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THE CONTACT PROBLEM FOR PIECEWISE-HOMOGENEOUS VISCOELASTIC PLATE WITH ELASTIC INCLUSION

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Dedicated to the memory of Professor Elene Obolashvili

Abstract. A piecewise-homogeneous viscoelastic plate, reinforced with a semi-infinite elastic inclusion, which meets the interface of two materials at a right angle and is loaded with normal forces is considered. The problem is reduced to a two-dimensional singular integro-differential equation with fixed singularity. Using the methods of the theory of analytic functions, the Carleman type problem for a strip is reduced to the Volterra integral equation, which is solved approximately by the method of successive approximations. The normal contact stresses along the contact line are determined and the behavior of contact stresses in the neighborhood of singular points is established.

Introduction

Exact and approximate solutions of static contact problems for different domains, reinforced with elastic thin inclusions, stringers and patches of variable rigidity were obtained earlier, and the behavior of the contact stresses at the ends of the contact line have been investigated [2,3,17,19–21]. The first fundamental problem was solved for a piecewise-homogeneous plane, when a crack of finite length approaches the interface of two bodies at the right angle [14], a similar problem was solved for a piecewise-homogeneous plane under the action of symmetrical normal stresses at the crack sides [7,22], and also a contact problem was solved for a piecewise-homogeneous plate with a semi-infinite and finite inclusion [8,9,12].

1. Statement of the Problem

Suppose the body occupies a complex plane $z = x + iy$, consisting of two dissimilar isotropic half-planes with viscoelastic properties [2,4,10,18]. The plane is reinforced with a semi-infinite elastic inclusion which is subjeted to the normal load of intensity $p_0(x,t)$. The function $p_0(x,t)$ satisfies Hölder's condition on an arbitrary finite segment of the interval $(0,+\infty)$ [16].

The half-planes $S_1 = \{z \mid \operatorname{Re} z > 0,\ z \notin l_1 = [0,\infty)\}$ and $S_2 = \{z \mid \operatorname{Re} z < 0\}$ are connected along the $Oy-$axis. The quantities and functions related to the half-planes S_k we denote by the index k $(k = 1,2)$, and the boundary values of the functions on the upper and lower edges of the inclusion are denoted by the signs $(+)$ and $(-)$, respectively (Figure 1).

The contact conditions along the interface are of the form

$$\sigma_x^{(1)} = \sigma_x^{(2)}, \quad \tau_{xy}^{(1)} = \tau_{xy}^{(2)}, \quad u_1 = u_2, \quad v_1 = v_2. \tag{1.1}$$

On the boundary of interaction of the elastic inclusion and half-plane S_1, the following conditions

$$\begin{gathered} \sigma_y^{(l)+} - \sigma_y^{(l)-} = p(x,t), \quad \tau_{xy}^{(l)+} - \tau_{xy}^{(l)-} = 0, \\ u_1^+ - u_1^- = 0, \quad \nu_1^+ = \nu_1^- = \nu(x,t), \end{gathered} \tag{1.2}$$

$$D_0 \frac{d^4 \nu_0(x,t)}{dx^4} = p_0(x,t) - p(x,t), \quad x > 0, \tag{1.3}$$

$$\nu_0(x,t) = \nu(x,t), \tag{1.4}$$

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$$\int_{l_1}[p(x,t)-p_0(x,t)]dx=0,\quad \int_{l_1}x[p(x,t)-p_0(x,t)]dx=0, \tag{1.5}$$

are valid, where (1.2) represents the jumps of the stress components and displacements of plate points on the contact line, (1.3) is the equation of bending of an elastic inclusion, (1.4) is the condition of rigid contact between the plate and the inclusion, (1.5) are equilibrium conditions of the inclusion. D_0 is bending rigidity of the inclusion material.

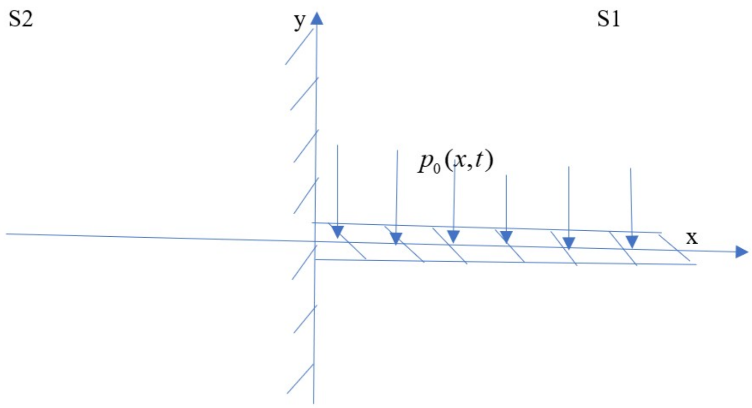

FIGURE 1

In the theory of viscoelatisity we have the formulas of Kolosov–Muskhelishvili's type [13]:

$$\sigma_y^{(k)}-i\tau_{xy}^{(k)}=\Phi_k\left(z,t\right)+\overline{\Phi_k\left(z,t\right)}+z\overline{\Phi_k'\left(z,t\right)}+\overline{\Psi_k\left(z,t\right)}, \tag{1.6}$$

$$(I-L)\left[\varkappa_k\Phi_k\left(z,t\right)-\overline{\Phi_k\left(z,t\right)}-z\overline{\Phi_k'\left(z,t\right)}-\overline{\Psi_k\left(z,t\right)}\right]=2\mu_k\left(n_k'+i\nu_k'\right), \tag{1.7}$$

where $(I-L)g_k\left(t\right)=g_k\left(t\right)-\int\limits_{t_0}^{t}E_k\frac{\partial}{\partial\tau}C_k\left(t,\tau\right)g_k\left(\tau\right)d\tau,\ 2\mu_k=\frac{E_k}{1+\nu_k},\ \varkappa_k=3-4\nu_k,$ (in the case of plane strain) or $\varkappa_k=(3-\nu_k)/(1+\nu_k)$ (in the case of generalized plane stress), $k=1,2$. $C_k(t,\tau)=\varphi_k(\tau)(1-e^{-\gamma(t-\tau)})$ and E_k are the creep measure and the Young module of the materials, respectively. Here, $\varphi_k(\tau)$ is known as the ageing function, and the function $(1-e^{-\gamma(t-\tau)})$ characterizes the hereditary properties of a materials, t_0 is the ageing of the material at the beginning of loading.

Besides, the Poisson plate coefficients for the elastic-instant deformation $v_k(t)$ and creep deformation $v_k(t,\tau)$ are the same and constant: $\nu_k(t)=\nu_k(t,\tau)=\nu_k=\text{const}$.

From relations (1.6), (1.7), we obtain the following boundary value problems of linear conjugation:

$$\Phi_1^+(x,t)-\Phi_1^-(x,t)=\frac{1}{\varkappa_1+1}p(x,t),$$

$$\Psi_1^+\left(x,t\right)-\Psi_1^-\left(x,t\right)=\frac{\varkappa_1-1}{\varkappa_1+1}\,p(x,t)-\frac{1}{\varkappa_1+1}\,xp'(x,t),\quad x>0.$$

The general solutions of these problems are represented as follows [15]:

$$\Phi_1\left(z,t\right)=A_1(z,t)+W_1(z,t),\quad \Psi_1\left(z,t\right)=B_1(z,t)+Q_1(z,t), \tag{1.8}$$

where

$$A_1(z,t)=\frac{1}{2\pi(\varkappa_1+1)i}\int\limits_0^\infty\frac{p(x,t)dx}{x-z},\quad B_1\left(z,t\right)=\frac{\varkappa_1-1}{2\pi(\varkappa_1+1)i}\int\limits_0^\infty\frac{p(x,t)dx}{x-z}$$

$$-\frac{1}{2\pi(\varkappa_1+1)i}\int\limits_0^\infty\frac{xp'(x,t)dx}{x-z}\,,$$

$W_1(z,t)$ and $Q_1(z,t)$ are unknown analytic functions in the half-plane S_1, which will be defined from the contact conditions (1.1) at the interface.

Using the methods of the theory of analytic functions (particularly, using the Cauchy theorems), the sought analytic functions are represented in the following form:

$$W_1(z,t) = -e_1 t_1 \int\limits_{l_1} \frac{xp'(x,t)dx}{x+z} + e_1 t_1 \int\limits_{l_1} \frac{xp(x,t)dx}{(x+z)^2} + e_1 t_1(\varkappa_1 - 1) \int\limits_{l_1} \frac{p(x,t)dx}{x+z},$$

$$\Phi_2(z,t) = h_3 t_1 \int\limits_{l_1} \frac{p(x,t)dx}{x-z},$$

$$Q_1(z,t) = -e_1 t_1 \int\limits_{l_1} \frac{x^2 p'(x,t)dx}{(x+z)^2} + m_1 t_1 \int\limits_{l_1} \frac{p(x,t)dt}{x+z} + e_1 t_1(\varkappa_1 - 1) \int\limits_{l_1} \frac{xp(x,t)dx}{(x+z)^2}$$

$$+e_1 t_1 z \int\limits_{l_1} \frac{p(x,t)dx}{(x+z)^2} + 2e_1 t_1 z \int\limits_{l_1} \frac{xp(x,t)dx}{(x+z)^3},$$

$$\Psi_2(z,t) = (h_3 - h_4) t_1 z \int\limits_{l_1} \frac{p(x,t)dx}{(x-z)^2} - h_4 t_1 \int\limits_{l_1} \frac{xp'(x,t)dx}{x-z} + (h_4(\varkappa_1 - 1) + m_1) t_1 \int\limits_{l_1} \frac{p(x,t)dt}{x-z}, \quad (1.9)$$

where $t_1 = \dfrac{1}{2\pi i(\varkappa_1+1)}, \quad e_1 = \dfrac{\mu_2-\mu_1}{\varkappa_1\mu_2+\mu_1}, \quad e_2 = \dfrac{\mu_2-\mu_1}{\varkappa_2\mu_1+\mu_2},$

$$m_1 = (\varkappa_1+1)\,\mu_2 \left[\frac{1}{\varkappa_2\mu_1+\mu_2} - \frac{1}{\varkappa_1\mu_2+\mu_1}\right] = h_2 - h_4,$$

$$m_2 = (\varkappa_2+1)\,\mu_1 \left[\frac{1}{\varkappa_2\mu_1+\mu_2} - \frac{1}{\varkappa_1\mu_2+\mu_1}\right] = h_3 - h_1,$$

$$h_1 = \frac{(\varkappa_2+1)\,\mu_1}{\varkappa_1\mu_2+\mu_1}, \quad h_2 = \frac{(\varkappa_1+1)\,\mu_2}{\varkappa_2\mu_1+\mu_2}, \quad h_3 = \frac{(\varkappa_2+1)\,\mu_1}{\varkappa_2\mu_1+\mu_2}, \quad h_4 = \frac{(\varkappa_1+1)\,\mu_2}{\varkappa_1\mu_2+\mu_1}.$$

Relations (1.8), (1.9) and (1.6), (1.7) result in

$$\begin{aligned}(\varkappa_1+1)\Phi_1(z,t) &= \frac{1}{2\pi i} \int\limits_{l_1} \frac{p(y,t)dy}{y-z} + \frac{e_1\varkappa_1}{2\pi i} \int\limits_{l_1} \frac{p(y,t)dy}{y+z},\\ 2\mu_1 \frac{d\nu(x,t)}{dx} &= (\varkappa_1+1)(I-L)\mathrm{Im}\Phi_1(x,t).\end{aligned} \quad (1.10)$$

It is required to determine the law of distribution of normal contact stresses $p(t,x)$ on the contact line, the asymptotic behavior of these stresses at the end of the inclusion.

To define the unknown contact stresses from (1.3), (1.10), we obtain the following two-dimensional integro-differential equation:

$$-D_0 \frac{d^4}{dx^4}(I-L)\left\{\frac{1}{4\pi\mu_1}\int\limits_0^\infty \frac{p(t,y)dy}{y-x} + \frac{e_1\varkappa_1}{4\pi\mu_1}\int\limits_0^\infty \frac{p(t,y)dy}{y+x}\right\} = p_0(t,x) - p(t,x), \quad x>0. \quad (1.11)$$

The inclusion equilibrium condition has the form

$$\int\limits_0^\infty [p(t,y) - p_0(t,y)]dy = 0, \quad \int\limits_0^\infty y[p(t,y) - p_0(t,y)]dy = 0. \quad (1.12)$$

Introducing the notation

$$\psi(t,x) = \int\limits_0^x ds \int\limits_0^s [p_0(t,y) - p(t,y)]dy, \quad f(t,x) = \int\limits_0^\infty \frac{p_0(t,y)dy}{y-x} + e_1\varkappa_1 \int\limits_0^\infty \frac{p_0(t,y)dy}{y+x},$$

from (1.11) and (1.12), we obtain the two-dimensional singular integro-differential equation with a fixed singularity

$$\frac{D_0}{4\pi\mu_1}(I-L)\left\{\int_0^\infty \frac{\psi''(t,y)dy}{y-x} + e_1\varkappa_1 \int_0^\infty \frac{\psi''(t,y)dy}{y+x}\right\}$$

$$= \int_0^x \psi(t,y)dy + (I-L)\frac{D_0}{4\pi\mu_1} f(t,x), \quad x>0, \tag{1.13}$$

and with the boundary conditions

$$\psi(t,0) = \psi(t,\infty) = 0, \quad \psi'(t,0) = \psi'(t,\infty) = 0. \tag{1.14}$$

(The constant of integration vanishes, since the rotation at the inclusion end is neglected, i.e., $\frac{d\nu_0(x,t)}{dx}|_{x=0} = 0$, the integral for $y = x$ is understood in the sense of the Cauchy principal value).

Suppose that the function $p_0(t,x)$ is continuous and integrable on the interval $(0,\infty)$ and satisfies the conditions

$$\int_0^\infty p_0(t,x)dx = 0, \quad p_0(t,x) = O(x^\varepsilon), \quad x \to 0+, \quad p_0(t,x) = O(x^{-2+\varepsilon}), \quad x \to \infty \tag{1.15}$$

(ε is an arbitrary small positive number).

2. Solution of the Integro-differential Equation

We are going to find a solution of problem (1.13) and (1.14) in the class of functions whose second derivative may have integrable singularities at the point $x = 0$ and which vanishes at infinity. The change of the variables $x = e^\xi$, $y = e^\xi$ yields

$$\frac{\lambda}{\pi}(1-L)\int_{-\infty}^\infty \left[\frac{1}{1-e^{\xi-\zeta}} + \frac{e_1\varkappa_1}{1+e^{\xi-\zeta}}\right]\left[\psi_0''(t,\zeta) - \psi_0'(t,\zeta)\right] e^{-2\zeta} d\zeta = \int_{-\infty}^\xi e^\tau \psi_0(t,\zeta)d\zeta$$

$$+\frac{\lambda}{\pi}(1-L)f_0(t,\xi), \quad |\xi| < \infty \quad \psi_0(t,\pm\infty) = 0, \quad \psi_0'(t,\pm\infty) = 0, \tag{2.1}$$

where $\psi_0(t,\xi) = \psi(t,e^\xi)$, $f_0(t,\xi) = f(t,e^\xi)$, $\lambda = \frac{D_0}{4\mu_1}$.

Owing to the generalized Fourier transformation [11] of both parts of equation (2.1), we obtain

$$\lambda(s-i)(s-2i)sG(s)(I-L)\Psi(t,s) = \Psi(t,s-3i) + F(t,s), \quad s = s_0 + i\varepsilon, \quad |s_0| < \infty, \tag{2.2}$$

where $G(s) = cth\pi s + \frac{e_1\varkappa_1}{sh\pi s}$,

$$\Psi(t,s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \psi_0(t,\xi)e^{is\xi}d\xi, \quad F(t,s) = \lambda(s-2i)G(s)(I-L)\widehat{P}_0(t,s-2i),$$

$$P_0(t,\xi) = p_0(t,e^\xi), \quad \hat{P}_0(t,s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty P_0(t,\xi)e^{i\xi\zeta}d\xi.$$

It follows from condition (1.15) that the function $\hat{P}_0(t,z)$ is analytic in the strip $-2+\varepsilon < \operatorname{Im} z < \varepsilon$, therefore the function $F(t,z)$ is analytic in the strip $\varepsilon < \operatorname{Im} z < 2+\varepsilon$ and exponentially vanishes at infinity.

The Carleman type problem for a strip is formulated as follows: find the function $\Psi^-(t,z)$, which is analytic in the strip $-3+\varepsilon < \operatorname{Im} z < 3+\varepsilon$ (with the exception of a finite number of points lying in the strip $\varepsilon < \operatorname{Im} z < 3+\varepsilon$, at which it has the first order poles), continuously extendable to the strip boundary, vanishing at infinity and satisfying condition (2.2) [5,6].

If we find the function $\Psi^-(t,z)$, holomorphic in the strip $-3+\varepsilon < \mathrm{Im} z < \varepsilon$ and continuously extendable to the strip boundary, vanishing at infinity and satisfying condition (2.2), then the solution of the Carleman type problem is the function

$$\Psi_0(t,z) = \begin{cases} (I-L)\Psi^-(t,z), & -3+\varepsilon < \mathrm{Im} z < \varepsilon, \\ \dfrac{\Psi^-(t,z-3i)+F(t,z)}{\lambda z(z-i)(z-2i)G(z)}, & \varepsilon < \mathrm{Im} z < 3+\varepsilon. \end{cases} \tag{2.3}$$

Representing the function $sG(s)(s-i)(s-2i)$ in the form

$$sG(s)(s-i)(s-2i)$$
$$= is\left(\mathrm{cth}\,\pi s + \frac{e_1\varkappa_1}{\mathrm{sh}\,\pi s}\right) th(\pi s/6)\frac{\sin(\pi/6)(s-3i)}{\mathrm{sh}(\pi s/6)}\,\frac{s-i}{s+2i}\,\frac{2s+3i}{2s-3i}(s^2+4)\,\frac{2s-3i}{2s+3i},$$

taking into account that the index of the function $G_0(s) = \left(\mathrm{cth}\,\pi s + \frac{e_1\varkappa_1}{\mathrm{sh}\,\pi s}\right) th(\pi s/6)\frac{s-i}{s+2i}\,\frac{2s+3i}{2s-3i}$ on the real axis is equal to zero, $G_0(\pm\infty)=1$ and the function $\ln G_0(s)$ is integrable on this axis, we obtain

$$G_0(s) = \frac{\mathrm{X}_0(s-3i)}{\mathrm{X}_0(s)}, \quad |s|<\infty, \tag{2.4}$$

where $X_0(z) = \exp\left\{\frac{1}{6i}\int\limits_{-\infty}^{\infty} \ln G_0(s)\,\mathrm{cth}\,\frac{\pi}{3}(s-z)ds\right\}$.

The function $X_0(z)$ is holomorphic in the strip, continuous on this boundary, bounded in the closed strip $-3 \le \ln z \le 0$ and vanishing at infinity.

Remark. Let us solve the following functional equations:

$$\chi(s-i\alpha) = \lambda(\beta+is)\chi(s), \quad \chi(s) = (\beta-is)\chi(s-i\alpha), \quad \alpha,\beta,\lambda>0.$$

By the Fourier transformation, we obtain respectively the first order differential equations

$$\tilde{\chi}'(y) + \left(\beta - \frac{1}{\lambda}e^{-xy}\right)\tilde{\chi}(y) = 0, \quad \tilde{\chi}'(y) + \left(e^{\alpha y} - \alpha - \beta\right)\tilde{\chi}(y) = 0,$$

the solutions of these differential equations and the inverse Fourier transform give

$$\chi(z) = M_1\int\limits_{-\infty}^{\infty}\exp\left\{-\beta s - \frac{1}{\lambda\alpha}e^{-\alpha s} - isz\right\}ds = M_1\lambda^{\frac{\beta+iz}{\alpha}}\alpha^{\frac{\beta+iz}{\alpha}-1}\Gamma\left(\frac{\beta+iz}{\alpha}\right),$$

$$\chi(z) = M_2\int\limits_{-\infty}^{\infty}\exp\left\{-\frac{1}{\alpha}e^{\alpha s} + (\alpha+\beta)s - isz\right\}ds = M_2\alpha^{\frac{\beta-iz}{\alpha}}\Gamma\left(\frac{\alpha+\beta-iz}{\alpha}\right),$$

where $\Gamma(z) = \int\limits_0^{\infty} e^{-\tau}\tau^{z-1}d\tau$ is the well-known Gamma-function, M_1 and M_2 are the constants.

As a result from the remark, the function $s^2+4 = (2-is)(2+is)$ can be written as follows:

$$s^2+4 = \frac{X_1(s-3i)}{X_1(s)}, \tag{2.5}$$

where $\mathrm{X}_1(z) = 3^{(2iz/3)-2}\frac{\Gamma((2+iz)/3)}{\Gamma((5-iz)/3)}$. Introducing the notation $\Psi_1(t,z) = \frac{iz\Psi^-(t,z)}{X_0(z)X_1(z)\,\mathrm{sh}(\pi z/6)(z+3i/2)}$, from (2.2), (2.4), (2.5), we get

$$\lambda(3+is)(I-L)\Psi_1(t,s)$$
$$= \Psi_1(t,s-3i) + \frac{iF(t,s)(3+is)}{X_0(s-3i)X_1(s-3i)ch(\pi s/6)(s-3i/2)}, \quad |s|<\infty. \tag{2.6}$$

Considering the relation

$$\lambda(3+is) = \frac{X_2(s-3i)}{\mathrm{X}_2(s)}, \tag{2.7}$$

where $X_2(z)=\lambda^{(3+iz)/3}3^{iz/3}\Gamma((3+iz)/3)$, condition (2.6) will take the following form:

$$(I-L)\Psi_2(t,s)=\Psi_2(t,s-3i)$$
$$+\frac{iF(t,s)(3+is)}{X_0(s-3i)X_1(s-3i)X_2(s-3i)ch(\pi s/6)(s-3i/2)},\quad |s|<\infty,$$

where $\Psi_2(t,s)=\frac{\Psi_1(t,s)}{X_2(s)}$.

As a result, the Carleman type boundary condition (2.2) is presented in the form

$$(I-L)\frac{\Psi^-(t,s)}{X(s)}=\frac{\Psi^-(t,s-3i)}{X(s-3i)}+P(t,s),\quad |s|<\infty, \tag{2.8}$$

where

$$X(z)=\frac{X_0(z)X_1(z)X_2(z)\operatorname{sh}\pi z/6}{iz}(z+3i/2),\quad P(t,s)=\frac{F(t,s)}{X(s-3i)}\,.$$

Since the function $X_0(z)$ is holomorpic in the strip $3<\operatorname{Im} z<0$ and bounded in the closed strip, using Stirling's formula for the Gamma function [1], we conclude that the function $X(z)$ for sufficiently large $|z|$ admits the following estimate:

$$X(z)=O(|s|^{-\omega-1/2}),\quad |s|\to\infty,\quad z=s+i\omega,\quad -3\le\omega\le 0. \tag{2.9}$$

Moreover, by choosing the constants M_1, M_2 mentioned in our Remark, the function $X(z)$ satisfies the condition

$$\lim_{|s|\to\infty}|\mathrm{X}(z)|\,|s|^{\omega+1/2}=1,\quad z=s+i\omega,\quad -3\le\omega\le 0.$$

The function $\Phi(t,z)=\frac{\Psi^-(t,z)}{X(z)}$ is holomorphic in the strip $-3<\ln z<0$, with the exception of the point $z=-3i/2$, at which it may have the first order pole. Applying the Fourier transformation to (2.8), we obtain the Volterra second order integral equation

$$\left[e^{-3w}+(I-L)\right]\hat{\Phi}(t,w)=\check{P}(t,w)+A(t)e^{-3w/2}, \tag{2.10}$$

where $\hat{\Phi}(t,w)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\Phi(t,s)e^{isw}ds$, $\check{P}(t,w)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}P(t,s)e^{isw}ds$, $A(t)$ is an unknown function.

The Volterra integral equation (2.10) is equivalent to the second order differential equation

$$\ddot{\hat{\Phi}}(t,w)+\gamma\alpha(t,w)\dot{\hat{\Phi}}(t,w)=g(t,w) \tag{2.11}$$

with the initial conditions

$$\hat{\Phi}(\tau_0,w)=[\check{P}(\tau_0,w)+A(\tau_0)e^{-3w/2}](1+e^{-3w})^{-1},$$
$$\dot{\hat{\Phi}}(\tau_0,w)=[\dot{\check{P}}(\tau_0,w)-\gamma E_1\varphi_1(\tau_0)\check{P}(\tau_0,w)(1+e^{-3w})^{-1}](1+e^{-3w})^{-1}$$
$$+[\dot{A}(\tau_0)-\gamma E_1\varphi_1(\tau_0)A(\tau_0)(1+e^{-3w})^{-1}]e^{-3w/2}(1+e^{-3w})^{-1},$$
$$\alpha(t,w)=1+E_1\varphi_1(t)(1+e^{-3w})^{-1},\quad g(t,w)=[g_0(t,w)+T(t)e^{-3w/2}](1+e^{-3w})^{-1}$$
$$g_0(t,w)=\ddot{\check{P}}(t,w)+\gamma\dot{\check{P}}(t,w),\quad T(t)=\ddot{A}(t)+\gamma\dot{A}(t),$$
$$\dot{\square}\equiv\frac{\partial}{\partial t},\qquad \ddot{\square}\equiv\frac{\partial}{\partial t}.$$

Integrating the differential equation (2.11) and fulfilling the initial conditions, we obtain the expression

$$\hat{\Phi}(t,w)=\left\{\check{P}(t,w)+A(t)e^{-3w/2}+F_1(t,\tau_0,w)+F_2(t,\tau_0,w)\right\}(1+e^{-3w})^{-1}, \tag{2.12}$$

where

$$F_1(t,\tau_0,w)=\gamma\check{P}(\tau_0,w)\varphi_1(\tau_0)(1+e^{-3w})^{-1}\int\limits_{\tau_0}^{t}\exp(-\gamma b(w,\tau,\tau_0))d\tau$$

$$-\gamma\int\limits_{\tau_0}^{t}\exp(-b(w,\tau,\tau_0))d\tau\int\limits_{\tau_0}^{\tau}(\alpha(q,w)-1)\exp(\gamma b(w,q,\tau_0))\bar{P}(q,w)dq,$$

$$F_2(t,\tau_0,w)=\gamma A(\tau_0)\varphi_1(\tau_0)(1+e^{-3w})^{-1}e^{-3w/2}\int\limits_{\tau_0}^{t}\exp(-\gamma b(w,\tau,\tau_0))d\tau$$

$$-e^{-3w/2}\gamma\int\limits_{\tau_0}^{t}\exp(-\gamma b(w,\tau,\tau_0))d\tau\int\limits_{\tau_0}^{\tau}(a(q,w)-1)\exp(\gamma b(w,q,\tau_0))\dot{A}(q)dq,$$

$$b(w,\tau,\tau_0)=\int\limits_{\tau_0}^{\tau}a(p,w)dp=(\tau-\tau_0)+E_1\psi_1(\tau,\tau_0)(1+e^{-3w})^{-1},\quad \psi_1(\tau,\tau_0)=\int\limits_{\tau_0}^{\tau}\varphi_1(p)dp.$$

By the inverse integral transformation of equality (2.12) and using the generalized Parseval's formula, we obtain

$$\Psi^-(t,z)=\frac{\mathrm{X}(z)}{3i}\sqrt{\frac{2}{\pi}}\int\limits_{-\infty}^{\infty}\frac{F(t,s)ds}{\mathrm{X}(s-3i)\,\mathrm{sh}\,\pi(s-z)/3}+\frac{A(t)X(z)}{ch\pi z/3}$$

$$+X(z)\gamma\varphi_1(\tau_0)\int\limits_{\tau_0}^{t}Q_1^{(1)}(\tau,z)d\tau-X(z)\gamma\int\limits_{\tau_0}^{t}d\tau\int\limits_{\tau_0}^{\tau}Q_2^{(1)}(\tau,q,z)dq+\mathrm{X}(z)\gamma\varphi_1(\tau_0)A(\tau_0)\int\limits_{\tau_0}^{t}\bar{Q}_1^{(2)}(\tau,z)d\tau$$

$$-\mathrm{X}(z)\gamma\int\limits_{\tau_0}^{t}d\tau\int\limits_{\tau_0}^{\tau}\tilde{Q}_2^{(2)}(\tau,q,z)\dot{A}(q)dq, \tag{2.13}$$

where

$$\tilde{Q}_1^{(1)}(\tau,z)=\int\limits_{-\infty}^{\infty}\frac{\exp(-\gamma b(w,\tau,\tau_0))\check{P}(\tau_0,w)e^{-iwz}dw}{(1+e^{-3w})^2},$$

$$\tilde{Q}_2^{(1)}(\tau,q,z)=\int\limits_{-\infty}^{\infty}\frac{\exp(-\gamma b(w,\tau,\tau_0))(\alpha(q,w)-1)\exp(\gamma b(w,q,\tau_0))\dot{\check{P}}(q,w)e^{-iwz}dw}{1+e^{-3w}},$$

$$\tilde{Q}_1^{(2)}(\tau,z)=\int\limits_{-\infty}^{\infty}\frac{\exp(-\gamma b(w,\tau,\tau_0))e^{-3w/2}e^{-iwz}dw}{(1+e^{-3w})^2},$$

$$\tilde{Q}_2^{(2)}(\tau,q,z)=\int\limits_{-\infty}^{\infty}\frac{\exp(-\gamma b(w,\tau,\tau_0))(\alpha(q,w)-1)\exp(\gamma b(w,q,\tau_q))e^{-3w/2}e^{-iwz}dw}{1+e^{-3w}}.$$

Since the function $G(z)$ has zeros at the points $z=\pm\frac{i}{\pi}\arccos(-e_1\varkappa_1)+2ki$, $|e_1\varkappa_1|<1$ and has the poles at the points $z=ki$, $k=0,\pm1,\pm2,\dots$, therefore the function $\Psi_0(t,z)$, represented by (2.3), is holomorphic in the strip $-3<\mathrm{Im}\,z<3$, with the exception of the points $z_0=y_0i$, $z_1=(2-y_0)i,\dots$, $y_0=\frac{1}{\pi}\arccos(-e_1\varkappa_1)$, $0<y_0<1$, at which it has the first order poles. It follows from (2.9), (2.13) that function $\Psi_0(t,z)$ exponentially vanishes at infinity.

To define the function $A(t)$, from formula (2.3), by satisfying the condition

$$\Psi^-(t,i(y_0-3))=0,\quad t\geq\tau_0,$$

we obtain the Volterra second order integral equation

$$\dot{A}(t)+\int\limits_{\tau_0}^{t}R(\tau,q)\dot{A}(q)dq=R_0(t), \tag{2.14}$$

where

$$R(t,q) = \gamma \cos\frac{\pi y_0}{3}\tilde{Q}_2^{(2)}(t,q,i(y_0-3)),$$

$$R_0(t) = \cos\frac{\pi y_0}{3}\Bigg[-\frac{1}{3i}\sqrt{\frac{2}{\pi}}\int\limits_{-\infty}^{\infty}\frac{\dot{F}(t,s)ds}{X(s-3i)\,\mathrm{sh}(\pi/3)(s-iy_0)}$$

$$+\gamma\varphi_1(\tau_0)[\tilde{Q}_1^{(1)}(t,i(\mathbf{y}_0-3)) + A(\tau_0)\tilde{Q}_1^{(2)}(t,i(\mathbf{y}_0-3))]\Bigg]$$

$$-\gamma\cos\frac{\pi y_0}{3}\int\limits_{\tau_0}^{t}\tilde{Q}_2^{(1)}(t,q,i(y_0-3))dq. \tag{2.15}$$

Equation (2.14) admits the application of the method of successive approximations.

Accordingly, the function $\Psi_0(t,z)$ has the pole (closest to the real axis in the strip $0 < \mathrm{Im}\, z < 3$) of first order at the point $z_1 = (2-y_0)i$.

Using the Cauchy formula and the residue theorem, applying the inverse Fourier transformation, we obtain [11]

$$\psi_0'(t,\ln x) = -\frac{i}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} s\Psi_0(t,s)e^{-is\ln x}ds = -\frac{ix^3}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}(s+3i)\Psi_q(t,s+3i)e^{-is\ln x}ds$$

$$+\sqrt{2\pi}res\left[s\Psi_0(t,s)e^{-is\ln x}\right]_{s=(2-y_0)i} = x^3q_1(t,x) + C_1(t)x^{2-y_0},$$

$$\psi_0''(t,\ln x) = -\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}s^2\Psi_0(t,s)e^{-is\ln x}ds = x^3q_2(t,x) + C_2(t)x^{2-y_0},$$

where $q_j(t,x), C_j(t)$ are the known functions and $q_j(t,0+) = q_j(t) \neq 0$, $j = 1;2$,

$$C_j(t) = \sqrt{2\pi}(-1)^j i^{j+1}\left[\frac{s^{j-1}\left[\Psi^-(t,s-3i) + F(t,s)\right]}{\left[\lambda(s-i)(s-2i)G(s)\right]_s'}\right]_{s=(2-y_0)i}, \quad j = 1;2, \quad (i^2 = -1).$$

Based on the formula $\psi''(t,x) = \frac{\psi_0''(t,\ln x) - \psi_0'(t,\ln x)}{x^2}$, for the sought function, we obtain the following estimate:

$$\psi''(t,x) = x^{-y_0}\tilde{C}(t) + x\tilde{q}(t,x),$$

where

$$\tilde{C}(t) = C_2(t) - C_1(t), \quad \tilde{q}(t,x) = q_2(t,x) - q_1(t,x), \quad y_0 = \frac{1}{\pi}\arccos(-e_1\varkappa_1), \quad 0 < y_0 < 1.$$

Therefore, the normal contact stresses in the neighborhood of the point $x = 0$ have the following behavior:

$$p(t,x) - p_0(t,x) = \psi''(t,x) = x^{-y_0}(\tilde{C}(t) + \varepsilon(t,x)),$$

where $\varepsilon(t,x) = x^{1+y_0}\tilde{q}(t,x)$ is a continuous function on the semi-axis $x \geq 0$, and

$$\varepsilon(t,x) = O(x^{1+y_0}), \quad x \to 0+.$$

With a similar reasoning, we can conclude that the normal contact stresses vanish at infinity with the power greater than three.

Conclusion. The following conclusions are valid:

a) If $e_1 < 0$, $(\mu_2 < \mu_1)$, then $0 < y_0 < 1/2$, therefore, the normal contact stresses have singularities of order less than $1/2$.

b) If $e_1 < 0$, $(\mu_2 > \mu_1)$, then $1/2 < y_0 < 1$, therefore, the normal contact stresses have integrable singularities of order greater than $1/2$.

c) If $e_1 = 0$, $(\mu_2 = \mu_1)$, then $y_0 = 1/2$ and the normal contact stresses have singularities of square root order.

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References

1. M. Abramowitz, I. Stegun (eds.), *Reference Book on Special Function.* (Russian) Nauka, Moscow, 1979.
2. V. M. Aleksandrov, S. M. Mkhitaryan, *Contact Problems for Bodies with thin Coverings and Layers.* (Russian) Nauka, Moscow, 1983.
3. Yu. A. Antipov, N. Kh. Arutynyanm, A contact problem for an elastic layer with cover plates in the presence of friction and adhesion. (Russian) *translated from Prikl. Mat. Mekh.* **57** (1993), no. 1, 137–147; *J. Appl. Math. Mech.* **57** (1993), no. 1, 159–170.
4. N. Kh. Arutyunyan, *Some Problems in the Theory of Creep.* Pergamon Press. Oxford. 1966.
5. R. D. Bantsuri, A certain boundary value problem in analytic function theory. (Russian) *Sakharth. SSR Mecn. Akad. Moambe* **73** (1974), 549–552.
6. R. D. Bantsuri, The contact problem for an anisotropic wedge with an elastic stringer. (Russian) *Dokl. Akad. Nauk SSSR* **222** (1975), no. 3, 568–571.
7. R. D. Bantsuri, The first fundamental problem for a piecewise homogeneous orthotropic plane with a cut perpendicular to the line of interface. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **91** (1978), no. 3, 569–572.
8. R. D. Bantsuri, N. N. Shavlakadze, The contact problem for a piecewise-homogeneous plane with a semi-infinite inclusion. (Russian) *translated from Prikl. Mat. Mekh.* **73** (2009), no. 4, 655–662; *J. Appl. Math. Mech.* **73** (2009), no. 4, 471–477.
9. R. D. Bantsuri, N. N. Shavlakadze, The contact problem for a piecewise-homogeneous plane with a finite inclusion. (Russian) *translated from Prikl. Mat. Mekh.* **75** (2011), no. 1, 133–139; *J. Appl. Math. Mech.* **75** (2011), no. 1, 93–97.
10. R. M. Christensen, *Theory of Viscoelasticity.* New York, Academic, 1971.
11. F. D. Gahov, Yu. I. Čerskiĭ, *Equations of Convolution Type.* (Russian) Nauka, Moscow, 1978.
12. Ts. Jamaspishvili, N. Shavlakadze, The boundary value problems for piecewise-homogeneous viscoelastic plate. *Rep. Enlarged Sess. Semin. I. Vekua Appl. Math.* **34** (2020), 34–37.
13. Ts. Jamaspishvili, Complex representation in the plane theory of viscoelasticity and its applications. *Trans. A. Razmadze Math. Inst.* **174** (2020), no. 3, 351–357.
14. A. Khrapkov, The first fundamental problem for a piecewise-homogeneous plane with a cut perpendicular to a straight section. *Prikl. Mat. I Mekh.* **32** (1968), no. 4, 647-669.
15. N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity.* Fundamental equations, plane theory of elasticity, torsion and bending. Translated by J. R. M. Radok. P. Noordhoff Ltd., Groningen, 1953.
16. N. I. Muskhelishvili, *Singular Integral Equations.* Boundary problems of function theory and their application to mathematical physics. (Russian) 2nd. revised ed., Fizmatgiz, Moscow, 1962; English translation: Wolters-Noordhoff Publishing, Groningen, 1967.
17. G. Ja. Popov, *Concentration of Elastic Stresses Near Punches, Cuts, thin Inclusion and Supports.* (Russian) Nauka, Moscow, 1983.
18. Yu. N. Rabotnov, *Elements of Continuum Mechanics with Memory.* (Russian) Izdat. Nauka, Moscow, 1977.
19. N. Shavlakadze, On singularities of contact stress upon tension and bending of plates with elastic inclusions. *Proc. A. Razmadze Math. Inst.* **120** (1999), 135–147.
20. N. Shavlakadze, The contact problems of the mathematical theory of elasticity for plates with an elastic inclusion. *Acta Appl. Math.* **99** (2007), no. 1, 29–51.
21. T. C. T. Ting, P. Schiavone, Uniform antiplane shear stress inside an anisotropic elastic inclusion of arbitrary shape with perfect or imperfect interface bonding. *Internat. J. Engrg. Sci.* **48** (2010), no. 1, 67–77.
22. A. V. Ungiadze, The first fundamental problem for a piecewise homogeneous plane that contains a semi-infinite crack crossing the boundary at right angles. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR* **81** (1986), 79–86.

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