pair of functions and $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. Then f is said to be deferred OPF statistical convergent to a real number L if for each $\varepsilon > 0$,

$$\lim_{u\to\infty} \frac{\mu_\Delta\left(A^f_\varepsilon \cap I_{p,q}(u)\right)}{\mu_\Delta\left(I_{p,q}(u)\right)} = 0,$$

or equivalently,

$$\lim_{u\to\infty} \frac{\mu_\Delta\left(\{t \in (p(u), q(u)]_\mathbb{T}\ :\ |f(t) - L| \geq \varepsilon\}\right)}{\mu_\Delta\left((p(u), q(u)]_\mathbb{T}\right)} = 0.$$

We denote it by $[DS^{p,q}_\mathbb{T}]-\lim_{t\to\infty} f(t) = L$. The set of all deferred OPF statistical convergent functions on $\mathbb{T}$ we denote by $[DS^{p,q}_\mathbb{T}]$.

Remark 2.1. (i) Taking $p, q : \mathbb{N} \to \mathbb{N}$ instead of $p, q : [1, \infty) \to [1, \infty)$, we get the definition for deferred statistical convergence introduced in [6].

(ii) If $p\left(u\right) = t_0 \geq 1,\ \ q\left(u\right) = t$ in Definition 2.2, then we get statistical convergence on $\mathbb{T}$ [17].

(iii) If $p\left(u\right) = k_{r-1},\ q(u) = k_r$ in Definition 2.2, where, (k_r) is a lacunary sequence, then we get lacunary statistical convergence on $\mathbb{T}$ defined in [19].

Definition 2.3. Let $(p,q) \in OPF$ be an arbitrary pair of functions and $0 < k < \infty$. Then a Δ-measurable function $f : \mathbb{T} \to \mathbb{R}$ is said to be strongly OPF deferred k-Cesàro summable to a real number L if

$$\lim_{u\to\infty} \frac{1}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)} \int\limits_{(p(u),q(u)]_{\mathbb{T}}} |f(s)-L|^k \Delta s = 0.$$

We denote it by $[DW^{p,q}_{\mathbb{T}}] - \lim\limits_{s\to\infty} f(s) = L$. The set of all strongly OPF deferred k-Cesàro summable functions on $\mathbb{T}$ we denote by $[DW^{p,q}_{\mathbb{T}}]$.

Remark 2.2. (i) Here we use strongly OPF deferred k-Cesàro summable functions in place of strongly OPF deferred p-Cesàro summable functions to avoid confusion with the use of OPF function (p,q).

(ii) If we take $p,q : \mathbb{N} \to \mathbb{N}$ instead of $p,q : [1,\infty) \to [1,\infty)$ and $k = 1$, then we get strongly deferred Cesàro summable functions introduced in [6].

(iii) If $p(u) = t_0 \geq 1, q(u) = t$, then we get strongly k-Cesàro summable functions on $\mathbb{T}$ [17].

(iv) If $p(u) = k_{r-1}, q(u) = k_r$, where (k_r) is a lacunary sequence, then we get strongly lacunary Cesàro summable functions on $\mathbb{T}$ defined in [18].

Definition 2.4. Let $(p,q) \in OPF$ be an arbitrary pair of functions and $f,\ g : \mathbb{T} \to \mathbb{R}$ be Δ-measurable functions. If the set $M = \{\, t \in \mathbb{T} : f(t) \neq g(t)\}$ has deferred OPF density 0 (zero), then f and g are called equivalent functions with respect to (p,q), denoted by $f \sim g$ (w.r.t. (p,q)).

Theorem 2.1. *Let $(p,q) \in OPF$ and $f \sim g$ (w.r.t. (p,q)). Then $f \in [DS^{p,q}_{\mathbb{T}}]$ if and only if $g \in [DS^{p,q}_{\mathbb{T}}]$.*

Proof. We assume that $f \in [DS^{p,q}_{\mathbb{T}}]$ and using the fact that $f \sim g$ (w.r.t. (p,q)), the set $M = \{t \in \mathbb{T} : f(t) \neq g(t)\}$ has zero deferred OPF density. Considering

$$A^g_\varepsilon = \{t \in \mathbb{T} : |g(t) - L| \geq \varepsilon\},$$

and $I_{p,q}(u) = (p(u), q(u)]_{\mathbb{T}}$, we have

$$A^g_\varepsilon \cap I_{p,q}(u) = (A^g_\varepsilon \cap M_{p,q}) \cup \left(A^g_\varepsilon \cap M^c_{p,q}\right),$$

where $M_{p,q} = I_{p,q}(u) \cap M$ and $M^c_{p,q} = I_{p,q}(u) \cap M^c$. Then we obtain

$$A^g_\varepsilon \cap I_{p,q}(u) \subset (A^g_\varepsilon \cap M_{p,q}) \cup \left(A^f_\varepsilon \cap I_{p,q}(u)\right).$$

This implies

$$\frac{\mu_\Delta\left(A^g_\varepsilon \cap I_{p,q}(u)\right)}{\mu_\Delta\left(I_{p,q}(u)\right)} \leq \frac{\mu_\Delta\left(A^g_\varepsilon \cap M_{p,q}\right)}{\mu_\Delta\left(I_{p,q}(u)\right)} + \frac{\mu_\Delta\left(A^f_\varepsilon \cap I_{p,q}(u)\right)}{\mu_\Delta\left(I_{p,q}(u)\right)}.$$

Taking $u \to \infty$, we get $g \in [DS^{p,q}_{\mathbb{T}}]$. The converse can also be proved in a similar manner. □

Theorem 2.2. *Let $(p,q) \in OPF$ and $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. Then $[DS^{p,q}_{\mathbb{T}}] - \lim\limits_{t\to\infty} f(t) = L$ if and only if there exists a Δ-measurable set $K \subset \mathbb{T}$ with deferred OPF density 1 and $\lim\limits_{t\to\infty (t\in K)} f(t) = L$.*

Proof. We consider the sets

$$K_j = \left\{t \in (p(u), q(u)]_{\mathbb{T}} : |f(t) - L| < \frac{1}{j}\right\}, \quad j = 1,\ 2,\ 3,\ \ldots .$$

From the hypothesis, we get $\delta_{\mathbb{T}}(K_j) = 1$, for each $j \in \mathbb{N}$. Also, from the construction of the sets, it can be seen that for all $j \in \mathbb{N}$, $K_{j+1} \subset K_j$ holds.

For $j = 1$, we can choose $t_1 \in K_1$. Since $\delta_{\mathbb{T}}(K_1) = 1$, there exists $t_2 \in K_2$ with $t_2 > t_1$ such that $\frac{\mu_\Delta(K_2(t))}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)} > \frac{1}{2}$ holds for each $t \geq t_2$ with $t \in \mathbb{T}$.

For $t_2 \in K_2$ and since $\delta_{\mathbb{T}}(K_2) = 1$, there exists $t_3 \in K_3$ with $t_3 > t_2$ such that $\frac{\mu_\Delta(K_3(t))}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)} > \frac{2}{3}$ holds for each $t \geq t_3$ with $t \in \mathbb{T}$.

If we continue this repeatedly, we get an increasing sequence $t_1 < t_2 < t_3 < \cdots$ such that $t_j \in K_j$ and for each $t \geq t_j$ and $t \in \mathbb{T}$,

$$\frac{\mu_\Delta(K_j(t))}{\mu_\Delta\left((p(u), q(u)]_\mathbb{T}\right)} > \frac{j-1}{j}, \text{ where } j \in \mathbb{N},$$

and $K_j(t) = \{s \in (p(u), t]_\mathbb{T} : s \in K_j\}$.
Using the sets K_j, we can construct a set K in the following manner:

If $t \in (p(u), t_1]_\mathbb{T}$, then $t \in K$.

If $t \in K_j \cap [t_1, t_{j+1}]_\mathbb{T}$, for $j = 1, 2, 3, \ldots$, then $t \in K$.

Hence, we obtain the set K as

$$K = \left\{t \in (p(u), q(u)]_\mathbb{T} : t \in (p(u), t_1]_\mathbb{T} \text{ or } t \in K_j \cap [t_1, t_{j+1}]_\mathbb{T}, \ j = 1, 2, 3, \ldots\right\}.$$

So,

$$\frac{\mu_\Delta(K(t))}{\mu_\Delta\left((p(u), q(u)]_\mathbb{T}\right)} \geq \frac{\mu_\Delta(K_j(t))}{\mu_\Delta\left((p(u), q(u)]_\mathbb{T}\right)} \geq \frac{j-1}{j}$$

holds for each $t \in [t_1, t_{j+1}]_\mathbb{T}$, $j \in \mathbb{N}$. This gives $\delta_\mathbb{T}(K) = 1$. Next, we show that $\lim\limits_{t\to\infty(t\in K)} f(t) = L$. Let $\varepsilon > 0$, then $\exists\ j \in \mathbb{N}$ such that $\frac{1}{j} < \varepsilon$ holds. Let $t \geq t_j$ with $t \in K$. Then $\exists\ n \geq j$ such that $t \in [t_n, t_{n+1}]_\mathbb{T}$. From the construction of K, we see that $t \in K_n$ and so, $|f(t) - L| < \frac{1}{n} \leq \frac{1}{j} < \varepsilon$, i.e., $|f(t) - L| < \varepsilon$ for each $t \in K$ with $t \geq t_j$. This gives $\lim\limits_{t\to\infty(t\in K)} f(t) = L$.

Conversely, from the given condition, for a given $\varepsilon > 0$, there exists a number $t_* \in (p(u), q(u)]_\mathbb{T}$ such that for any $t \geq t_*$ with $t \in K$, we obtain $|f(t) - L| < \varepsilon$.
Hence, if we take $A(\varepsilon) = \{t \in (p(u), q(u)]_\mathbb{T} : |f(t) - L| \geq \varepsilon\}$ and $B = K \cap (t_*, q(u)]_\mathbb{T}$, then we get $A(\varepsilon) \subset (p(u), q(u)]_\mathbb{T} \backslash B$. Using the facts that

$$K = (K \cap (p(u), t_*]_\mathbb{T}) \cup B, \quad \delta_\mathbb{T}(K) = 1,$$

and $\delta_\mathbb{T}(K \cap (p(u), t_*]_\mathbb{T}) = 0$, for the boundedness, we have $\delta_\mathbb{T}(B) = 1$ and so, $\delta_\mathbb{T}(A(\varepsilon)) = 0$. This completes the proof. □

Theorem 2.3. *Let $(p, q) \in OPF$ and $f, g : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. Then the following statements hold:*

(i) *If $[DS_\mathbb{T}^{p,q}] - \lim\limits_{t\to\infty} f(t) = L$ and $c \in \mathbb{R}$, then $[DS_\mathbb{T}^{p,q}] - \lim\limits_{t\to\infty} cf(t) = cL$,*

(ii) *If $[DS_\mathbb{T}^{p,q}] - \lim\limits_{t\to\infty} f(t) = L_1$ and $[DS_\mathbb{T}^{p,q}] - \lim\limits_{t\to\infty} g(t) = L_2$, then*

$$[DS_\mathbb{T}^{p,q}] - \lim_{t\to\infty} [f(t) + g(t)] = L_1 + L_2,$$

(iii) *If $[DW_\mathbb{T}^{p,q}] - \lim\limits_{t\to\infty} f(t) = L$ and $c \in \mathbb{R}$, then $[DW_\mathbb{T}^{p,q}] - \lim\limits_{t\to\infty} cf(t) = cL$,*

(iv) *If $[DW_\mathbb{T}^{p,q}] - \lim\limits_{t\to\infty} f(t) = L_1$ and $[DW_\mathbb{T}^{p,q}] - \lim\limits_{t\to\infty} g(t) = L_2$, then*

$$[DW_\mathbb{T}^{p,q}] - \lim_{t\to\infty} [f(t) + g(t)] = L_1 + L_2.$$

Proof. (i) Since $[DS_\mathbb{T}^{p,q}] - \lim\limits_{t\to\infty} f(t) = L$, so,

$$\lim_{u\to\infty} \frac{\mu_\Delta\left(\{t \in (p(u), q(u)]_\mathbb{T} : |f(t) - L| \geq \varepsilon\}\right)}{\mu_\Delta\left((p(u), q(u)]_\mathbb{T}\right)} = 0.$$

Therefore

$$\lim_{u\to\infty} \frac{\mu_\Delta\left(\{t \in (p(u), q(u)]_\mathbb{T} : |cf(t) - cL| \geq \varepsilon\}\right)}{\mu_\Delta\left((p(u), q(u)]_\mathbb{T}\right)}$$
$$= \lim_{u\to\infty} \frac{\mu_\Delta\left(\left\{t \in (p(u), q(u)]_\mathbb{T} : |f(t) - L| \geq \frac{\varepsilon}{c}\right\}\right)}{\mu_\Delta\left((p(u), q(u)]_\mathbb{T}\right)}.$$

Considering $\frac{\varepsilon}{c} = \varepsilon'$, we get our desired result.

(ii) Since $[DS_{\mathbb{T}}^{p,q}]-\lim_{t\to\infty} f(t)=L_1$ and $[DS_{\mathbb{T}}^{p,q}]-\lim_{t\to\infty} g(t)=L_2$ so,

$$\lim_{u\to\infty}\frac{\mu_\Delta\left(\{t\in(p(u),q(u)]_{\mathbb{T}}:|f(t)-L_1|\geq\varepsilon\}\right)}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)}=0$$

and

$$\lim_{u\to\infty}\frac{\mu_\Delta\left(\{t\in(p(u),q(u)]_{\mathbb{T}}:|g(t)-L_2|\geq\varepsilon\}\right)}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)}=0,$$

respectively. Therefore

$$\begin{aligned}&\lim_{u\to\infty}\frac{\mu_\Delta\left(\{t\in(p(u),q(u)]_{\mathbb{T}}:|\{f(t)+g(t)\}-(L_1+L_2)|\geq\varepsilon\}\right)}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)}\\&=\lim_{u\to\infty}\frac{\mu_\Delta\left(\{t\in(p(u),q(u)]_{\mathbb{T}}:|\{f(t)-L_1\}+\{g(t)-L_2\}|\geq\varepsilon\}\right)}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)}\\&\leq\lim_{u\to\infty}\frac{\mu_\Delta\left(\{t\in(p(u),q(u)]_{\mathbb{T}}:|f(t)-L_1|\geq\varepsilon\}\right)}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)}\\&+\lim_{u\to\infty}\frac{\mu_\Delta\left(\{t\in(p(u),q(u)]_{\mathbb{T}}:|g(t)-L_2|\geq\varepsilon\}\right)}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)}.\end{aligned}$$

This completes the proof.

(iii) Since $[DW_{\mathbb{T}}^{p,q}]-\lim_{t\to\infty} f(t)\ =L$, so,

$$\lim_{u\to\infty}\frac{1}{\mu_\Delta\left((p\left(u\right),q(u)]_{\mathbb{T}}\right)}\int\limits_{(p(u),q(u)]_{\mathbb{T}}}\left|f\left(t\right)-L\right|^k\Delta t=0.$$

Therefore

$$\begin{aligned}&\lim_{u\to\infty}\frac{1}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)}\int\limits_{(p(u),q(u)]_{\mathbb{T}}}\left|cf(t)-cL\right|^k\Delta t\\&=\lim_{u\to\infty}\frac{|c|^k}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)}\int\limits_{(p(u),q(u)]_{\mathbb{T}}}\left|f(t)-L\right|^k\Delta t.\end{aligned}$$

This completes the proof.

(iv) Since $[DW_{\mathbb{T}}^{p,q}]-\lim_{t\to\infty} f(t)\ =L_1$ and $[DW_{\mathbb{T}}^{p,q}]-\lim_{t\to\infty} g(t)=L_2$, so,

$$\lim_{u\to\infty}\frac{1}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)}\int\limits_{(p(u),q(u)]_{\mathbb{T}}}\left|f(t)-L_1\right|^k\Delta t=0$$

and

$$\lim_{u\to\infty}\frac{1}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)}\int\limits_{(p(u),q(u)]_{\mathbb{T}}}\left|g(t)-L_2\right|^k\Delta t=0,$$

respectively. Therefore

$$\begin{aligned}&\lim_{u\to\infty}\frac{1}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)}\int\limits_{(p(u),q(u)]_{\mathbb{T}}}\left|\{f(t)+g(t)\}-\{L_1+L_2\}\right|^k\Delta t\\&\leq\lim_{u\to\infty}\frac{1}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)}\int\limits_{(p(u),q(u)]_{\mathbb{T}}}\left|f(t)-L_1\right|^k\Delta t\\&\quad+\lim_{u\to\infty}\frac{1}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)}\int\limits_{(p(u),q(u)]_{\mathbb{T}}}\left|g(t)-L_2\right|^k\Delta t.\end{aligned}$$

This completes the proof. □

Next, we need a lemma which gives the Markov inequality in deferred sense on time scales.

Lemma 2.1. *Let $(p,q) \in OPF$ and $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. For $\varepsilon > 0$, consider the set $K = \{s \in (p(u), q(u)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\}$, then*

$$\mu_\Delta(K) \leq \frac{1}{\varepsilon} \int_K |f(s) - L| \Delta s \leq \frac{1}{\varepsilon} \int_{(p(u),q(u)]_{\mathbb{T}}} |f(s) - L| \Delta s.$$

Proof. For all $s \in (p(u), q(u)]_{\mathbb{T}}$ and $\varepsilon > 0$, we have

$$0 \leq \varepsilon \chi_K(s) \leq |f(s) - L| \, \chi_K(s) \leq |f(s) - L| \, .$$

This gives

$$\varepsilon \int_K \Delta s \leq \int_K |f(s) - L| \Delta s \leq \int_{(p(u),q(u)]_{\mathbb{T}}} |f(s) - L| \Delta s.$$

So, we obtain

$$\varepsilon \; \mu_\Delta(K) \leq \int_K |f(s) - L| \Delta s \leq \int_{(p(u),q(u)]_{\mathbb{T}}} |f(s) - L| \Delta s.$$

This produces the required result. $\square$

Theorem 2.4. *Let $(p,q) \in OPF$ and $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function, $L \in \mathbb{R}$ and $0 < k < \infty$. Then*

(i) $[DW_{\mathbb{T}}^{p,q}] - \lim_{t\to\infty} f(t) = L$, *then* $[DS_{\mathbb{T}}^{p,q}] - \lim_{t\to\infty} f(t) = L$,

(ii) $[DS_{\mathbb{T}}^{p,q}] - \lim_{t\to\infty} f(t) \; = L$ *and f is a bounded function, then* $[DW_{\mathbb{T}}^{p,q}] - \lim_{t\to\infty} f(t) \; = L$.

Proof. (i) Suppose that $[DW_{\mathbb{T}}^{p,q}] - \lim_{t\to\infty} f(t) \; = L$. For a given $\varepsilon > 0$, we consider the set

$$K = \{s \in (p(u), q(u)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\} \, .$$

Then from Lemma 2.1, we get

$$\varepsilon^k \mu_\Delta(K) \leq \int_{(p(u),q(u)]_{\mathbb{T}}} |f(s) - L|^k \Delta s.$$

Dividing both sides of this inequality by $\mu_\Delta\left((p(u), q(u)]_{\mathbb{T}}\right)$ and taking limits as $u \to \infty$, we get

$$\lim_{u\to\infty} \frac{\mu_\Delta(K)}{\mu_\Delta\left((p(u), q(u)]_{\mathbb{T}}\right)} \; \leq \frac{1}{\varepsilon^k} \lim_{u\to\infty} \frac{1}{\mu_\Delta\left((p(u), q(u)]_{\mathbb{T}}\right)} \int_{(p(u),q(u)]_{\mathbb{T}}} |f(s) - L|^k \Delta s.$$

From the given assumption, the RHS of this inequality vanishes. So, we obtain

$$\lim_{u\to\infty} \frac{\mu_\Delta(K)}{\mu_\Delta\left((p(u), q(u)]_{\mathbb{T}}\right)} = 0.$$

This implies $[DS_{\mathbb{T}}^{p,q}] - \lim_{t\to\infty} f(t) = L$.

(ii) Suppose that $[DS_{\mathbb{T}}^{p,q}] - \lim_{t\to\infty} f(t) = L$ and f is bounded. Then there exists a real number M such that $|f(s)| \leq M$ for all $s \in (p(u), q(u)]_{\mathbb{T}}$, and

$$\lim_{u\to\infty} \frac{\mu_\Delta(K)}{\mu_\Delta\left((p(u), q(u)]_{\mathbb{T}}\right)} = 0,$$

where $K = \{s \in (p(u), q(u)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\}$, which was defined in the previous proof. Now,

$$\begin{aligned}\int\limits_{(p(u),q(u)]_{\mathbb{T}}} |f(s)-L|^k \Delta s &= \int\limits_{K} |f(s)-L|^k \Delta s + \int\limits_{(p(u),q(u)]_{\mathbb{T}}\setminus K} |f(s)-L|^k \Delta s \\ &\leq (M+|L|)^k \int\limits_{K} \Delta s + \varepsilon^k \int\limits_{(p(u),q(u)]_{\mathbb{T}}} \Delta s \\ &\leq (M+|L|)^k \ \mu_\Delta(K) + \varepsilon^k \ \mu_\Delta\left((p(u), q(u)]_{\mathbb{T}}\right).\end{aligned}$$

This gives

$$\lim_{u\to\infty} \frac{1}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)} \int\limits_{(p(u),q(u)]_{\mathbb{T}}} |f(s)-L|^k \Delta s \leq (M+|L|)^k \lim_{u\to\infty} \frac{\mu_\Delta(K)}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)} + \varepsilon^k.$$

ε being arbitrary, the RHS of this inequality vanishes. So, we obtain

$$\lim_{u\to\infty} \frac{1}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)} \int\limits_{(p(u),q(u)]_{\mathbb{T}}} |f(s)-L|^k \Delta s = 0.$$

This proves our theorem. □

Theorem 2.5. *Let $(p,q),(r,s) \in OPF$ be two pairs of functions such that $p(u) \leq r(u) < s(u) \leq q(u)$ holds for all $u \in [1,\infty)$. Let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. If*

$$\lim_{u\to\infty} \frac{\mu_\Delta\left((p(u),r(u)]_{\mathbb{T}}\right)}{\mu_\Delta\left((r(u),s(u)]_{\mathbb{T}}\right)} = 0 \quad \text{and} \quad \lim_{u\to\infty} \frac{\mu_\Delta\left((s(u),q(u)]_{\mathbb{T}}\right)}{\mu_\Delta\left((r(u),s(u)]_{\mathbb{T}}\right)} = 0 \tag{2.1}$$

and f is bounded, then $f \in [DS_{\mathbb{T}}^{r,s}]$ implies $f \in [DW_{\mathbb{T}}^{p,q}]$.

Proof. Suppose that $[DS_{\mathbb{T}}^{r,s}] - \lim_{t\to\infty} f(t) = L$. Since f is bounded, there exists a number $M > 0$ such that $|f(t)-L| \leq M$. So, we get the following:

$$\begin{aligned}&\frac{1}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)} \int\limits_{(p(u),q(u)]_{\mathbb{T}}} |f(t)-L|^k \Delta t \\ &= \frac{1}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)} \left\{ \int\limits_{(p(u),r(u)]_{\mathbb{T}}} |f(t)-L|^k \Delta t + \int\limits_{(r(u),s(u)]_{\mathbb{T}}} |f(t)-L|^k \Delta t + \int\limits_{(s(u),q(u)]_{\mathbb{T}}} |f(t)-L|^k \Delta t \right\} \\ &\leq \frac{1}{\mu_\Delta\left((r(u),s(u)]_{\mathbb{T}}\right)} \left\{ \int\limits_{(p(u),r(u)]_{\mathbb{T}}} |f(t)-L|^k \Delta t + \int\limits_{(r(u),s(u)]_{\mathbb{T}}} |f(t)-L|^k \Delta t + \int\limits_{(s(u),q(u)]_{\mathbb{T}}} |f(t)-L|^k \Delta t \right\} \\ &\leq \frac{M^k}{\mu_\Delta\left((r(u),s(u)]_{\mathbb{T}}\right)} \left\{ \int\limits_{(p(u),r(u)]_{\mathbb{T}}} \Delta t + \int\limits_{(s(u),q(u)]_{\mathbb{T}}} \Delta t \right\} + \frac{1}{\mu_\Delta\left((r(u),s(u)]_{\mathbb{T}}\right)} \int\limits_{(r(u),s(u)]_{\mathbb{T}}} |f(t)-L|^k \Delta t \\ &= \frac{M^k}{\mu_\Delta\left((r(u),s(u)]_{\mathbb{T}}\right)} \left\{ \sigma\left(r(u)\right) - \sigma\left(p(u)\right) + \sigma\left(q(u)\right) - \sigma(s(u)) \right\} \\ &+ \frac{1}{\mu_\Delta\left((r(u),s(u)]_{\mathbb{T}}\right)} \left\{ \int\limits_{\{t\in(r(u),s(u)]_{\mathbb{T}}:|f(t)-L|\geq\varepsilon\}} |f(t)-L|^k \Delta t + \int\limits_{\{t\in(r(u),s(u)]_{\mathbb{T}}:|f(t)-L|<\varepsilon\}} |f(t)-L|^k \Delta t \right\} \\ &\leq \frac{M^k \left\{ \mu_\Delta\left((p(u),r(u)]_{\mathbb{T}}\right) + \mu_\Delta\left((s(u),q(u)]_{\mathbb{T}}\right) \right\}}{\mu_\Delta\left((r(u),s(u)]_{\mathbb{T}}\right)}\end{aligned}$$

$$+\frac{M^k}{\mu_\Delta\left((r(u),s(u)]_{\mathbb{T}}\right)}\mu_\Delta\left(\{t\in(r(u),s(u)]_{\mathbb{T}}:|f(t)-L|\geq\varepsilon\}\right)$$
$$+\frac{\varepsilon^k}{\mu_\Delta\left((r(u),s(u)]_{\mathbb{T}}\right)}\mu_\Delta\left((r(u),s(u)]_{\mathbb{T}}\right).$$

Taking limits as $u\to\infty$ and using condition (2.1) and the fact that $[DS_{\mathbb{T}}^{r,s}]-\lim_{t\to\infty}f(t)=L$, we get $[DW_{\mathbb{T}}^{p,q}]-\lim_{t\to\infty}f(t)=L$. This completes the proof. □

Theorem 2.6. *Let $(p,q),(r,s)\in OPF$ be two pairs of functions with the condition*

$$p(u)\leq r(u)<s(u)\leq q(u) \tag{2.2}$$

such that

$$\lim_{u\to\infty}\inf\frac{\mu_\Delta\left((r(u),s(u)]_{\mathbb{T}}\right)}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)}>0 \tag{2.3}$$

holds. Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function. Then

(i) *If $[DS_{\mathbb{T}}^{p,q}]-\lim_{t\to\infty}f(t)=L$, then $[DS_{\mathbb{T}}^{r,s}]-\lim_{t\to\infty}f(t)=L$,*

(ii) *If $[DW_{\mathbb{T}}^{p,q}]-\lim_{t\to\infty}f(t)=L$, then $[DW_{\mathbb{T}}^{r,s}]-\lim_{t\to\infty}f(t)=L$.*

Proof. (i) Suppose that $[DS_{\mathbb{T}}^{p,q}]-\lim_{t\to\infty}f(t)=L$. Using (2.2), for any $\varepsilon>0$, we have

$$\{t\in(r(u),s(u)]_{\mathbb{T}}:|f(t)-L|\geq\varepsilon\}\subseteq\{t\in(p(u),q(u)]_{\mathbb{T}}:|f(t)-L|\geq\varepsilon\}.$$

This implies

$$\frac{1}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)}\mu_\Delta\left(\{t\in(r(u),s(u)]_{\mathbb{T}}:|f(t)-L|\geq\varepsilon\}\right)$$
$$\leq\frac{1}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)}\mu_\Delta\left(\{t\in(p(u),q(u)]_{\mathbb{T}}:|f(t)-L|\geq\varepsilon\}\right),$$

or

$$\frac{\mu_\Delta\left((r(u),s(u)]_{\mathbb{T}}\right)}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)}\frac{1}{\mu_\Delta\left((r(u),s(u)]_{\mathbb{T}}\right)}\mu_\Delta\left(\{t\in(r(u),s(u)]_{\mathbb{T}}:|f(t)-L|\geq\varepsilon\}\right)$$
$$\leq\frac{1}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)}\mu_\Delta\left(\{t\in(p(u),q(u)]_{\mathbb{T}}:|f(t)-L|\geq\varepsilon\}\right).$$

Using the assumption $[DS_{\mathbb{T}}^{p,q}]-\lim_{t\to\infty}f(t)=L$ and condition (2.3), we get

$$\lim_{u\to\infty}\frac{1}{\mu_\Delta\left((r(u),s(u)]_{\mathbb{T}}\right)}\mu_\Delta\left(\{t\in(r(u),s(u)]_{\mathbb{T}}:|f(t)-L|\geq\varepsilon\}\right)=0.$$

Hence $[DS_{\mathbb{T}}^{r,s}]-\lim_{t\to\infty}f(t)=L$.

(ii) Suppose that $[DW_{\mathbb{T}}^{p,q}]-\lim_{t\to\infty}f(t)=L$. Since $(r(u),s(u)]_{\mathbb{T}}\subseteq(p(u),q(u)]_{\mathbb{T}}$, we have

$$\int_{(r(u),s(u)]_{\mathbb{T}}}|f(s)-L|^k\Delta s\leq\int_{(p(u),q(u)]_{\mathbb{T}}}|f(s)-L|^k\Delta s.$$

This implies

$$\frac{1}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)}\int_{(r(u),s(u)]_{\mathbb{T}}}|f(s)-L|^k\Delta s\leq\frac{1}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)}\int_{(p(u),q(u)]_{\mathbb{T}}}|f(s)-L|^k\Delta s,$$

or

$$\frac{\mu_\Delta\left((r(u),s(u)]_{\mathbb{T}}\right)}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)}\frac{1}{\mu_\Delta\left((r(u),s(u)]_{\mathbb{T}}\right)}\int\limits_{(r(u),s(u)]_{\mathbb{T}}}|f(s)-L|^k\Delta s$$
$$\leq\frac{1}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)}\int\limits_{(p(u),q(u)]_{\mathbb{T}}}|f(s)-L|^k\Delta s.$$

Using the assumption $[DW_{\mathbb{T}}^{p,q}]-\lim\limits_{t\to\infty}f(t)=L$ and condition (2.3), we get

$$\lim_{u\to\infty}\frac{1}{\mu_\Delta\left((r(u),s(u)]_{\mathbb{T}}\right)}\int\limits_{(r(u),s(u)]_{\mathbb{T}}}|f(s)-L|^k\Delta s=0.$$

Hence $[DW_{\mathbb{T}}^{r,s}]-\lim\limits_{t\to\infty}f(t)\ =L$. This proves the theorem. □

Theorem 2.7. *Let $(p,q),(r,s)\in OPF$ be two pairs of functions such that*

$$p(u)<r(u)<s(u)<q(u)$$

holds for all $u\in[1,\infty)$. If $f:\mathbb{T}\to\mathbb{R}$ is a Δ-measurable function such that $f\in[DS_{\mathbb{T}}^{p,r}]\cap[DS_{\mathbb{T}}^{r,s}]\cap[DS_{\mathbb{T}}^{s,q}]$ and if f is bounded, then $f\in[DS_{\mathbb{T}}^{p,q}]$ and $f\in[DW_{\mathbb{T}}^{p,q}]$.

Proof. For any $\varepsilon>0$, we have

$$\begin{aligned}\{t\in(p(u),q(u)]_{\mathbb{T}}:|f(t)-L|\geq\varepsilon\}&=\{t\in(p(u),r(u)]_{\mathbb{T}}:|f(t)-L|\geq\varepsilon\}\\&\cup\{t\in(r(u),s(u)]_{\mathbb{T}}:|f(t)-L|\geq\varepsilon\}\\&\cup\{t\in(s(u),q(u)]_{\mathbb{T}}:|f(t)-L|\geq\varepsilon\}.\end{aligned}$$

From this equality, we get the following:

$$\begin{aligned}\frac{\mu_\Delta\left(\{t\in(p(u),q(u)]_{\mathbb{T}}:|f(t)-L|\geq\varepsilon\}\right)}{\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)}&\leq\frac{\mu_\Delta\left(\{t\in(p(u),r(u)]_{\mathbb{T}}:|f(t)-L|\geq\varepsilon\}\right)}{\mu_\Delta\left((p(u),r(u)]_{\mathbb{T}}\right)}\\&+\frac{\mu_\Delta\left(\{t\in(r(u),s(u)]_{\mathbb{T}}:|f(t)-L|\geq\varepsilon\}\right)}{\mu_\Delta\left((r(u),s(u)]_{\mathbb{T}}\right)}\\&+\frac{\mu_\Delta\left(\{t\in(s(u),q(u)]_{\mathbb{T}}:|f(t)-L|\geq\varepsilon\}\right)}{\mu_\Delta\left((s(u),q(u)]_{\mathbb{T}}\right)}.\end{aligned}$$

This is possible because $\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)\geq\mu_\Delta\left((p(u),r(u)]_{\mathbb{T}}\right)$, $\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)\geq\mu_\Delta\left((r(u),s(u)]_{\mathbb{T}}\right)$ and $\mu_\Delta\left((p(u),q(u)]_{\mathbb{T}}\right)\geq\mu_\Delta\left((s(u),q(u)]_{\mathbb{T}}\right)$.

From the above inequality, using $f\in[DS_{\mathbb{T}}^{p,r}]\cap[DS_{\mathbb{T}}^{r,s}]\cap[DS_{\mathbb{T}}^{s,q}]$, we get $f\in[DS_{\mathbb{T}}^{p,q}]$.

For the second part of the theorem, we use the boundedness condition of f.

Since $f\in[DS_{\mathbb{T}}^{p,r}]\cap[DS_{\mathbb{T}}^{r,s}]\cap[DS_{\mathbb{T}}^{s,q}]$, f is bounded. So, $f\in[DW_{\mathbb{T}}^{p,r}]\cap[DW_{\mathbb{T}}^{r,s}]\cap[DW_{\mathbb{T}}^{s,q}]$ by Theorem 2.4. Now,

$$\int\limits_{(p(u),q(u)]_{\mathbb{T}}}|f(t)-L|^k\Delta t=\int\limits_{(p(u),r(u)]_{\mathbb{T}}}|f(t)-L|^k\Delta t+\int\limits_{(r(u),s(u)]_{\mathbb{T}}}|f(t)-L|^k\Delta t+\int\limits_{(s(u),q(u)]_{\mathbb{T}}}|f(t)-L|^k\Delta t.$$

This implies

$$\begin{aligned}\frac{1}{(p(u),q(u)]_{\mathbb{T}}}\int\limits_{(p(u),q(u)]_{\mathbb{T}}}|f(t)-L|^k\Delta t&\leq\frac{1}{(p(u),r(u)]_{\mathbb{T}}}\int\limits_{(p(u),r(u)]_{\mathbb{T}}}|f(t)-L|^k\Delta t\\&+\frac{1}{(r(u),s(u)]_{\mathbb{T}}}\int\limits_{(r(u),s(u)]_{\mathbb{T}}}|f(t)-L|^k\Delta t\\&+\frac{1}{(s(u),q(u)]_{\mathbb{T}}}\int\limits_{(s(u),q(u)]_{\mathbb{T}}}|f\ (t)-L|^k\Delta t.\end{aligned}$$

Using $f \in [DW_{\mathbb{T}}^{p,r}] \cap [DW_{\mathbb{T}}^{r,s}] \cap [DW_{\mathbb{T}}^{s,q}]$, we get $f \in [DW_{\mathbb{T}}^{p,q}]$.

This completes the proof. □

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