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HARMONIC NEUMANN FUNCTIONS FOR FINITE PARQUETING-REFLECTION DOMAINS

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Dedicated to the memory of Professor Elene Obolashvili

Abstract. The harmonic Green and Neumann functions are indispensable for solving boundary value problems for the Poisson equations. Their existence and properties are well known and included in all relevant textbooks. But explicit expressions are rarely offered except for simple domains like balls and half-spaces, although for practical problems in physics and engineering, explicit representations of solutions for boundary value problems are important. The parqueting-reflection principle offers a method for determining these fundamental solutions to the Laplace operator for certain circular polygons in the plane. In particular, the Neumann functions are problematic as they depend on some density function on the boundary of the domain. Different density functions determine various Neumann functions. However, the density function determines the related Neumann function uniquely. An example is offered showing how various Neumann functions create different solutions to the Neumann boundary value problem and different related solvability conditions. Knowing a Neumann function, one can calculate its density. An advantage of the method is that a possible density function becomes available prior to the Neumann function. As a side effect, the conformal invariance of the Neumann function is shown as a consequence of its invariance under inversive transformations.

1. Parqueting-reflection Principle

The explicit construction of certain kernel functions for the formulas for representation of complex smooth functions in circular polygons of the complex plane becomes possible due to the parqueting-reflection principle [12]. Such kernels are, e.g., the analytic Schwarz kernels, harmonic Green and Neumann functions [16,17]. The principle consists in reflecting the circular domain D, bounded by sections of circles and straight lines, so that the continued reflections at the boundary arcs cover the complex plane completely without overlapping. The trace of a point $z \in D$ in this procedure leads to some (formal) meromorphic functions $P(z,\zeta)$, $Q(z,\zeta)$ in $\mathbb{C}$, depending on z as a parameter, having a simple pole at the point z and either simple zeros or simple poles at the other trace points. $\log|P(z,\zeta)|^2$, $\log|Q(z,\zeta)|^2$ serve respectively as harmonic Green and Neumann functions for the domain D. This procedure works well in case of finite parqueting-reflection domains, i.e., circular domains providing a finite set of reflected domains that cover the plane. For infinite parqueting-reflections domains [1,9,11,16], convergence problems occur for the meromorphic functions leading to the Neumann function as determined just by (infinitely many) simple poles [16,18–21]. Convergence guaranteeing factors have to be introduced not changing the polar behavior and contributing only a harmonic alteration of the Neumann function. Because of the relative arbitrariness of the alteration, no normalization condition for the Neumann function can be guaranteed. Hence the Neumann function is not uniquely defined, in general. The uniqueness, however, is not essential for solving the Neumann boundary value problem for the Poisson equation. Examples are available, e.g., in [5–7,10,11,14].

The Neumann function is characterized by its properties [1–3].

Definition 1.1 (Neumann function)**.** A real-valued function $N_1(z,\zeta)$ is called harmonic Neumann function for a regular domain D, if it satisfies for any $\zeta \in D$ the properties:

- $N_1(\,\cdot\,,\zeta)$ is harmonic in $D \setminus \{\zeta\}$ and continuously differentiable in $\overline{D} \setminus \{\zeta\}$;

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- $h_2(z,\zeta) = N_1(z,\zeta) + \log|\zeta - z|^2$ is harmonic in $D \times D$;
- $\partial_{\nu_z} N_1(z,\zeta) = \sigma(s)$ for $z = z(s) \in \partial D$, ∂_{ν_z} is the outward normal derivative on ∂D and s is the arc length parameter of ∂D. The density function σ is a real-valued, piecewise continuous function of s with finite mass $\int\limits_{\partial D} \sigma(s)ds$;
- $\int\limits_{\partial D} \sigma(s_z) N_1(z,\zeta)\, ds_z = 0$ (normalization condition).

Remark 1.1. Naturally, the domain D is required to be regular, i.e., its boundary is piecewise smooth. If the density function $\sigma(s)$ is known, then the Neumann function is uniquely defined (see below Section 3, Lemma 3.2). But there is some freedom in prescribing $\sigma(s)$ [15]. While the Green function for a domain is uniquely defined, the Neumann function may vary and hence, different solutions to the Neumann boundary value problem for the Poisson equation are available under various solvability conditions. An example is given in Section 4.

2. Finite Parqueting-reflection Domains

A detailed discussion of the subject in this section is available from the Ph.D. thesis [16] (see also [17]). Some basics are extracted here.

Generalized circles are circles and lines on the (extended) complex plane $\mathbb{C}_\infty$. They are uniquely described as

$$az\overline{z} + \overline{b}z + b\overline{z} + c = 0, \quad a, c \in \mathbb{R}, \;\; b \in \mathbb{C}, \;\; ac - |b|^2 < 0. \tag{2.1}$$

For lines $a = 0$, circles are given for $a \neq 0$ and they are expressible as

$$\left| z + \frac{b}{a} \right|^2 = \frac{|b|^2 - ac}{a^2}\,.$$

The use of homogeneous coordinate representation is convenient for commonly describing circles and lines (see [13,16]). Replacing z by $[z : w]$, representation (2.1) becomes

$$az\overline{z} + \overline{b}z\overline{w} + b\overline{z}w + cw\overline{w} = 0, \quad [z : w] \neq [0 : 0].$$

Obviously, this relation is characterized by the 2×2 Hermitian matrix

$$A = \begin{pmatrix} a & \overline{b} \\ b & c \end{pmatrix}$$

with $\det A < 0$. Hence

$$H^- = \{A : A \in GL_2(\mathbb{C}),\; A^* = A, \det A < 0\}$$

corresponds to the set of circles and lines in the complex plane $\mathbb{C}$, where A^* is the complex conjugate transpose of A. Two elements A and B from H^- correspond to the same curve in $\mathbb{C}$ if they are related via $A = \lambda B$ with some non-zero $\lambda \in \mathbb{R}$. This fact is helpful in further treatment and corresponds to an equivalence relation in H^-. A point $\zeta \in \mathbb{C}$ is reflected at the generalized circle (2.1) onto ζ_{re}, satisfying the relation

$$a\zeta_{re}\overline{\zeta} + \overline{b}\zeta_{re} + b\overline{\zeta} + c = 0.$$

Hence

$$\zeta_{re} = -\frac{b\overline{\zeta} + c}{a\overline{\zeta} + \overline{b}}\,.$$

Using homogeneous coordinates, this reflection can be described by introducing the product rule

$$[z : w] \begin{pmatrix} a & b \\ c & d \end{pmatrix} = [az + cw : bz + dw]$$

and the convention $\overline{[z : w]} = [\overline{z} : \overline{w}]$. Introducing the $2 \times 2-$matrix

$$P = \begin{pmatrix} 0 & 1 \\ -1 & 0, \end{pmatrix}$$

the reflected point $[\zeta_{re} : \omega_{re}]$ at the circle (2.1) of the point $[\zeta : \omega]$ can be calculated (see [13, 16]) via

$$[\zeta_{re} : \omega_{re}] = [-b\overline{\zeta} - c\overline{\omega} : a\overline{\zeta} + \overline{b}\overline{\omega}] = \overline{[\zeta : \omega]AP}.$$

The reflection at A is denoted by R_A. Reflections leave the set of circles and lines invariant. In particular, the image of a circle B at a circle A is the circle $AB^{-1}A$. (For a proof see [13, 16]). Thus for $A = B$, obviously, A remains unchanged. Reflections preserve the magnitude of angles, but reverse their orientation. A consequence is the relation $AB^{-1}A = B$, where the equality is meant in the sense of H^- as equality of the respective equivalent classes, in case A and B are orthogonal to one another. The composition of two reflections is a Möbius transformation. With regard to the composition, all reflections form a group, the group of inversive transformations, with the Möbius group as a subgroup.

Lemma 2.1 (Consecutive reflections)**.** *For two circles or lines A_0, A_1, let A_{k+1} be the reflection of A_{k-1} at A_k, $A_{k+1} = A_k A_{k-1}^{-1} A_k$ (and A_{k-1} the reflection of A_{k+1} at A_k, $A_{k-1} = A_k A_{k+1}^{-1} A_k$), then*

$$A_{k+1} = A_0[A_0^{-1}A_1]^{k+1} = A_1[A_0^{-1}A_1]^k \text{ for } k \in \mathbb{Z}.$$

The proof is simple inductive. A consequence is

$$A_k A_l^{-1} A_m = A_{k-l+m} \text{ for } k, l, m \in \mathbb{Z},$$

in particular, $A_{-k}A_0^{-1}A_k = A_0$ for any $k \in \mathbb{Z}$. Thus, the reflection at the circle $A_{-k}A_0^{-1}A_k$ is achieving the same as reflection at A_0 itself, i.e., $R_{A_0} = R_{A_{-k}} \circ R_{A_0} \circ R_{A_k}$ for any $k \in \mathbb{Z}$. This can be written as

$$R_{A_{-1}} \circ R_{A_{-2}} \circ \cdots \circ R_{A_{-k}} \circ R_{A_0} \circ R_{A_k} \circ R_{A_{k-1}} \circ \cdots \circ R_{A_1} = R_{A_0}.$$

Definition 2.1. Two domains D_1, D_n, $n \in \mathbb{N}$, are called reflective congruent, if there exists a set of domains $\{D_k : 1 \le k \le n-1\}$ such that D_{k+1} is the reflection of D_k for $1 \le k \le n-1$.

Two possibilities occur. Either an even number of reflections is needed, then the two domains are conformally equivalent by a Möbius transformation of the form

$$\varphi(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0,$$

or an odd number of reflections is required, so that an inverse orientation– reversing transformation of the form

$$\phi(z) = \frac{\alpha\overline{z} + \beta}{\gamma\overline{z} + \delta}, \quad \alpha\delta - \beta\gamma \neq 0,$$

connects the two domains.

Definition 2.2. A circular domain D, whose boundary consists of finitely many circular arcs, provides a parqueting of the complex plane $\mathbb{C}_\infty$ via reflection, if there exist a finite or countable index set I and reflective congruent domains D_ν, $\nu \in I$ to D, satisfying

$$D_\nu \cap D_\mu = \emptyset \text{ for } \nu \neq \mu, \quad \bigcup_{\nu \in I} \overline{D_\nu} = \mathbb{C}_\infty.$$

Definition 2.3. If the index set J of the boundary decomposition $\partial D = \bigcup_{j \in J} C_j$ of a circular domain D is finite, where C_j are circle arcs, the reflection at which are denoted by r_j, then their generated group of inverse transformations is called the inverse group of D, denoted by

$$\mathrm{Inv}(D) = < r_j : j \in J > .$$

The subset

$$M(D) = \{T \in \mathrm{Inv}(D) : T \text{ product of even many } r_j\,'s\}$$

of Möbius transformations form a subgroup and the complementary subset is given as

$$\mathrm{Inv}(D) \setminus M(D) = r_j M(D) = M(D) r_j,$$

for any $j \in J$.

Definition 2.4. For $z \in D$, the set $T(z) = N(z) \cup P(z)$ with

$$N(z) = \{\varphi(z) : \varphi \in M(D)\}, \quad P(z) = \{\phi(z) : \phi \in \mathrm{Inv}(D) \setminus M(D)\}$$

is the trace of $z \in D$ under the reflections from $\mathrm{Inv}(D)$ in $\mathbb{C}_\infty$.

In case $z \in C_j$, since $z = r_j(z)$, therefore $\varphi(z) = \varphi(r_j(z))$ for all $\varphi \in M(D)$.

3. Neumann Function for Finite Parqueting Domains

Let D be a finite parqueting domain with an inverse group $\mathrm{Inv}(D)$ and Möbius transformation subgroup $M(D)$. Denoting $D_j = \varphi_j(D)$, $\widehat{D}_j = \phi_j(D)$, $\varphi_j \in M(D)$, $\phi_j \in \mathrm{Inv}(D) \setminus M(D)$, $j \in J$, and for $z \in D$

$$z_j = \varphi_j(z) = \frac{a_j z + b_j}{c_j z + d_j}, \quad \widehat{z}_j = \phi_j(z) = \frac{\alpha_j \overline{z} + \beta_j}{\gamma_j \overline{z} + \delta_j},$$

we introduce their denominators

$$\mathrm{den}(z_j) = c_j z + d_j, \quad \mathrm{den}(\widehat{z}_j) = \gamma_j \overline{z} + \delta_j.$$

To define a Neumann function for the finite parqueting domain D from the trace points T of the point $z \in D$, we use the function

$$\prod_{j\in J} |(\zeta - \widehat{z}_j)(\zeta - z_j)|^2.$$

Lemma 3.1. *The normal derivatives at the circular arc C, $a\zeta\overline{\zeta} + \overline{b}\zeta + b\overline{\zeta} + c = 0$, from ∂D for the function* $\log|(\zeta - \widehat{z})(\zeta - z)|^2$, *$\zeta$, $z \in D$, $a\widehat{z}\overline{z} + \overline{b}\widehat{z} + b\overline{z} + c = 0$, are given as:*

$$\partial_{\nu_\zeta} \log|(\zeta - \widehat{z})(\zeta - z)|^2 = \frac{2a}{r}, \ \text{ for } \ \zeta \in C, \ \ z \in \mathbb{C} \setminus C,$$
$$\partial_{\nu_z} \log|(\zeta - \widehat{z})(\zeta - z)|^2 = 0, \quad \text{ for } \ z \in C, \ \ \zeta \in \mathbb{C} \setminus C,$$
$$\partial_{\nu_z} \log|(\zeta - \widehat{z})(\zeta - z)|^2 = \frac{2}{r}\Big[a - \frac{a\zeta + b}{\zeta - z} - \overline{\frac{a\zeta + b}{\zeta - z}}\Big] \ \text{ for } \ z, \zeta \in C.$$

Proof. Observing $a\zeta\overline{\zeta} + \overline{b}\zeta + b\overline{\zeta} + c = 0$ for $\zeta \in C$, the relation

$$\frac{a\zeta + b}{\zeta - \widehat{z}} = \frac{(a\zeta + b)(a\overline{z} + \overline{b})}{\zeta(a\overline{z} + \overline{b}) + b\overline{z} + c} = \frac{(a\zeta + b)(a\overline{z} + \overline{b})}{\overline{z}(a\zeta + b) + \overline{b}\zeta + c} = \overline{\frac{az + b}{z - \zeta}} = a - \overline{\frac{a\zeta + b}{\zeta - z}}$$

provides

$$\partial_{\nu_\zeta} \log|(\zeta - \widehat{z})(\zeta - z)|^2 = \frac{2}{r}\mathrm{Re}\Big[a - \overline{\frac{a\zeta + b}{\zeta - z}} + \frac{a\zeta + b}{\zeta - z}\Big] = \frac{2a}{r},$$

for $\zeta \in C, \ z \in C$.

For $z \in C$, where $\widehat{z} = z$, we have

$$\partial_{\nu_z} \log|(\zeta - \widehat{z})(\zeta - z)|^2 = \frac{2}{r}\mathrm{Re}\Big[\overline{\frac{az + b}{\zeta - \widehat{z}}}\frac{r^2}{(az + b)^2} - \frac{az + b}{\zeta - z}\Big] = \frac{2}{r}\mathrm{Re}\Big[\overline{\frac{az + b}{\zeta - z}} - \frac{az + b}{\zeta - z}\Big] = 0$$

for $z \in C, \ \zeta \in \mathbb{C} \setminus C$. Here, for the next to the last equality

$$\frac{r^2}{(\overline{\zeta - \widehat{z}})(az + b)} = \overline{\frac{az + b}{\zeta - \widehat{z}}}$$

and $\widehat{z} = z$ are used.

Finally, for $z, \zeta \in C$, from

$$\overline{\frac{az + b}{\zeta - \widehat{z}}}\,\frac{(az + b)(a\zeta + b)}{(az + b)(a\zeta + b)} = \frac{r^2(a\zeta + b)}{r^2(z - \zeta)}$$

follows

$$\partial_{\nu_z} \log|(\zeta - \widehat{z})(\zeta - z)|^2 = \frac{2}{r}\mathrm{Re}\Big[\overline{\frac{az + b}{\zeta - \widehat{z}}} - \frac{az + b}{\zeta - z}\Big] = \frac{2}{r}\mathrm{Re}\Big[a - \frac{a\zeta + b}{\zeta - z} - \overline{\frac{a\zeta + b}{\zeta - z}}\Big]. \qquad \square$$

Theorem 3.1. *A Neumann function for the finite circular parqueting domain D and the trace points z_j, $\widehat{z}_j$, $j \in J$ for $z \in D$ is given by*

$$N_1(z, \zeta) = -\log\prod_{j\in J} |(\zeta - z_j)(\zeta - \widehat{z}_j)\,\mathrm{den}(z_j)\,\mathrm{den}(\widehat{z}_j)|^2.$$

Proof. Obviously, $(\zeta - z_j)\,\mathrm{den}(z_j)$, $(\zeta - \widehat{z_j})\,\mathrm{den}(\widehat{z_j})$ are linear functions of ζ and either of z or $\overline{z}$. The only zero of the product involved in D is $\zeta = z$. All the other trace points lie outside D. Thus, $N_1(z,\zeta)$ is harmonic in both its variables up to the point $\zeta = z$ in $D \times D$. Also, $N_1(z,\zeta) + \log|\zeta - z|^2$ is harmonic in $D \times D$. On the boundary arc C the normal derivative for $z \in D$ is

$$\partial_{\nu_\zeta} N_1(z,\zeta) = -\sum_{j\in J} \partial_{\nu_\zeta} \log|(\zeta - z_j)(\zeta - \widehat{z_j})|^2 = -\sum_{j\in J} \frac{2a}{r} = -\frac{2a}{r}|J|.$$

Here $|J|$ denotes the cardinality of the finite index set J. □

Remark 3.1. A characteristic for the Neumann function is that it has a piecewise constant normal derivative at the boundary. For the unit disk $D = \mathbb{D} = \{z\overline{z} - 1 < 0\}$, this value is obviously -2, for the upper half-unit disk $\mathbb{D}^+ = \{i(z - \overline{z}) < 0 < 1 - z\overline{z}\}$, the constant is again -2, for the upper half-unit circle and 0 for the real axis $z - \overline{z} = 0$, since for this straight line $a = c = 0$, $b = -i$, i.e., $r = 1$. In addition, the factor $|J|$ needs to be observed. For rectilinear sections ($a = 0$) the value is always 0, whereas for circular sections, besides the radius, the coefficient a also makes a difference. But the constant values are always non-positive. The Neumann function is not uniquely defined by the three properties of the theorem. In [15], it is mentioned that the Neumann functions exist for properly prescribed density functions on the boundary.

Lemma 3.2. *For prescribed density function $\sigma(z)$, the Neumann function is uniquely defined.*

Proof. For proving this, let $N_k(z,\zeta)$, $k = 1, 2$, be two such Neumann functions and $h = N_1 - N_2$ be their difference, a harmonic function in $z \in D\backslash\{\zeta\}$, $\zeta \in D$, satisfying on ∂D the relation $\partial_{\nu_z} h(z,\zeta) = 0$. Representing it by the Neumann representation formula (see, e.g., [4,8] and Theorem 4.1 below), with regard to say N_1 gives

$$h(z,\zeta) = -\frac{1}{4\pi} \int_{\partial D} h(\widetilde{\zeta},\zeta)\partial_{\nu_{\widetilde{\zeta}}} N_1((\widetilde{\zeta},\zeta) ds_{\widetilde{\zeta}} = 0,$$

where the last equality follows from the both normalization conditions. □

A verification of such a normalization condition for general finite parqueting domains seems not to be available.

If a fundamental solution of the Laplacian does satisfy the first three conditions for the Neumann function, an arbitrary harmonic function in the variable ζ may be added without altering these three conditions. This altered function still serves to solve the Neumann boundary value problem for the Poisson equation. However, the solution is defined only up to an arbitrary additive constant. Only the normalization of the Neumann function serves to determine this constant via a proper additional side condition (see [7, 17]).

The possibility of adding harmonic functions in the variable ζ to some Neumann function can be used to create a Neumann function, symmetric in its variables. It may also happen that the Neumann problem, in general an over-determined boundary value problem and hence only conditionally solvable, is unconditionally and uniquely solvable [8]. As an example, the upper half-unit disc $\mathbb{D}^+$ will be used in the next subsection.

The Neumann function, as is the Green function, is known to be conformal invariant. In fact, as the Green function, it is even invariant under inversive transformations. This is shown in [16] for the Green function.

Lemma 3.3. *The Neumann function is invariant under inverse transformations.*

Proof. Let

$$w(z) = \frac{a\overline{z} + b}{c\overline{z} + d}, \quad ad - bc \neq 0,$$

map the domain Ω onto the domain D of the complex plane and $N_{1,D}(w,\omega)$ be the harmonic Neumann function for D. Then

$$N_1(z,\zeta) = N_{1,D}(w(z), w(\zeta))$$

turns out to be the harmonic Neumann function for Ω.

From

$$\begin{aligned}\partial_{\overline{z}} N_{1,D}(w(z), w(\zeta)) &= \partial_{w(z)} N_{1,D}(w(z), w(\zeta)) \partial_{\overline{z}} w(z),\\ \partial_z \partial_{\overline{z}} N_{1,D}(w(z), w(\zeta)) &= \partial_{\overline{w(z)}} \partial_{w(z)} N_{1,D}(w(z), w(\zeta)) w_{\overline{z}}(z) \overline{w_{\overline{z}}(z)}\\ &\quad + \partial_{w(z)} N_{1,D}(w(z), w(\zeta)) \partial_z \partial_{\overline{z}} w(z) = 0,\end{aligned}$$

$N_1(z,\zeta)$ is seen to be harmonic in $\Omega \setminus \{\zeta\}$.

The right-hand side of

$$N_1(z,\zeta) + \log|\zeta - z|^2) = N_{1,D}(w(z), w(\zeta)) + \log|w(\zeta) - w(z)|^2 - \log\Big|\frac{w(\zeta) - w[z]}{\zeta - z}\Big|^2$$

is a harmonic function in both its variables z and ζ in Ω as

$$\frac{w(\zeta) - w(z)}{\zeta - z} = \frac{ad - bc}{(c\overline{\zeta} + d)(c\overline{z} + d)}\,.$$

On $\partial\Omega = w[\partial D]$, the outer normal derivative for $N_1(z,\zeta)$ is given in

$$\begin{aligned}\sigma_\Omega(z) dS_z &= \partial_{\nu_z} N_1(z,\zeta) dS_z = -i[\partial_z N_1(z,\zeta) dz - \partial_{\overline{z}} N_1(z,\zeta) d\overline{z}]\\ &= -i[\partial_{\overline{w(z)}} N_{1,D}(w(z), w(\zeta)) \partial_z \overline{w(z)} dz - \partial_{w(z)} N_{1,D}(w(z), w(\zeta)) \partial_{\overline{z}} w(z) d\overline{z}]\\ &= i[\partial_{w(z)} N_{1,D}(w(z), w(\zeta)) dw(z) - \partial_{\overline{w(z)}} N_{1,D}(w(z), w(\zeta)) d\overline{w(z)}]\\ &= -\partial_{\nu_{w(z)}} N_{1,D}(w(z), w(\zeta)) ds_{w(z)} = -\sigma_D(w(z)) ds_{w(z)} = -\sigma_D(w) ds_w.\end{aligned}$$

S_z, s_w denote the arc length parameters on $\partial\Omega$, ∂D, respectively. Also,

$$\int_{\partial\Omega} \sigma_\Omega(z) dS_z = -\int_{\partial\Omega} \sigma_D(w(z)) ds_{w(z)} = \int_{\partial D} \sigma_D(w) ds_w$$

is a finite number. Similarly, the normalization condition

$$\begin{aligned}\int_{\partial\Omega} \sigma_\Omega(z) N_1(z,\zeta) dS_z &= -\int_{\partial\Omega} \sigma_D(w(z)) N_{1,D}(w(z), w(\zeta)) ds_{w(z)}\\ &= \int_{\partial D} \sigma_D(w) N_{1,D}(w,\omega) ds_w = 0\end{aligned}$$

is verified. □

Corollary 3.1. *The Neumann function is conformally invariant.*

4. Neumann Problem in $\mathbb{D}^+$

As $\mathbb{D}^+ = \{|z| < 1, 0 < \text{Im} z\}$ is a finite parqueting domain, its Neumann function is

$$N_1(z,\zeta) = -\log|(\zeta - z)(\overline{\zeta} - z)(1 - z\overline{\zeta})(1 - z\zeta)|^2.$$

Also,

$$\widetilde{N}_1(z,\zeta) = 2\log|z\zeta|^2 + N_1(z,\zeta)$$

is a Neumann function for $\mathbb{D}^+$. Their properties are listed here:

Lemma 4.1.

$$\partial_{\nu_z} N_1(z,\zeta) = \begin{cases} -4, & |z|=1, \quad 0<\mathrm{Im}z, \quad |\zeta|<1, \quad 0\le \mathrm{Im}\zeta, \\ 0, & \mathrm{Im}z=0, \quad |z|<1, \quad |\zeta|<1, \quad 0<\mathrm{Im}\zeta, \end{cases}$$

$$(z\partial_z+\overline{z}\partial_{\overline{z}})N_1(z,\zeta) = 2\Big[\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta-z}}-2+\frac{\zeta}{\zeta-\overline{z}}+\frac{\overline{\zeta}}{\overline{\zeta}-z}-2\Big],$$

$$|z|<1, \quad 0<\mathrm{Im}z, \quad |\zeta|=1, \quad 0\le\mathrm{Im}\zeta,$$

$$-i(\partial_z-\partial_{\overline{z}})N_1(z,\zeta) = 2i(z-\overline{z})\Big[\frac{1}{|\zeta-z|^2}+\frac{\zeta^2}{|1-z\zeta|^2}\Big],$$

$$0<\mathrm{Im}z, \quad |z|<1, \quad \zeta=\overline{\zeta}, \quad |\zeta|<1;$$

$$\int\limits_{\partial\mathbb{D}^+}\sigma(s_z)N_1(z,\zeta)ds_z = 0.$$

$$\partial_{\nu_z} \widetilde{N}_1(z,\zeta) = \begin{cases} 0, & |z|=1, \quad 0<\mathrm{Im}z, \quad |\zeta|<1, \quad 0\le \mathrm{Im}\zeta, \\ 0, & \mathrm{Im}z=0, \quad |z|<1, \quad |\zeta|\le 1, \quad 0<\mathrm{Im}\zeta, \end{cases}$$

$$(z\partial_z+\overline{z}\partial_{\overline{z}})\widetilde{N}_1(z,\zeta) = 2\Big[\frac{\zeta}{\zeta-z}+\frac{\overline{\zeta}}{\overline{\zeta-z}}-1+\frac{\zeta}{\zeta-\overline{z}}+\frac{\overline{\zeta}}{\overline{\zeta}-z}-1\Big],$$

$$|z|<1, \quad 0<\mathrm{Im}z, \quad |\zeta|=1, \quad 0<\mathrm{Im}\zeta,$$

$$-i(\partial_z-\partial_{\overline{z}})\widetilde{N}_1(z,\zeta) = 2i(z-\overline{z})\Big[\frac{1}{|\zeta-z|^2}+\frac{\zeta^2}{|1-z\zeta|^2}+\frac{1}{|z|^2}\Big],$$

$$0<\mathrm{Im}z, \quad |z|<1, \quad \zeta=\overline{\zeta}, \quad |\zeta|<1:$$

$$\int\limits_{\partial\mathbb{D}^+}\widetilde{\sigma}(s_z)\widetilde{N}_1(z,\zeta)ds_z = 0.$$

Proof. On the basis of

$$z\partial_z N_1(z,\zeta) = \frac{z}{\zeta-z}+\frac{z}{\overline{\zeta}-z}+\frac{z\overline{\zeta}}{1-z\overline{\zeta}}+\frac{z\zeta}{1-z\zeta},$$

$$z\partial_z \log|z|^2 = 1$$

and

$$\partial_{\nu_z} = z\partial_z+\overline{z}\partial_{\overline{z}} \text{ on } |z|=1,$$

$$\partial_{\nu_z} = -i(\partial_z-\partial_{\overline{z}}) \text{ on } z=\overline{z},$$

the differential relations follow. The normalization condition for N_1 is seen owing to

$$\int\limits_{\partial\mathbb{D}^+}\sigma(s_z)N_1(z,\zeta)ds_z = 4\int\limits_{|z|=1,0<\mathrm{Im}z}\log|(\zeta-z)(\overline{\zeta-z})(1-z\overline{\zeta})(1-z\zeta)|^2\frac{dz}{iz}$$

$$=\frac{16}{i}\int\limits_{|z|=1}\log|1-z\overline{\zeta}|^2\frac{dz}{z}=0.$$

The second normalization condition is obvious as the density $\widetilde{\sigma}$ is identically zero. □

Theorem 4.1. *Let $D\subset\mathbb{C}$ be a regular domain and $N_1(z,\zeta)$ be a harmonic Neumann function for D, then any $w\in C^2(D;\mathbb{C})\cap C^1(\overline{D};\mathbb{C})$ can be represented as*

$$w(z) = -\frac{1}{4\pi}\int\limits_{\partial D}\{w(\zeta)\partial_{\nu_\zeta}N_1(z,\zeta)-\partial_{\nu_\zeta}w(\zeta)N_1(z,\zeta)\}ds_\zeta-\frac{1}{\pi}\int\limits_D w_{\zeta\overline{\zeta}}(\zeta)N_1(z,\zeta)d\xi d\eta.$$

A proof is available, e.g., in [3,8]. This representation formula provides candidates for the solutions of the

Neumann boundary value problem for the Poisson equation: Find a solution to the Poisson equation $\partial_z\partial_{\overline{z}}w = f$ in D satisfying $\partial_{\nu_z}w = \gamma$ on ∂D for given $f \in L_p(D;\mathbb{C})$, $2 < p$, $\gamma \in C(\partial D;\mathbb{C})$.

In general, this problem is conditionally solvable and the solution is determined up to an additive constant, which can be determined by an additional side condition.

Both Neumann functions lead to possible solutions to the Neumann problem for $\mathbb{D}^+$.

Theorem 4.2. *The Neumann problem for $\mathbb{D}^+$ is solvable if and only if*

$$\frac{1}{4\pi i}\int\limits_{|\zeta|<1,0<\mathrm{Im}\zeta}\gamma(\zeta)\frac{d\zeta}{\zeta} = \frac{1}{\pi}\int\limits_{\mathbb{D}^+}f(\zeta)d\xi d\eta.$$

The solution is then uniquely determined by the side condition

$$\frac{1}{\pi i}\int\limits_{|z|=1,0<\mathrm{Im}z}w(z)\frac{dz}{z} = c_0$$

with given $c_0 \in \mathbb{C}$, in the form

$$\begin{aligned} w(z) =& c_0 + \frac{1}{4\pi i}\int\limits_{|\zeta|=1,0<\mathrm{Im}\zeta}\gamma(\zeta)N_1(z,\zeta)\frac{d\zeta}{\zeta} \\ &+ \frac{1}{4\pi}\int\limits_{-1}^{1}\gamma(t)N_1(z,t)dt - \frac{1}{\pi}\int\limits_{\mathbb{D}^+}f(\zeta)N_1(z,\zeta)d\xi d\eta. \end{aligned} \tag{4.1}$$

Theorem 4.3. *The Neumann problem for $\mathbb{D}^+$ is unconditionally solvable without any side condition in the form*

$$\begin{aligned} w(z) =& \frac{1}{4\pi i}\int\limits_{|\zeta|=1,0<\mathrm{Im}\zeta}\gamma(\zeta)\widetilde{N}_1(z,\zeta)\frac{d\zeta}{\zeta} \\ &+ \frac{1}{4\pi}\int\limits_{-1}^{1}\gamma(t)\widetilde{N}_1(z,t)dt - \frac{1}{\pi}\int\limits_{\mathbb{D}^+}f(\zeta)\widetilde{N}_1(z,\zeta)d\xi d\eta. \end{aligned} \tag{4.2}$$

Proof of Theorems 4.2 and 4.3. Since the Neumann functions are fundamental solutions to the Laplace operator, the line integrals are harmonic functions, while the area integrals contribute particular solutions to the Poisson equation with homogeneous boundary data. As the area integrals contribute solutions to the Poisson equation satisfying the related homogeneous boundary condition, it is necessary to show that only the boundary integrals satisfy the boundary condition, and the side condition has to be verified.

1. From (4.2), for $z \in \mathbb{D}^+$, it follows that

$$\begin{aligned} (z\partial_z + \overline{z}\partial_{\overline{z}})w(z) =& \frac{1}{2\pi i}\int\limits_{\substack{|\zeta|=1,\\ 0<\mathrm{Im}\,\zeta}}\gamma(\zeta)\left[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\overline{\zeta} - z}} - 1 + \frac{\overline{\zeta}}{\overline{\zeta} - z} + \frac{\zeta}{\zeta - \overline{z}} - 1\right]\frac{d\zeta}{\zeta} \\ &+ \frac{1}{2\pi}\int\limits_{-1}^{1}\gamma(t)\left[2 + \frac{z}{t-z} + \frac{\overline{z}}{t-\overline{z}} + \frac{zt}{1-zt} + \frac{\overline{z}t}{1-\overline{z}t}\right]dt \\ &- \frac{1}{\pi}\int\limits_{\mathbb{D}^+}f(\zeta)(z\partial_z + \overline{z}\partial_{\overline{z}})N_1(z,\zeta)d\xi d\eta, \end{aligned}$$

so, for $|\zeta_0| = 1$, $0 < \mathrm{Im}\,\zeta_0$,

$$\partial_\nu w(\zeta_0) = \lim_{z\to\zeta_0}(z\partial_z + \overline{z}\partial_{\overline{z}})w(z) = \gamma(\zeta_0),$$

because for $0 < \operatorname{Im}\zeta$,

$$\frac{\overline{\zeta}}{\overline{\zeta} - z} + \frac{\zeta}{\zeta - \overline{z}} - 1 = \frac{1 - |z|^2}{|\overline{\zeta} - z|^2},$$

and for $|t| < 1$, $t = \overline{t}$,

$$2 + \frac{z}{t - z} + \frac{\overline{z}}{t - \overline{z}} + \frac{zt}{1 - zt} + \frac{\overline{z}t}{1 - \overline{z}t} = (1 - |z|^2)\left[\frac{t}{(t - z)(1 - \overline{z}t)} + \frac{t}{(t - \overline{z})(1 - zt)}\right].$$

2. Similarly, from (4.2), for $z \in \mathbb{D}^+$, it follow that

$$- i(\partial_z - \partial_{\overline{z}})w(z) = -\frac{1}{4\pi}\int\limits_{\substack{|\zeta|=1,\\ 0<\operatorname{Im}\zeta}} \gamma(\zeta)\left[\frac{2}{z} - \frac{2}{\overline{z}}\right.$$

$$\left. + \frac{1}{\zeta - z} - \frac{1}{\overline{\zeta - z}} + \frac{1}{\overline{\zeta} - z} - \frac{1}{\zeta - \overline{z}} + \frac{\zeta}{1 - z\zeta} - \frac{\overline{\zeta}}{1 - \overline{z\zeta}} + \frac{\overline{\zeta}}{1 - z\overline{\zeta}} - \frac{\zeta}{1 - \overline{z}\zeta}\right]\frac{d\zeta}{\zeta}$$

$$+ \frac{z - \overline{z}}{2\pi i}\int\limits_{-1}^{1} \gamma(t)\left[\frac{1}{|t - z|^2} + \frac{t^2}{|1 - zt|^2} - \frac{1}{|z|^2}\right]dt$$

$$+ \frac{i}{\pi}\int\limits_{\mathbb{D}^+} f(\zeta)(\partial_z - \partial_{\overline{z}})N_1(z, \zeta)d\xi d\eta.$$

Hence, for $z \in \mathbb{D}^+$, $|t_0| < 1$, $t_0 = \overline{t_0}$,

$$\partial_\nu w(t_0) = \lim_{z \to t_0}[-i(\partial_z - \partial_{\overline{z}})]w(z) = \gamma(t_0),$$

as for $0 < \operatorname{Im}\zeta$,

$$\frac{2}{z} - \frac{2}{\overline{z}} + \frac{1}{\zeta - z} - \frac{1}{\overline{\zeta - z}} + \frac{1}{\overline{\zeta} - z} - \frac{1}{\zeta - \overline{z}} + \frac{\zeta}{1 - z\zeta} - \frac{\overline{\zeta}}{1 - \overline{z\zeta}} + \frac{\overline{\zeta}}{1 - z\overline{\zeta}} - \frac{\zeta}{1 - \overline{z}\zeta} = (z - \overline{z})$$

$$\times\left[\frac{1}{(\zeta - z)(\zeta - \overline{z})} + \frac{1}{(\overline{\zeta} - z)(\overline{\zeta - z})} + \frac{\zeta^2}{(1 - z\zeta)(1 - \overline{z}\zeta)} + \frac{\overline{\zeta}^2}{(1 - \overline{z\zeta})(1 - z\overline{\zeta})} - \frac{2}{|z|^2}\right]$$

and for $|t_0| < 1$, $t_0 = \overline{t_0}$,

$$\lim_{z \to t_0}\left[\frac{t^2}{|1 - zt|^2} - \frac{1}{|z|^2}\right] = \frac{t^2}{|1 - t_0 t|^2} - \frac{1}{t_0^2}.$$

Thus, Theorem 4.3 is proved. The difference in the treatment of Theorem 4.2 is the alteration in the Poisson kernel on the upper unit circle by subtracting 2 and the difference in the modified boundary behavior of the normal derivative on the upper half of the unit circle, causing the solvability condition. Multiplying (4.1) by $\sigma(z)$ and integrating over $\partial\mathbb{D}^+$ verifies the side condition based on the normalization condition. □

References

1. M. Akel, H. Begehr, A. Mohammed, Integral representations in the complex plane and iterated boundary value problems. *Rocky Mountain J. Math.* **52** (2022), no. 2, 381–413.
2. H. Begehr, *Complex Analytic Methods for Partial Differential Equations*. An introductory text. World Scientific Publishing Co., Inc., River Edge, NJ, 1994.
3. H. Begehr, Boundary value problems in complex analysis. I, II. *Bol. Asoc. Mat. Venez.* **12** (2005), 65–85; 217–250.
4. H. Begehr, S. Burgumbayeva, B. Shupeyeva, Remark on Robin problem for Poisson equation. *Complex Var. Elliptic Equ.* **62** (2017), no. 10, 1589–1599.
5. H. Begehr, S. Burgumbayeva, A. Dauletkulova, H. Lin, Harmonic Green functions for the Almaty apple. *Complex Var. Elliptic Equ.* **65** (2020), no. 11, 1814–1825.
6. H. Begehr, H. Lin, H. Liu, B. Shupeyeva, An iterative real sequence based on $\sqrt{2}$ and $\sqrt{3}$ providing a plane parqueting and harmonic Green functions. *Complex Var. Elliptic Equ.* **66** (2021), no. 6-7, 988–1013.
7. H. Begehr, H. Lin, H. Liu, B. Shupeyeva, Harmonic Neumann function for a planar circular rectangle-a case study. *Int. J. Rep. Kernels* **1** (2022), no. 1, 1–33.

8. H. Begehr, T. Vaitekhovich, Harmonic boundary value problems in half disc and half ring. *Funct. Approx. Comment. Math.* **40** (2009), part 2, 251–282.
9. H. Begehr, T. Vaitekhovich, Green functions, reflections, and plane parqueting. *Eurasian Math. J.* **1** (2010), 17–31; **2** (2011), 139–142.
10. H. Begehr, T. Vaitekhovich, How to find harmonic Green functions in the plane. *Complex Var. Elliptic Equ.* **56** (2011), no. 12, 1169–1181.
11. H. Begehr, T. Vaitekhovich, Harmonic Dirichlet problem for some equilateral triangle. *Complex Var. Elliptic Equ.* **57** (2012), no. 2-4, 185–196.
12. H. Begehr, T. Vaitekhovich, The parqueting-reflection principle for constructing Green function. In: *Analytic Methods of Analysis and Differential Equations*, 11–20, AMADE-2012 (edited by S.V. Rogosin, M.V. Dubatovskaya), Cambridge Sci. Publ., Cottenham, 2013.
13. D. Brannan, M. Esplen, J. Gray, *Geometry.* Second edition. Cambridge, Cambridge Univ. Press, 2012.
14. E. Gaertner, *Basic Complex Boundary Value Problems in the Upper Half Plane.* Ph.D. thesis, FU Berlin, 2006, `https://refubium.fu-berlin.de/handle/fub188/5020`.
15. W. Haack, W. Wendland, *Lectures on Partial and Pfaffian Differential Equations.* Translated by E. R. Dawson. English translation edited by W. N. Everitt. Oxford etc.: Pergamon Press, 1972.
16. H. Lin, *Parqueting-reflection Principle and Boundary Value Problems in Some Circular Polygons.* Ph.D. thesis, FU Berlin, 2021, `https://refubium.fu-berlin.de/handle/fub188/33226`.
17. H. Lin, Harmonic Green and Neumann functions for domains bounded by two intersecting circular arcs. *Complex Var. Elliptic Equ.* **67** (2022), no. 1, 79–95.
18. B. Shupeyeva, *Some Basic Boundary Value Problems for Complex Partial Differential Equations in Quarter Ring And Half Hexagon.* Ph.D. thesis, FU Berlin, 2013; `https://refubium.fu-berlin.de/handle/fub188/8588`.
19. T. Vaitsiakhovich, *Boundary Value Problems for Complex Partial Differential Equations in a Ring Domain.* Ph.D. thesis, FU Berlin, 2008, `https://refubium.fu-berlin.de/handle/fub188/7004`.
20. T. Vaitsiakhovich, Boundary value problems to first order complex partial differential equations in a ring domain. *Integral Transf. Special Funct.* **19** (2008), no. 3, 211–233.
21. Y. Wang, *Boundary Value Problems for Complex Partial Differetial Equations in Fan-Shaped Domains.* Ph.D. thesis, FU Berlin, 2011 `https://refubium.fu-berlin.de/handle/fub188/10134`.

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