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UNIQUENESS THEOREMS IN THE STEADY VIBRATION PROBLEMS OF THE MOORE–GIBSON–THOMPSON THERMOELASTICITY FOR MATERIALS WITH VOIDS

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Dedicated to the memory of Professor Elene Obolashvili

Abstract. In this paper, the linear Moore–Gibson–Thompson thermoelasticity model for materials with voids is considered. The governing equations of motion and steady vibrations of this model are proposed. The basic system of equations of steady vibrations is expressed in terms of the displacement vector, temperature change and volume fraction of pores. The radiation conditions are established and the first Green's identity is obtained. Finally, on the basis of this identity, the uniqueness theorems for classical solutions of boundary value problems of steady vibrations in the theory under consideration are proved.

1. Introduction

In the past century, mathematical theories of porous media have been developed and intensively studied. These theories are mainly based on Darcy's law for fluid flows in pores or on the concept of the volume fraction of the pore network. Namely, Biot [2] presented a theory of poroelasticity based on Darcy's law in which the governing quasi-static equations are written with respect to the displacement vector and the change of fluid pressure in the pores. Nowadays, this theory has been generalized to take into account various mechanical effects and material structures. The main results obtained in this direction of research and an extensive review of the references are given in the books by Cheng [9], Selvadurai and Suvorov [38], Straughan [40, 42], Svanadze [46] and Wang [60].

On the other hand, on the basis of the concept of pore volume fraction, the theory of elastic materials with voids was introduced by Nunziato and Cowin [11, 35]. The basic equations of this theory involve the displacement vector field and the change in the pore volume fraction. The Nunziato-Cowin theory was generalized by Ieşan [18] and based on the classical Fourier law of heat conduction, he proposed the theory of thermoelasticity for materials with voids. Over the past three decades, more general models of the theories of elasticity and thermoelasticity for materials with voids have been introduced and intensively investigated by several authors. The basic results in the theories for materials with single voids can be found in the books by Ciarletta and Ieşan [10], Ieşan [19], Straughan [40], while the main conclusions in the theories for materials with multiple voids are given in the series of papers by Fernández and Quintanilla [12], Ieşan and Quintanilla [20, 21], Kumar and Vohra [26, 27], Svanadze [43–45], Tsagareli [56] (see also references therein).

Moreover, Svanadze [47, 48] presented the mathematical models of elasticity and thermoelasticity for single porosity materials, which simultaneously take into account Darcy's law and the concept of volume fraction. These related theories have been generalized to solids with double porosity [49, 50] and triple porosity [51], and a wide class of problems has been studied in the papers by Bitsadze [4], Mikelashvili [31–33], Svanadze [52], and Tsagareli [57].

In the second half of the past century, the non-Fourier laws of heat conduction and relevant theories of generalized thermoelasticity began to be proposed and intensively investigated. Namely, Cattaneo [5, 6] and Vernotte [59] presented a hyperbolic heat conduction equation by introducing a positive relaxation parameter. On the basis of Cattaneo–Vernotte equation, Lord and Shulman [29]

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developed a generalized thermoelasticity theory, which is an extension of the Biot [3] classical theory of thermoelasticity based on the Fourier law.

Furthermore, Green and Naghdi [14–16] presented three theories of generalized thermoelasticity based on an entropy-balance equation and in which the thermal displacement variable is introduced. The linear version of the first of these theories is similar to the Biot classical thermoelasticity. The second theory (GN type II thermoelasticity) does not involve energy dissipation and the third (GN type III thermoelasticity) proposes a more general one.

Recently, Quintanilla [36] used the Moore–Gibson–Thompson [34, 55] equation and developed a new theory of thermoelasticity (called the MGT thermoelasticity), which turned out to be more general than the aforementioned thermoelasticity theories. Accordingly, this theory has attracted much attention from researches and various important problems of this theory are currently being investigated. For details, see the series of papers by Bazarra et al. [1], Florea and Bobe [13], Jangid and Mukhopadhyay [23], Marin et al. [30], Quintanilla [37], Singh and Mukhopadhyay [39], Svanadze [53,54] and references therein.

A wide historical information on the non-Fourier heat conduction laws is given in the papers by Joseph and Preziosi [24,25]. An extensive review of the literature and the basic results obtained in the generalised theories of thermoelasticity can be found in the books by Ignaczak and Ostoja–Starzewski [22], Straughan [41] and in the papers of Chandrasekharaiah [7,8], Hetnarski and Ignaczak [17].

The aims of this paper are twofold. The first task is to introduce a linear mathematical model of the MGT thermoelasticity for materials with voids, and the second is to prove the uniqueness theorems for the classical solutions of the basic internal and external boundary value problems (BVPs) of steady vibrations of this model.

This paper is articulated as follows. In Section 2, we present the governing equations of motion and steady vibrations of the linear model of the MGT thermoelasticity for materials with voids. The basic system of equations of steady vibrations is expressed in terms of the displacement vector, temperature change and pores volume fraction. In Section 3, the radiation conditions (the conditions at infinity) are established and the basic internal and external BVPs are formulated. In Section 4, the first Green's identity for the steady vibration equations is obtained. Finally, in Section 5, on the basis of this identity, the uniqueness theorems for classical solutions of the BVPs of steady vibrations in the theory under consideration are proved.

2. Basic Equations

We consider a porous material occupying the region Ω of the Euclidean three-dimensional space $\mathbb{R}^3$, whose skeleton is an isotropic and homogeneous elastic solid and the pores are filled with a fluid. Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point of $\mathbb{R}^3$ and let t denote a time variable, $t \geq 0$.

In what follows, we assume that subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, repeated indices are summed over the range (1,2,3), vectors and matrices will be marked with bold letters, functions and vectors that depend on $\mathbf{x}$ and t will be denoted with the "hat" symbol, and a superposed dot denotes differentiation with respect to t.

Let $\hat{\mathbf{u}}$ be the displacement vector in a solid skeleton, $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$, $\hat{\varphi}$ be the change of the volume fraction of pores from the initial configuration, $\hat{\theta}$ be the temperature measured from some constant absolute temperature $T_0(> 0)$, and $\hat{\vartheta}$ be the thermal displacement variable satisfying the condition [14]

$$\dot{\hat{\vartheta}} = \hat{\theta}. \tag{2.1}$$

Following Ieşan [18] and Quintanilla [36], the governing system of field motion equations in the linear theory of MGT thermoelasticity for materials with voids is composed of the next four sets of equations:

1. *Equations of motion*

$$\hat{t}_{lj,j} = \rho\left(\ddot{\hat{u}}_l - \hat{\mathcal{F}}_l\right), \qquad \hat{\sigma}_{j,j} + \hat{\xi} = \rho_1 \ddot{\hat{\varphi}} - \rho \hat{s}_1, \quad l = 1, 2, 3, \tag{2.2}$$

where $\hat{t}_{lj}$ is the component of the total stress tensor, $\rho(>0)$ is the reference mass density, $\hat{\boldsymbol{\mathcal{F}}} = (\hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2, \hat{\mathcal{F}}_3)$ is the body force per unit mass; $\hat{\sigma}_j, \hat{s}_1$ and $\rho_1(>0)$ are the component of the equilibrated stress, the extrinsic equilibrated body force and the coefficient of the equilibrated inertia of the pore network, respectively; $\hat{\xi}$ is the intrinsic equilibrated body force defined as

$$\hat{\xi} = -b\hat{e}_{rr} - \alpha_1\hat{\varphi} + \varepsilon_1\hat{\theta}, \tag{2.3}$$

$\hat{e}_{lj}$ is the component of the strain tensor and defined by

$$\hat{e}_{lj} = \frac{1}{2}(\hat{u}_{l,j} + \hat{u}_{j,l}), \quad l,j = 1,2,3. \tag{2.4}$$

2. *Constitutive equations*

$$\hat{t}_{lj} = 2\mu\hat{e}_{lj} + \lambda\hat{e}_{rr}\delta_{lj} + (b\hat{\varphi} - \varepsilon_0\hat{\theta})\delta_{lj}, \qquad \hat{\sigma}_l = \alpha\hat{\varphi}_{,l}, \quad l,j = 1,2,3, \tag{2.5}$$

where λ and μ are the Lamé constants, ε_0 is the thermal expansion coefficient, $\varepsilon_0 \neq 0$, b, α, α_1 and ε_1 are the constitutive coefficients of the porous material, $\alpha > 0$, δ_{lj} is the Kronecker delta.

3. *Heat transfer equation*

$$\text{div}\hat{\mathbf{q}} = -T_0\rho\dot{\hat{\eta}} - \rho\hat{s}_2, \tag{2.6}$$

where $\hat{\mathbf{q}} = (\hat{q}_1, \hat{q}_2, \hat{q}_3)$ is the heat flux vector and $\hat{s}_2$ is the heat source, $\hat{\eta}$ is the entropy per unit mass defined as

$$\rho\hat{\eta} = c\hat{\theta} + \varepsilon_0\hat{e}_{rr} + \varepsilon_1\hat{\varphi}, \tag{2.7}$$

and $c(>0)$ is the thermal capacity.

4. *MGT equation*

$$\tau\dot{\hat{q}}_l + \hat{q}_l = -\left(k^*\hat{\vartheta}_{,l} + k\hat{\theta}_{,l}\right), \quad l = 1,2,3, \tag{2.8}$$

where the non-negative constants k^*, k and τ are the conductivity rate parameter, the thermal conductivity and the relaxation parameter, respectively.

Substituting equations (2.1), (2.3)–(2.5), (2.7) and (2.8) into (2.2) and (2.6), we obtain the following system of equations of motion in the linear theory of MGT thermoelasticity for materials with voids expressed in terms of the displacement vector field $\hat{\mathbf{u}}$, the change in the volume fraction $\hat{\varphi}$ of the pore network and the change in temperature $\hat{\theta}$ of the porous material:

$$\begin{gathered}
\mu\Delta\hat{\mathbf{u}} + (\lambda+\mu)\nabla\text{div}\hat{\mathbf{u}} + b\nabla\hat{\varphi} - \varepsilon_0\nabla\hat{\theta} = \rho\left(\ddot{\hat{\mathbf{u}}} - \hat{\boldsymbol{\mathcal{F}}}\right),\\
\alpha\Delta\hat{\varphi} - \alpha_1\hat{\varphi} - b\,\text{div}\hat{\mathbf{u}} + \varepsilon_1\hat{\theta} = \rho_1\ddot{\hat{\varphi}} - \rho\hat{s}_1,\\
k^*\Delta\hat{\theta} + k\Delta\dot{\hat{\theta}} - T_0M\left(c\dot{\hat{\theta}} + \varepsilon_0\text{div}\dot{\hat{\mathbf{u}}} + \boldsymbol{\varepsilon_1}\dot{\hat{\varphi}}\right) = -\rho M\hat{s}_2,
\end{gathered} \tag{2.9}$$

where Δ is the Laplacian operator, and M is the differential operator defined as $M = \frac{\partial}{\partial t} + \tau\frac{\partial^2}{\partial t^2}$.

If $\hat{\mathbf{u}}, \hat{\varphi}, \hat{\theta}, \hat{\boldsymbol{\mathcal{F}}}, \hat{s}_1, \hat{s}_2$ are postulated to have a harmonic time variation, that is,

$$\left\{\hat{\mathbf{u}}, \hat{\varphi}, \hat{\theta}, \hat{\boldsymbol{\mathcal{F}}}, \hat{s}_1, \hat{s}_2\right\}(\mathbf{x},t) = \text{Re}\left[\{\mathbf{u}, \varphi, \theta, \boldsymbol{\mathcal{F}}, s_1, s_2\}(\mathbf{x})\,e^{-i\omega t}\right],$$

then system (2.9) is reduced to the following system of equations of steady vibrations:

$$\begin{gathered}
(\mu\Delta + \rho\omega^2)\mathbf{u} + (\lambda+\mu)\nabla\text{div}\mathbf{u} + b\nabla\varphi - \varepsilon_0\nabla\theta = -\rho\boldsymbol{\mathcal{F}},\\
(\alpha\Delta + \eta)\varphi - b\,\text{div}\mathbf{u} + \varepsilon_1\theta = -\rho_1 s_1,\\
(k_0\Delta + mc)\theta + m\varepsilon_0\text{div}\mathbf{u} + m\varepsilon_1\varphi = i\omega(1 - i\omega\tau)\rho s_2,
\end{gathered} \tag{2.10}$$

where $\omega(>0)$ is the oscillation frequency and

$$\eta = \rho_1\omega^2 - \alpha_1, \qquad k_0 = k^* - i\omega k \neq 0, \qquad m = T_0\omega^2(1 - i\omega\tau).$$

Now, we consider special cases of the parameters k, k^*, τ and from (2.10), we obtain the systems of equations of the different thermoelasticity theories for materials with voids. Clearly, only the last equation of the system (2.10) will change by changing these parameters. We will have the following cases:

1. Let

$$k - \tau k^* \neq 0. \tag{2.11}$$

In this case, we have the following 4 versions:

(i) If $k^* = \tau = 0$, then from the last equation of (2.10) it follows that

$$k\Delta\theta + i\omega T_0(c\theta + \varepsilon \mathrm{div}\mathbf{u} + \gamma p) = -\rho s_2.$$

Consequently, from (2.10), we get the system of steady vibration equations of Ieşan's [18] theory of thermoelasticity for materials with voids based on the Fourier classical law.

(ii) If $k^* = 0$ and $\tau > 0$, then the last equation of (2.10) is replaced by

$$k\Delta\theta + i\omega T_0(1 - i\omega\tau)(c\theta + \varepsilon \mathrm{div}\mathbf{u} + \gamma p) = (i\omega\tau - 1)\rho s_2.$$

Clearly, we get the system of equations of thermoelasticity based on the Cattaneo–Vernotte law of heat conduction. Obviously, this system of equations is the extension of Lord–Shulman [29] equations of thermoelasticity for materials with voids.

(iii) If $k^* > 0$ and $\tau = 0$, then the last equation of (2.10) can be expressed as

$$k_0\Delta\theta + m_1(c\theta + \varepsilon \mathrm{div}\mathbf{u} + \gamma p) = i\omega\rho s_2.$$

In this case, we have the system of steady vibration equations of thermoelasticity for materials with voids based on the Green–Naghdi type III equation of heat conduction.

(iv) If $k^* > 0$ and $\tau > 0$, then we have system (2.10) of the MGT thermoelasticity for materials with voids.

2. Let $k - \tau k^* = 0$. In this case, if $k^* > 0$, then the last equation of (2.10) reduces now to

$$k^*\Delta\theta + m_1(c\theta + \varepsilon \mathrm{div}\mathbf{u} + \gamma p) = -i\omega\rho s_2,$$

where $m_1 = T_0\omega^2$. Consequently, from (2.10), we obtain the system of steady vibration equations of thermoelasticity for materials with voids based on the Green-Naghdi type II equation of heat conduction.

The purpose of this article is to prove the uniqueness theorems for the classical solutions of the basic BVPs of steady vibrations in the theories of MGT thermoelasticity for materials with voids under condition (2.11).

Remark 1. The theory of thermoelasticity for materials with voids based on the Green-Naghdi type II equation of heat conduction is specific (this is a theory of thermoelasticity without energy dissipation, the BVPs of steady vibrations are reduced to Fredholm integral equations of the second kind with a symmetric kernel, etc.), therefore the uniqueness of the solutions to the steady vibration problems of this theory will be investigated in another work.

3. Basic Boundary Value Problems

In this section, we introduce the class of five-component regular vector functions, in which we prove the uniqueness of classical solutions to the basic BVPs of steady vibrations of the MGT thermoelasticity for materials with voids. Then, we establish the radiation conditions and finally, we formulate the 3D internal and external BVPs of steady vibrations.

Throughout this paper, we assume that the following conditions

$$\mu > 0, \qquad 3\lambda + 2\mu > 0, \qquad k - \tau k^* \neq 0, \qquad \eta c - \varepsilon_1^2 \neq 0 \tag{3.1}$$

are fulfilled. We will need the following matrix differential operator $\mathbf{A}(\mathbf{D_x}) = (A_{lj}(\mathbf{D_x}))_{5\times 5}$, where

$$A_{lj}(\mathbf{D_x}) = (\mu\Delta + \rho\omega^2)\delta_{lj} + (\lambda + \mu)\frac{\partial^2}{\partial x_l \partial x_j}, \qquad A_{l4}(\mathbf{D_x}) = -A_{4l}(\mathbf{D_x}) = b\frac{\partial}{\partial x_l},$$

$$A_{l5}(\mathbf{D_x}) = -\varepsilon_0\frac{\partial}{\partial x_l}, \qquad A_{44}(\mathbf{D_x}) = \alpha\Delta + \eta, \qquad A_{45}(\mathbf{D_x}) = \varepsilon_1, \qquad A_{5l}(\mathbf{D_x}) = m\varepsilon_0\frac{\partial}{\partial x_l},$$

$$A_{54}(\mathbf{D_x}) = m\varepsilon_1, \qquad A_{55}(\mathbf{D_x}) = k_0\Delta + mc, \qquad \mathbf{D_x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right), \quad l, j = 1, 2, 3.$$

Obviously, system (2.10) we can be rewritten in the form

$$\mathbf{A}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \tag{3.2}$$

where $\mathbf{U} = (\mathbf{u}, \varphi, \theta)$ and $\mathbf{F} = (-\rho\boldsymbol{\mathcal{F}}, -\rho_1 s_1, i\omega(1 - i\omega\tau)\rho s_2)$ are the five-component vector functions, and $\mathbf{x} \in \Omega$.

Let $\mathbf{B}(\mathbf{D_x})$ be the following matrix differential operator:

$$\mathbf{B}(\mathbf{D_x}) = (B_{lj}(\mathbf{D_x}))_{3\times 3} = \begin{pmatrix} \mu_0\Delta + \rho\omega^2 & -b\Delta & m\varepsilon_0\Delta \\ b & \alpha\Delta + \eta & m\varepsilon_1 \\ \varepsilon_0 & \varepsilon_1 & k_0\Delta + mc \end{pmatrix}_{3\times 3},$$

where $\mu_0 = \lambda + 2\mu$. Let us introduce the notation

$$\Lambda(\Delta) = \frac{1}{\mu_0 \alpha k_0} \det \mathbf{B}(\Delta) = \prod_{j=1}^{3} (\Delta + \zeta_j^2),$$

where ζ_1^2, ζ_2^2 and ζ_3^2 are the roots of the equation $\Lambda(-\chi) = 0$ (with respect to χ).

We assume that the values ζ_1, ζ_2, ζ_3 and ζ_4 are distinct, $\mathrm{Im}\zeta_l > 0$ for negative or complex number ζ_l^2 and $\zeta_l > 0$ for $\zeta_l^2 > 0$ $(l = 1, 2, 3)$. Here, $\zeta_4 = \sqrt{\rho\omega^2\mu^{-1}}$.

Let S be the surface of a finite domain Ω^+ in $\mathbb{R}^3$, $S \in C^{1,\nu}$, $0 < \nu \le 1$, $\overline{\Omega^+} = \Omega^+ \cup S$, $\Omega^- = \mathbb{R}^3 \backslash \overline{\Omega^+}$, $\overline{\Omega^-} = \Omega^- \cup S$, $\mathbf{n}(\mathbf{z})$ be the external (with respect to Ω^+) unit normal vector to S at $\mathbf{z}$, $\mathbf{n} = (n_1, n_2, n_3)$, and $\frac{\partial}{\partial \mathbf{n}}$ be the derivative along the vector $\mathbf{n}$.

Definition 3.1. Vector function $\mathbf{U} = (U_1, U_2, \dots, U_5)$ is called *regular* in Ω^- (or Ω^+) if

(i) $U_l \in C^2(\Omega^-) \cap C^1(\overline{\Omega^-})$ (or $U_l \in C^2(\Omega^+) \cap C^1(\overline{\Omega^+})$);

(ii) $\mathbf{U}(\mathbf{x}) = \sum_{j=1}^{4} \mathbf{U}^{(j)}(\mathbf{x})$, $\mathbf{U}^{(j)} = (U_1^{(j)}, U_2^{(j)}, \dots, U_5^{(j)})$, $U_l^{(j)} \in C^2(\Omega^-) \cap C^1(\overline{\Omega^-})$;

(iii) $(\Delta + \zeta_j^2) U_l^{(j)}(\mathbf{x}) = 0$ and

$$\left(\frac{\partial}{\partial|\mathbf{x}|} - i\zeta_j\right) U_l^{(j)}(\mathbf{x}) = e^{i\zeta_j|\mathbf{x}|} o(|\mathbf{x}|^{-1}) \quad \text{for} \quad |\mathbf{x}| \gg 1, \tag{3.3}$$

where $U_4^{(4)} = U_5^{(4)} = 0$, $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$, $j = 1, 2, 3, 4$, $l = 1, 2, \dots, 5$.

It is worth noting that relation (3.3) ensures (for details, see Vekua [58])

$$U_l^{(j)}(\mathbf{x}) = e^{i\zeta_j|\mathbf{x}|} O(|\mathbf{x}|^{-1}) \quad \text{for} \quad |\mathbf{x}| \gg 1, \tag{3.4}$$

where $j = 1, 2, 3, 4$, $l = 1, 2, \dots, 5$.

Relations (3.3) and (3.4) are the radiation conditions in the linear theory of MGT thermoelasticity for materials with voids.

Let us introduce the matrix differential operator $\mathbf{R}(\mathbf{D_x}, \mathbf{n})$, where

$$\begin{gathered} \mathbf{R}(\mathbf{D_x}, \mathbf{n}) = (R_{lj}(\mathbf{D_x}, \mathbf{n}))_{5\times 5}, \qquad R_{lj} = \mu\delta_{lj}\frac{\partial}{\partial \mathbf{n}} + \mu n_j \frac{\partial}{\partial x_l} + \lambda n_l \frac{\partial}{\partial x_j}, \\ R_{l4} = b n_l, \qquad R_{l5} = -\varepsilon_0 n_l, \qquad R_{44} = \alpha\frac{\partial}{\partial \mathbf{n}}, \qquad R_{55} = k_0 \frac{\partial}{\partial \mathbf{n}}, \\ R_{4j} = R_{45} = R_{5j} = R_{54} = 0, \quad l, j = 1, 2, 3. \end{gathered} \tag{3.5}$$

The basic internal and external BVPs of steady vibrations in the linear theory of MGT thermoelasticity for materials with voids can be formulated as follows:

Find a regular solution to (3.2) for $\mathbf{x} \in \Omega^+$ satisfying the boundary condition

$$\lim_{\Omega^+ \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z}) \tag{3.6}$$

in the internal *Problem* $(I)^+_{\mathbf{F},\mathbf{f}}$,

$$\lim_{\Omega^+ \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{R}(\mathbf{D_x}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x}) \equiv \{\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z}) \tag{3.7}$$

in the internal *Problem* $(II)^+_{\mathbf{F},\mathbf{f}}$.

Find a regular solution to (3.2) for $\mathbf{x} \in \Omega^-$ satisfying the boundary condition

$$\lim_{\Omega^- \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z}) \tag{3.8}$$

in the external *Problem* $(I)^-_{\mathbf{F},\mathbf{f}}$,

$$\lim_{\Omega^- \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{R}(\mathbf{D_x}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x}) \equiv \{\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z}) \tag{3.9}$$

in the external *Problem* $(II)^-_{\mathbf{F},\mathbf{f}}$. Here $\mathbf{F}$ and $\mathbf{f}$ are the prescribed five-component vector functions.

4. Green's Identity

In this section, Green's identity of the theory under consideration are established. Based on this identity, in the next section we prove the uniqueness theorems.

The scalar product of two vectors $\boldsymbol{\phi} = (\phi_1, \phi_2, \dots, \phi_l)$ and $\boldsymbol{\psi} = (\psi_1, \psi_2, \dots, \psi_l)$ is denoted by $\boldsymbol{\phi} \cdot \boldsymbol{\psi} = \sum\limits_{j=1}^{l} \phi_j \overline{\psi_j}$, where $\overline{\psi_j}$ is the complex conjugate of ψ_j.

We introduce the following notation:

$$\begin{aligned}
W^{(0)}(\mathbf{u}, \mathbf{u}') &= \frac{1}{3}(3\lambda + 2\mu)\mathrm{div}\mathbf{u}\,\mathrm{div}\overline{\mathbf{u}'} + \frac{\mu}{2} \sum_{l,j=1; l \neq j}^{3} (u_{l,j} + u_{j,l})(\overline{u'_{l,j}} + \overline{u'_{j,l}}) \\
&\quad + \frac{\mu}{3} \sum_{l,j=1}^{3} \left(\frac{\partial u_l}{\partial x_l} - \frac{\partial u_j}{\partial x_j}\right)\left(\frac{\partial \overline{u'_l}}{\partial x_l} - \frac{\partial \overline{u'_j}}{\partial x_j}\right), \\
W^{(1)}(\mathbf{U}, \mathbf{u}') &= W^{(0)}(\mathbf{u}, \mathbf{u}') - \rho\omega^2 \mathbf{u} \cdot \mathbf{u}' + (b\varphi - \varepsilon_0 \theta)\mathrm{div}\overline{\mathbf{u}'}, \\
W^{(2)}(\mathbf{U}, \varphi') &= \alpha \nabla\varphi \cdot \nabla\varphi' - \eta\varphi\overline{\varphi'} + (b\,\mathrm{div}\mathbf{u} - \varepsilon_1\theta)\overline{\varphi'}, \\
W^{(3)}(\mathbf{U}, \theta') &= k_0 \nabla\theta \cdot \nabla\theta' - m(c\theta + \varepsilon_0 \mathrm{div}\mathbf{u} + \varepsilon_1 \varphi)\overline{\theta'},
\end{aligned} \tag{4.1}$$

where $\mathbf{U}' = (\mathbf{u}', \varphi', \theta')$, $\mathbf{u}' = (u'_1, u'_2, u'_3)$.

We have the following result.

Lemma 4.1. *If* $\mathbf{U} = (\mathbf{u}, p, \theta)$ *is a regular vector in* Ω^+, $u'_j, p', \theta' \in C^1(\Omega^+) \cap C(\overline{\Omega^+})$, $j = 1, 2, 3$, *then*

$$\begin{aligned}
&\int_{\Omega^+} \left[\mathbf{A}^{(1)}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x}) \cdot \mathbf{u}'(\mathbf{x}) + W^{(1)}(\mathbf{U}, \mathbf{u}')\right] d\mathbf{x} = \int_S \mathbf{R}^{(1)}(\mathbf{D_z}, \mathbf{n})\mathbf{U} \cdot \mathbf{u}'\, d_{\mathbf{z}}S, \\
&\int_{\Omega^+} \left[\mathbf{A}^{(2)}(\mathbf{D_x})\mathbf{U}(\mathbf{x})\overline{\varphi'(\mathbf{x})} + W^{(2)}(\mathbf{U}, \varphi')\right] d\mathbf{x} = \alpha \int_S \frac{\partial\varphi}{\partial\mathbf{n}}\, \overline{\varphi'} d_{\mathbf{z}}S,] \\
&\int_{\Omega^+} \left[\mathbf{A}^{(3)}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\,\overline{\theta'(\mathbf{x})} + W^{(3)}(\mathbf{U}, \theta')\right] d\mathbf{x} = k_0 \int_S \frac{\partial\theta}{\partial\mathbf{n}}\, \overline{\theta'} d_{\mathbf{z}}S,
\end{aligned} \tag{4.2}$$

where $\mathbf{u}' = (u'_1, u'_2, u'_3)$ *and*

$$\begin{aligned}
\mathbf{A}^{(1)}(\mathbf{D_x}) &= \left(A^{(1)}_{lj}(\mathbf{D_x})\right)_{3\times5}, \qquad A^{(1)}_{lj}(\mathbf{D_x}) = A_{lj}(\mathbf{D_x}), \\
\mathbf{A}^{(2)}(\mathbf{D_x}) &= \left(A^{(2)}_{1j}(\mathbf{D_x})\right)_{1\times5}, \qquad A^{(2)}_{1j}(\mathbf{D_x}) = A_{4j}(\mathbf{D_x}), \\
\mathbf{A}^{(3)}(\mathbf{D_x}) &= \left(A^{(3)}_{1j}(\mathbf{D_x})\right)_{1\times5}, \qquad A^{(3)}_{1j}(\mathbf{D_x}) = A_{5j}(\mathbf{D_x}), \\
\mathbf{R}^{(1)}(\mathbf{D_x}, \mathbf{n}) &= \left(R^{(1)}_{lj}(\mathbf{D_x}, \mathbf{n})\right)_{3\times5}, \qquad R^{(1)}_{lj}(\mathbf{D_x}, \mathbf{n}) = R_{lj}(\mathbf{D_x}, \mathbf{n}), \\
&l = 1, 2, 3, \qquad j = 1, 2, \dots, 5.
\end{aligned}$$

Proof. Clearly, on the basis of our notation, we can write the first Green's identity of the classical theory of elasticity in the form (see Kupradze et al. [28])

$$\int\limits_{\Omega^+} \Big[\mathbf{A}^{(0)}(\mathbf{D_x})\,\mathbf{u}(\mathbf{x})\cdot\mathbf{u}'(\mathbf{x}) + W^{(0)}(\mathbf{u},\mathbf{u}')\Big]\,d\mathbf{x} = \int\limits_S \mathbf{R}^{(0)}(\mathbf{D_z},\mathbf{n})\mathbf{u}(\mathbf{z})\cdot\mathbf{u}'(\mathbf{z})\,d_{\mathbf{z}}S, \tag{4.3}$$

where

$$\begin{gathered}\mathbf{A}^{(0)}(\mathbf{D_x}) = \Big(A^{(0)}_{lj}(\mathbf{D_x})\Big)_{3\times 3}, \qquad A^{(0)}_{lj}(\mathbf{D_x}) = A_{lj}(\mathbf{D_x}),\\ \mathbf{R}^{(0)}(\mathbf{D_x},\mathbf{n}) = \Big(R^{(0)}_{lj}(\mathbf{D_x},\mathbf{n})\Big)_{3\times 3}, \qquad R^{(0)}_{lj}(\mathbf{D_x},\mathbf{n}) = R_{lj}(\mathbf{D_x},\mathbf{n}),\\ l,j=1,2,3.\end{gathered} \tag{4.4}$$

Now, using the well-known identity (see, e.g., [28])

$$\int\limits_{\Omega^+} \Big[\nabla\varphi(\mathbf{x})\cdot\mathbf{u}'(\mathbf{x}) + \varphi(\mathbf{x})\mathrm{div}\,\overline{\mathbf{u}'(\mathbf{x})}\Big]\,d\mathbf{x} = \int\limits_S \varphi(\mathbf{z})\mathbf{n}(\mathbf{z})\cdot\mathbf{u}'(\mathbf{z})\,d_{\mathbf{z}}S$$

and relation (4.1), from (4.3), we can obtain the first identity of (4.2).

Furthermore, in a similar manner, in view of (4.1), from the relation

$$\int\limits_{\Omega^+} \Big[\Delta\varphi(\mathbf{x})\,\overline{\varphi'(\mathbf{x})} + \nabla\varphi(\mathbf{x})\cdot\nabla\varphi'(\mathbf{x})\Big]\,d\mathbf{x} = \int\limits_S \frac{\partial\varphi(\mathbf{z})}{\partial\mathbf{n}(\mathbf{z})}\,\overline{\varphi'(\mathbf{z})}\,d_{\mathbf{z}}S,$$

we can obtain the second and third identities of (4.2). □

With the help of Lemma 1 and the radiation conditions (3.3) and (3.4) we have the following consequence.

Lemma 4.2. *If* $\mathbf{U} = (\mathbf{u},\varphi,\theta)$ *is a regular vector in* Ω^-, $\mathbf{u}',\varphi',\theta' \in C^1(\Omega^-)\cap C(\overline{\Omega^-})$ *and* $\mathbf{U}' = (\mathbf{u}',\varphi',\theta')$ *satisfies condition* (3.3)*, then*

$$\begin{gathered}\int\limits_{\Omega^-} \Big[\mathbf{A}^{(1)}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\mathbf{u}'(\mathbf{x}) + W^{(1)}(\mathbf{U},\mathbf{u}')\Big]\,d\mathbf{x} = -\int\limits_S \mathbf{R}^{(1)}(\mathbf{D_z},\mathbf{n})\mathbf{U}\cdot\mathbf{u}'\,d_{\mathbf{z}}S,\\ \int\limits_{\Omega^-} \Big[\mathbf{A}^{(2)}(\mathbf{D_x})\mathbf{U}(\mathbf{x})\overline{\varphi'(\mathbf{x})} + W^{(2)}(\mathbf{U},\varphi')\Big]\,d\mathbf{x} = -\alpha\int\limits_S \frac{\partial\varphi}{\partial\mathbf{n}}\,\overline{\varphi'}d_{\mathbf{z}}S,\\ \int\limits_{\Omega^-} \Big[\mathbf{A}^{(3)}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\,\overline{\theta'(\mathbf{x})} + W^{(3)}(\mathbf{U},\theta')\Big]\,d\mathbf{x} = -k_0\int\limits_S \frac{\partial\theta}{\partial\mathbf{n}}\,\overline{\theta'}d_{\mathbf{z}}S.\end{gathered} \tag{4.5}$$

Lemmas 1 and 2 lead to the following two theorems.

Theorem 4.1. *If* $\mathbf{U} = (\mathbf{u},\varphi,\theta)$ *is a regular vector in* Ω^+, $\mathbf{U}' = (\mathbf{u}',\varphi',\theta') \in C^1(\Omega^+)\cap C(\overline{\Omega^+})$*, then*

$$\int\limits_{\Omega^+} [\mathbf{A}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\mathbf{U}'(\mathbf{x}) + W(\mathbf{U},\mathbf{U}')]\,d\mathbf{x} = \int\limits_S \mathbf{R}(\mathbf{D_z},\mathbf{n})\mathbf{U}(\mathbf{z})\cdot\mathbf{U}'(\mathbf{z})\,d_{\mathbf{z}}S, \tag{4.6}$$

where the matrix differential operator $\mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))$ *is defined by* (3.5) *and*

$$W(\mathbf{U},\mathbf{U}') = W^{(1)}(\mathbf{U},\mathbf{u}') + W^{(2)}(\mathbf{U},\varphi') + W^{(3)}(\mathbf{U},\theta').$$

Theorem 4.2. *If* $\mathbf{U} = (\mathbf{u},\varphi,\theta)$ *and* $\mathbf{U}' = (\mathbf{u}',\varphi',\theta')$ *are regular vectors in* Ω^-*, then*

$$\int\limits_{\Omega^-} [\mathbf{A}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\mathbf{U}'(\mathbf{x}) + W(\mathbf{U},\mathbf{U}')]\,d\mathbf{x} = -\int\limits_S \mathbf{R}(\mathbf{D_z},\mathbf{n})\mathbf{U}(\mathbf{z})\cdot\mathbf{U}'(\mathbf{z})\,d_{\mathbf{z}}S. \tag{4.7}$$

Formulas (4.6) and (4.7) are the first Green's identity in the linear theory of MGT thermoelasticity for materials with voids in the domains Ω^+ and Ω^-, respectively.

5. Uniqueness Theorems

In this section, we prove the uniqueness theorems for the classical solutions of the basic BVPs of steady vibrations in the linear theory of MGT thermoelasticity for materials with voids.

We have the following uniqueness theorems.

Theorem 5.1. *Two regular solutions of the internal BVP* $(I)^+_{\mathbf{F},\mathbf{f}}$ *may differ only for an additive vector* $\mathbf{U} = (\mathbf{u}, \varphi, \theta)$, *where*

$$\theta(\mathbf{x}) = 0, \tag{5.1}$$

and the vector $\mathbf{V} = (\mathbf{u}, \varphi)$ *is a regular solution of the following system of homogeneous equations:*

$$\begin{gathered}(\mu\Delta + \rho\omega^2)\mathbf{u} + \left[b - (\lambda+\mu)\frac{\varepsilon_1}{\varepsilon_0}\right]\nabla\varphi = \mathbf{0},\\ \left(\alpha\Delta + \eta + \frac{b\varepsilon_1}{\varepsilon_0}\right)\varphi = 0,\end{gathered} \tag{5.2}$$

satisfying the homogeneous boundary condition

$$\{\mathbf{V}(\mathbf{z})\}^+ = \mathbf{0} \quad \text{for } \mathbf{z} \in S. \tag{5.3}$$

Moreover, the homogeneous BVPs $(I)^+_{\mathbf{0},\mathbf{0}}$ *and* (5.2), (5.3) *have the same eigenfrequencies.*

Proof. We suppose that there are two regular solutions of the BVP (3.2), (3.6). Then their difference $\mathbf{U}$ is a regular solution of the internal homogeneous BVP $(I)^+_{\mathbf{0},\mathbf{0}}$. This means that $\mathbf{U}$ is a regular solution of the system of homogeneous equations

$$\mathbf{A}(\mathbf{D}_\mathbf{x})\mathbf{U}(\mathbf{x}) = \mathbf{0} \tag{5.4}$$

for $\mathbf{x} \in \Omega^+$ satisfying the homogeneous boundary condition

$$\{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{0}, \tag{5.5}$$

where $\mathbf{z} \in S$.

By virtue of (5.4) and (5.5), from (4.2) for $\mathbf{U}' = \mathbf{U}$ it follows that

$$\begin{gathered}\int\limits_{\Omega^+} W^{(1)}(\mathbf{U}, \mathbf{u})d\mathbf{x} = 0, \qquad \int\limits_{\Omega^+} W^{(2)}(\mathbf{U}, \varphi)d\mathbf{x} = 0,\\ \int\limits_{\Omega^+} W^{(3)}(\mathbf{U}, \theta)d\mathbf{x} = 0.\end{gathered} \tag{5.6}$$

Keeping in mind relations (4.1), we get

$$\begin{aligned} W^{(0)}(\mathbf{u}, \mathbf{u}) &= \frac{1}{3}(3\lambda + 2\mu)\,|\mathrm{div}\mathbf{u}|^2 + \frac{\mu}{2}\sum_{l,j=1;l\neq j}^{3} |u_{l,j} + u_{j,l}|^2\\ &\quad + \frac{\mu}{3}\sum_{l,j=1}^{3}\left|\frac{\partial u_l}{\partial x_l} - \frac{\partial u_j}{\partial x_j}\right|^2,\\ W^{(1)}(\mathbf{U}, \mathbf{u}) &= W^{(0)}(\mathbf{u}, \mathbf{u}) - \rho\omega^2|\mathbf{u}|^2 + (b\,\varphi - \varepsilon_0\theta)\mathrm{div}\overline{\mathbf{u}},\\ W^{(2)}(\mathbf{U}, \varphi) &= \alpha|\nabla\varphi|^2 - \eta|\varphi|^2 + (b\,\mathrm{div}\mathbf{u} - \varepsilon_1\theta)\overline{\varphi},\\ W^{(3)}(\mathbf{U}, \theta) &= k_0|\nabla\theta|^2 - mc|\theta|^2 - m(\varepsilon_0\mathrm{div}\mathbf{u} + \varepsilon_1\varphi)\overline{\theta}. \end{aligned} \tag{5.7}$$

On the basis of (5.7), we can write

$$\begin{aligned}
&\mathrm{Im}W^{(1)}(\mathbf{U},\mathbf{u}) = \mathrm{Im}[(b\,\varphi-\varepsilon_0\theta)\mathrm{div}\overline{\mathbf{u}}],\\
&\mathrm{Im}W^{(2)}(\mathbf{U},\varphi) = \mathrm{Im}[(b\,\mathrm{div}\mathbf{u}-\varepsilon_1\theta)\overline{\varphi}],\\
&\mathrm{Re}W^{(3)}(\mathbf{U},\theta) = k^*|\nabla\theta|^2 - m_1 c|\theta|^2 - m_1\mathrm{Re}\left[(\varepsilon_0\mathrm{div}\mathbf{u}+\varepsilon_1\varphi)\,\overline{\theta}\right]\\
&\qquad - \omega\tau m_1 \mathrm{Im}\left[(\varepsilon_0\mathrm{div}\mathbf{u}+\varepsilon_1\varphi)\,\overline{\theta}\right],\\
&\mathrm{Im}W^{(3)}(\mathbf{U},\theta) = -\omega k|\nabla\theta|^2 + \omega\tau m_1 c|\theta|^2 - m_1\mathrm{Im}\left[(\varepsilon_0\mathrm{div}\mathbf{u}+\varepsilon_1\varphi)\,\overline{\theta}\right]\\
&\qquad + \omega\tau m_1\mathrm{Re}\left[(\varepsilon_0\mathrm{div}\mathbf{u}+\varepsilon_1\varphi)\,\overline{\theta}\right].
\end{aligned} \tag{5.8}$$

Obviously, from (5.8), we have

$$\begin{aligned}
&\mathrm{Im}W^{(1)}(\mathbf{U},\mathbf{u}) + \mathrm{Im}W^{(2)}(\mathbf{U},\varphi) = \mathrm{Im}[(b\,\varphi-\varepsilon_0\theta)\mathrm{div}\overline{\mathbf{u}}] + \mathrm{Im}[(b\,\mathrm{div}\mathbf{u}-\varepsilon_1\theta)\overline{\varphi}]\\
&\qquad = -\mathrm{Im}\left[\varepsilon_0\theta\mathrm{div}\overline{\mathbf{u}}+\varepsilon_1\theta\,\overline{\varphi}\right] = \mathrm{Im}\left[(\varepsilon_0\mathrm{div}\mathbf{u}+\varepsilon_1\varphi)\overline{\theta}\right].\\
&\omega\tau\mathrm{Re}W^{(3)}(\mathbf{U},\theta) + \mathrm{Im}W^{(3)}(\mathbf{U},\theta) = \omega(\tau k^* - k)|\nabla\theta|^2\\
&\qquad -(1+\omega^2\tau^2)m_1\mathrm{Im}\left[(\varepsilon_0\mathrm{div}\mathbf{u}+\varepsilon_1\varphi)\overline{\theta}\right].
\end{aligned} \tag{5.9}$$

From (5.9) it follows that

$$\tilde{W}(\mathbf{U}) = \omega(k-\tau k^*)|\nabla\theta|^2, \tag{5.10}$$

where

$$\begin{aligned}
\tilde{W}(\mathbf{U}) = &-(1+\omega^2\tau^2)m_1\left[\mathrm{Im}W^{(1)}(\mathbf{U},\mathbf{u}) + \mathrm{Im}W^{(2)}(\mathbf{U},\varphi)\right]\\
&- \left[\omega\tau\mathrm{Re}W^{(3)}(\mathbf{U},\theta) + \mathrm{Im}W^{(3)}(\mathbf{U},\theta)\right].
\end{aligned}$$

On the other hand, from (5.6), we get

$$\int\limits_{\Omega^+} \tilde{W}(\mathbf{U})d\mathbf{x} = 0 \tag{5.11}$$

and consequently, using the assumption (3.1) and relation (5.10), from (5.11), we can write $\nabla\theta(\mathbf{x}) = \mathbf{0}$, i.e.,

$$\theta(\mathbf{x}) = c_0 = \mathrm{const} \tag{5.12}$$

for $\mathbf{x}\in\Omega^+$. Afterwards, in view of the homogeneous boundary condition (5.5), from (5.12), we get equation (5.1).

Now, taking into account (5.1), from (5.4) follows the system of equations

$$\begin{aligned}
(\mu\Delta+\rho\omega^2)\mathbf{u} + (\lambda+\mu)\nabla\mathrm{div}\mathbf{u} + b\nabla\varphi = \mathbf{0},\\
(\alpha\Delta+\eta)\varphi - b\,\mathrm{div}\mathbf{u} = 0,\\
\varepsilon_0\mathrm{div}\mathbf{u}+\varepsilon_1\varphi = 0.
\end{aligned} \tag{5.13}$$

Obviously, from (5.13), we obtain system (5.2). Moreover, from (5.5) we get the homogeneous boundary condition (5.3).

In addition, it is easy to see that the homogeneous BVPs $(I)^+_{\mathbf{0},\mathbf{0}}$ and (5.2), (5.3) have the same eigenfrequencies. □

Theorem 5.2. *Two regular solutions of the internal BVP* $(II)^+_{\mathbf{F},\mathbf{f}}$ *may differ only for an additive vector* $\mathbf{U}=(\mathbf{u},\varphi,\theta)$*, where* θ *satisfies condition* (5.1)*, the vector* $\mathbf{V}=(\mathbf{u},\varphi)$ *is a regular solution of the system of homogeneous equations* (5.2) *for* $\mathbf{x}\in\Omega^+$ *satisfying the homogeneous boundary condition*

$$\left\{\mathbf{R}^{(2)}(\mathbf{D}_\mathbf{z},\mathbf{n}(\mathbf{z}))\mathbf{V}(\mathbf{z})\right\}^+ = \mathbf{0} \quad \textit{for}\ \ \mathbf{z}\in S, \tag{5.14}$$

where the matrix differential operator $\mathbf{R}^{(2)}(\mathbf{D}_\mathbf{z},\mathbf{n}(\mathbf{z}))$ *is defined by*

$$\mathbf{R}^{(2)}(\mathbf{D}_\mathbf{x},\mathbf{n}) = \left(R^{(2)}_{lj}(\mathbf{D}_\mathbf{x},\mathbf{n})\right)_{4\times4}, \quad R^{(2)}_{lj}(\mathbf{D}_\mathbf{x},\mathbf{n}) = R_{lj}(\mathbf{D}_\mathbf{x},\mathbf{n}),\ \ l,j=1,2,3,4.$$

In addition, the homogeneous BVPs $(II)^+_{\mathbf{0},\mathbf{0}}$ *and* (5.2), (5.14) *have the same eigenfrequencies.*

Proof. Let us say that the BVP (3.2), (3.7) has two regular solutions. Then their difference $\mathbf{U}$ is a regular solution of the internal homogeneous BVP $(II)^+_{\mathbf{0},\mathbf{0}}$. This means that $\mathbf{U}$ is a regular solution of the homogeneous system of equations (5.5) for $\mathbf{x} \in \Omega^+$ satisfying the homogeneous boundary condition

$$\{\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^+ = \mathbf{0}, \tag{5.15}$$

where $\mathbf{z} \in S$.

In a similar manner, as in the previous theorem, we obtain the relations of (5.12). Then, using (5.12), we can rewrite (5.5) as

$$\begin{gathered}(\mu\Delta + \rho\omega^2)\mathbf{u} + (\lambda + \mu)\nabla \mathrm{div}\mathbf{u} + b\nabla\varphi = \mathbf{0},\\ (\alpha\Delta + \eta)\varphi - b\,\mathrm{div}\mathbf{u} + \varepsilon_1 c_0 = 0,\\ cc_0 + \varepsilon_0 \mathrm{div}\mathbf{u} + \varepsilon_1\varphi = 0.\end{gathered} \tag{5.16}$$

Applying the operator div to the first equation of (5.16), we get

$$\begin{gathered}(\mu_0\Delta + \rho\omega^2)\mathrm{div}\mathbf{u} + b\Delta\varphi = 0,\\ -b\,\mathrm{div}\mathbf{u} + (\alpha\Delta + \eta)\varphi + \varepsilon_1 c_0 = 0,\\ \varepsilon_0 \mathrm{div}\mathbf{u} + \varepsilon_1\varphi + cc_0 = 0.\end{gathered} \tag{5.17}$$

It follows from (5.17) that

$$\det\begin{pmatrix}\mu_0\Delta + \rho\omega^2 & b\Delta & 0\\ -b & \alpha\Delta + \eta & \varepsilon_1\\ \varepsilon_0 & \varepsilon_1 & c\end{pmatrix}_{3\times 3} c_0 = 0.$$

This equation now reduces to $\rho\omega^2(\eta c - \varepsilon_1^2)c_0 = 0$. By virtue of (3.1), we find that $c_0 = 0$ and consequently, we get equation (5.1).

Furthermore, taking into account (5.1), from (5.16), we obtain system (5.2). Therewith, the boundary condition (5.15) implies relation (5.14).

Finally, it is clear that the homogeneous BVPs $(II)^+_{\mathbf{0},\mathbf{0}}$ and (5.2), (5.14) have the same eigenfrequencies. □

Theorem 5.3. *The external BVP* $(K)^-_{\mathbf{F},\mathbf{f}}$ *has one regular solution, where* $K = I, II$.

Proof. We suppose that there are two regular solutions of problem (3.2), (3.8) (and (3.2), (3.9)), then their difference $\mathbf{U}$ is a regular solution of the external homogeneous BVP $(K)^+_{\mathbf{0},\mathbf{0}}$ $(K = I, II)$). Consequently, $\mathbf{U}$ is a regular solution of (5.4) for $\mathbf{x} \in \Omega^-$ satisfying the homogeneous boundary condition

$$\{\mathbf{U}(\mathbf{z})\}^- = \mathbf{0} \tag{5.18}$$

for $K = I$ and

$$\{\mathbf{R}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z})\}^- = \mathbf{0} \tag{5.19}$$

for $K = II$. In view of (5.4), (5.18) and (5.19), from (4.5), we have

$$\int\limits_{\Omega^-} W^{(1)}(\mathbf{U}, \mathbf{u})d\mathbf{x} = 0, \qquad \int\limits_{\Omega^-} W^{(2)}(\mathbf{U}, \varphi)d\mathbf{x} = 0, \qquad \int\limits_{\Omega^-} W^{(3)}(\mathbf{U}, \theta)d\mathbf{x} = 0. \tag{5.20}$$

In a similar way as in Theorem 3, from (5.20) we obtain the relations (5.12) for $\mathbf{x} \in \Omega^-$. Afterwards, by virtue of the radiation condition (3.4), from (5.12), we get relation (5.1) for $\mathbf{x} \in \Omega^-$. Therefore, from (5.13), we have system (5.2) in Ω^-.

On the basis of (5.2) and the boundary conditions (5.18) and (5.19) the function φ satisfies the Helmholtz equation

$$\left(\alpha\Delta + \eta + \frac{b\varepsilon_1}{\varepsilon_0}\right)\varphi(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega^-, \tag{5.21}$$

and the homogeneous boundary condition

$$\{\varphi(\mathbf{z})\}^- = 0 \tag{5.22}$$

for $K = I$ and

$$\left\{\frac{\partial\varphi(\mathbf{z})}{\partial\mathbf{n}(\mathbf{z})}\right\}^{-} = 0 \tag{5.23}$$

for $K = II$. Keeping in mind the radiation conditions (3.3) and (3.4), the BVPs (5.21), (5.22) and (5.21), (5.23) have only trivial solution, i.e.,

$$\varphi(\mathbf{x}) \equiv 0, \qquad \mathbf{x} \in \Omega^{-}. \tag{5.24}$$

Now, taking into account (5.24), the first equation of system (5.2) reduces to

$$(\Delta + \zeta_4^2)\mathbf{u}(\mathbf{x}) = \mathbf{0} \qquad \text{for } \mathbf{x} \in \Omega^{-}, \tag{5.25}$$

and from the radiation conditions (3.3) and (3.4), we obtain

$$u_l(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad \left(\frac{\partial}{\partial|\mathbf{x}|} - i\zeta_4\right) u_l(\mathbf{x}) = o(|\mathbf{x}|^{-1}), \quad l = 1, 2, 3, \tag{5.26}$$

for $|\mathbf{x}| \gg 1$. Moreover, the boundary conditions (5.18) and (5.19) reduce to

$$\{\mathbf{u}(\mathbf{z})\}^{-} = \mathbf{0} \tag{5.27}$$

and

$$\left\{\mathbf{R}^{(0)}(\mathbf{D}_{\mathbf{z}}, \mathbf{n})\mathbf{u}(\mathbf{z})\right\}^{-} = \mathbf{0}, \tag{5.28}$$

respectively. Here, the matrix differential operator $\mathbf{R}^{(0)}(\mathbf{D}_{\mathbf{z}}, \mathbf{n}(\mathbf{z}))$ is defined by (4.4).

Obviously, the BVPs (5.25), (5.27) and (5.25), (5.28) with the radiation conditions (5.26) have only trivial solution, i.e., $\mathbf{u}(\mathbf{x}) \equiv \mathbf{0}$ for $\mathbf{x} \in \Omega^{-}$. Therefore, $\mathbf{U}(\mathbf{x}) \equiv \mathbf{0}$ for $\mathbf{x} \in \Omega^{-}$. Hence, we have the desired result. □

6. Conclusion

1. In the present paper, the linear theory of MGT thermoelasticity for materials with voids is considered and the following results are obtained:

(i) The governing equations of motion and steady vibrations of this theory are proposed. The basic system of equations of steady vibrations is expressed in terms of the displacement vector, the changes of temperature and pores volume fraction.

(ii) The radiation conditions are established and the first Green's identity is obtained.

(iii) The uniqueness theorems for classical solutions of the internal and external BVPs of steady vibrations in the theory under consideration are proved.

2. By virtue of the results of this paper it is possible to prove:

(i) the existence theorems for classical solutions of the BVPs of steady vibrations in the theory of MGT thermoelaticity for materials with voids by using the potential method and the theory of singular integral equation;

(ii) the uniqueness theorems in the MGT thermoelasticity for materials with multiple voids.

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