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ON THE LIQUID FLOWS BETWEEN TWO INFINITELY DISTANT ROTATING POROUS CYLINDERS

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Dedicated to the memory of Professor Elene Obolashvili

Abstract. In this paper, the problem of a viscous liquid flows through two rotating permeable cylinders in the presence of a transverse pressure gradient is considered. The existence of flows in the case of a large gap between the cylinders for an infinitely distant outer cylinder is established.

1. The well-know Georgian scientist D. Dolidze in his work (see [2]), when considering the flows of a viscous incompressible fluid in the exterior of a rotating porous cylinder under the condition of velocity damping at infinity, has come to the paradoxical conclusion that a laminar flow is violated for certain values of the problem parameters when the unique solvability of the initial-boundary value problem is proved. As is known, these flows are of particular interest, as the study of the interaction of a fluid and a permeable porous cylinder is important for many applications such as filtration systems, complex engineering devices, and also for biomedical applications. 40 years later, after the publication of D. Dolidze's work (after his death), the famous Russian mathematician and mechanic V. Yudovich, studying the flow of an incompressible viscous fluid in a circular ring, as well as in the limiting case of a circle exterior (see [10]), noted that the flows of a viscous fluid in a circle exterior considered first by D. Dolidze, calling these flows "Dolidze's flows", are real under certain conditions and can be observed in experiments.

The aim of this paper is to show the existence of viscous fluid flows in the outer part of a rotating permeable cylinder by using the statements proposed by V. Yudovich when considering Dolidze's problem, taking into account the influence of a transversal pressure gradient maintained by pumping the fluid through an annular space of the cylinders. In the case of a large gap between the cylinders, the pressure gradients may lead to complex interactions between internal and external flows, which in turn may lead to unexpected effects onto an external flow, strengthening or suppressing certain flow regimes.

The viscous incompressible flow with a constant transverse pressure gradient between rigid and permeable cylinders and their stability have been studied by many authors (see, [1,3–9] and references therein).

2. We consider the viscous, incompressible fluid between the permeable cylinders of radii R_1, R_2 ($R_2 > R_1$), rotating with constant angular velocities Ω_1, Ω_2, respectively. It is assumed that the flow is under the action of a constant transverse pressure gradient in the azimuthal direction and the fluid inflow through one cylinder is equal to the fluid outflow through the other one.

We use the two-dimensional system of Navier–Stokes equations and the equation of continuity in the cylindrical coordinates r, θ. Let us assume that the following boundary conditions:

$$R_1 u\big|_{r=R_1} = R_2 u\big|_{r=R_2} = s, \quad s = \text{const},$$
$$v\big|_{r=R_i} = \Omega_i R_i \quad (i = 1, 2)$$

are satisfied, where $V(u, v)$ is a velocity vector, $2\pi s$ is the liquid flow through the cylinders walls for $r = R_1$ or $r = R_2$.

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This problem admits an exact unique solution $V_0(u_0, v_0)$ and pressure p_0 (see e.g., [5–7]):

$$u_0 = \frac{s}{r}\,, \tag{1}$$

$$v_0 = \begin{cases} \dfrac{K}{\varkappa}\Big(-r + ar^{\varkappa+1} + \dfrac{b}{r}\Big) + Ar^{\varkappa+1} + \dfrac{B}{r}\,, & \varkappa \neq -2, \\ \dfrac{K}{2}\Big(r + \dfrac{a'}{r}\ln r + \dfrac{b'}{r}\Big) + \dfrac{A'\ln r}{r} + \dfrac{B'}{r}\,, & \varkappa = -2, \end{cases} \tag{2}$$

$$\frac{\partial p_0}{\partial r} = \frac{s^2}{r^3} + \frac{v^2}{r}\,, \tag{3}$$

where $\varkappa = \frac{s}{\nu}$ is a radial Reynolds number (for $\varkappa > 0$, the liquid flows through the inner cylinder, while for $\varkappa < 0$, through the outer one), ν is the kinematic viscosity, $K = \frac{1}{2\rho\nu}\,(\frac{\partial p}{\partial\theta})_0$, $(\frac{\partial p}{\partial\theta})_0$ is the constant transverse pressure gradient, v_0 is the solution of the inhomogeneous parabolic equation:

$$\frac{d^2v}{dr^2} + \frac{1-\varkappa}{r}\frac{dv}{dr} - \frac{1+\varkappa}{r^2}\,v = \frac{1}{\rho\nu r}\Big(\frac{\partial p}{\partial\theta}\Big)_0, \tag{4}$$

$$a = \frac{R_2^2 - R_1^2}{R_2^{\varkappa+2} - R_1^{\varkappa+2}}\,, \quad b = \frac{R_1^2R_2^2(R_2^{\varkappa} - R_1^{\varkappa})}{R_2^{\varkappa+2} - R_1^{\varkappa+2}}\,, \tag{5}$$

$$a' = \frac{R_1^2 - R_2^2}{\ln R}\,, \quad b' = \frac{R_2^2\,\ln R_1 - R_1^2\,\ln R_2}{\ln R}\,, \tag{6}$$

$$A = \frac{\Omega_2R_2^2 - \Omega_1R_1^2}{R_2^{\varkappa+2} - R_1^{\varkappa+2}}\,, \quad B = \frac{R_1^2R_2^2(R_2^{\varkappa}\Omega_1 - R_1^{\varkappa}\Omega_2)}{R_2^{\varkappa+2} - R_1^{\varkappa+2}}\,, \tag{7}$$

$$A' = \frac{\Omega_2R_2^2 - \Omega_1R_1^2}{\ln R}\,, \quad B' = \frac{\Omega_1R_1^2\ln R_2 - \Omega_2R_2^2\ln R_1}{lnR}\,. \tag{8}$$

Solution (1)–(3) itself is a stationary flow when the liquid flows over the surfaces of cylinders with a constant pressure gradient along the cylinder circuference.

Consider the external boundary value problem for equation (4) with the boundary conditions

$$v\big|_{r=R_1} = \Omega_1R_1, \quad v\big|_{r=\infty} = 0. \tag{9}$$

If $\varkappa + 1 > 0$, the flow is divergent or convergent, but for low intensity, it follows from the condition at infinity that in (2) $A = 0$ and $K = 0$. In this case, taking into account condition (9), we find that

$$v = \frac{\Omega_1R_1^2}{r} \ \text{ as } \ r \to \infty. \tag{10}$$

Therefore, as in D. Dolidze's problem, we have no interaction between the radial and azimuthal velocity components.

If $\varkappa + 1 < 0$, then from conditions (9), we obtain

$$\begin{aligned} v &= \frac{K}{\varkappa}\Big[\frac{R_1^2 - r^2}{r} + aR_1^{\varkappa+1}\Big(\Big(\frac{r}{R_1}\Big)^{\varkappa+1} - \frac{R_1}{r}\Big)\Big] \\ &\qquad + \frac{\Omega_1R_1^2}{r} + AR_1^{\varkappa+1}\Big[\Big(\frac{r}{R_1}\Big)^{\varkappa+1} - \frac{R_1}{r}\Big], \ \ \varkappa \neq -2, \\ v &= \frac{K}{2}\Big(\frac{r^2 - R_1^2}{r} + \frac{a'}{r}\ln\frac{r}{R_1}\Big) + \frac{\Omega_1R_1^2}{r} + \frac{A'}{r}\ln\frac{r}{R_1}\,, \ \ \varkappa = -2. \end{aligned} \tag{11}$$

Thus, for $\varkappa + 1 > 0$, the external boundary value problem for equation (4) under conditions (9) has, like Dolidze's problem, one unique solution (10), where for $\varkappa + 1 < 0$, there exists a family of solutions (11) with arbitrary parameters A and a, while for $\varkappa = -2$, there is a family of solutions (2) with arbitrary parameters A' and a'.

3. As V. Yudovich noted [10], the flows in unbounded domains have physical meaning, if and only if they are obtained as limits under an unlimited extension of bounded domains, and it is fundamentally important to understand which boundary condition on a part of the boundary tending to infinity is responsible for the given external flow. There are the cases for which the influence of the infinitely distant part of the boundary does not disappear during the passage to the limit.

Following the reasoning of V. Yudovich [10], we pass in (2) to the limit as $R_2 \to +\infty$ and assume that all the remaining parameters R_1, Ω_1, Ω_2, $\varkappa$, K and the argument r are fixed. In (5)–(8), assuming $R_2 \to \infty$ for $\varkappa > 0$, we have

$$v(r) \to \frac{K}{\varkappa}\Big(-r + \frac{R_1^2}{r}\Big) + \frac{\Omega_1 R_1^2}{r}\,.$$

Consequently, unlike D. Dolidze's problem, we have the interaction of the radial and azimuthal components of velocity.

Thus we have obtained the limiting profile that changes the intensity of linear vortex formed by the rotation of the interior cylinder;

Consider the case for $\varkappa < 0$ (converging flow) and pass to the limit, as $R_2 \to \infty$ in the coefficient (5)–(8). We have the following asymptotics:

$$A \sim \Omega_2 R_2^{-\varkappa}, \quad B \sim -\Omega_2 R_2^{-\varkappa}, \quad a \sim R_2^{-\varkappa}, \quad b \sim -R_2^{-\varkappa} \;\; (-2 < \varkappa < 0),$$
$$A \sim -\Omega_2 R_2^{2}, \quad B \sim \Omega_2 R_2^{2}, \quad a \sim -R_2^{2}, \quad b \sim R_2^{2} \;\; (\varkappa < -2).$$

Respectively, asymptotically as $R_2 \to \infty$, for large r, we have

$$\begin{aligned} v(r) &\sim \frac{K}{\varkappa}\Big[-r + R_2^{-\varkappa} r^{\varkappa+1}\Big(1 - \frac{1}{r^{\varkappa+2}}\Big)\Big] \\ &\qquad + \Omega_2 R_2^{-\varkappa} r^{\varkappa+1}\Big(1 - \frac{1}{r^{\varkappa+2}}\Big), \quad \varkappa + 2 > 0, \\ v(r) &\sim \frac{K}{\varkappa}\Big[-r + R_2^2 \frac{1}{r}\Big(1 - r^{\varkappa+2}\Big)\Big] + \frac{\Omega_2 R_2^2}{r}(1 - r^{\varkappa+2}), \quad \varkappa + 2 < 0. \end{aligned} \tag{12}$$

In case $\varkappa = -2$, as $R_2 \to \infty$, we have

$$v(r) \sim \frac{K}{2}\Big(r + \frac{R_2^2}{r \ln R} \ln \frac{r}{R_1}\Big) + \frac{\Omega_2 R_2^2 \ln \frac{r}{R_1}}{r \ln R}\,. \tag{13}$$

As is seen from (12), (13), the behavior of the limiting profile depends on the following parameters:

$$\gamma_{-\varkappa} = R_2^{-\varkappa} r^{\varkappa+1}\Big(\frac{K}{\varkappa} + \Omega_2\Big), \quad \varkappa + 2 > 0, \tag{14}$$

$$\gamma_2 = \frac{R_2^2}{r}\Big(\frac{K}{\varkappa} + \Omega_2\Big), \quad \varkappa + 2 < 0, \tag{15}$$

$$\gamma_2^1 = \frac{R_2^2 \ln \frac{r}{R_1}}{r \ln R}\Big(\frac{K}{2} + \Omega_2\Big), \quad \varkappa = -2. \tag{16}$$

If these parameters remain constant as $R_2 \to \infty$ (here, $\Omega_2 \to 0$ and $K \to 0$), then there exists a finite limit $v_\infty(r)$ of a solution of the boundary value problem. We have

$$v_\infty(r) = \begin{cases} \dfrac{\Omega_1}{r} + \gamma_{-\varkappa}\Big(1 - \dfrac{1}{r^{\varkappa+2}}\Big) - r\,\dfrac{K}{\varkappa}\,, & \varkappa + 2 > 0, \\[2ex] \Omega_1 r^{\varkappa+1} + \dfrac{\gamma_2}{r}\Big(1 - r^{\varkappa+2}\Big) - r\,\dfrac{K}{\varkappa}\,, & \varkappa + 2 < 0, \\[2ex] \dfrac{\Omega_1}{r} + \gamma_2^1 \ln r + \dfrac{K}{2}\, r, & \varkappa = -2. \end{cases} \tag{17}$$

Therefore, the limit behavior of the profile $v_\infty(r)$ (17) depends on the parameters $\gamma_{-\varkappa}$ for $\varkappa + 2 > 0$ (14), γ_2 for $\varkappa + 2 < 0$ (15) and on the parameter γ_2^1 for $\varkappa = -2$ (16). Different regimes differ in the values of the parameters $\gamma_{-\varkappa}$, γ_2, γ_2^1. Just as for the Dolidze's problem, the values of these parameters describe the influence of the infinitely distant boundary; each time, with the chosen limit transitions, a unique limit regime is obtained.

References

1. S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*. International Series of Monographs on Physics. Clarendon Press, Oxford, 1961.
2. D. Dolidze, Rotation of a porous cylinder in liquid. (Russian) *Trudy Tbiliss. Gos. Univ.* **84** (1961), 39–44.
3. T. H. Hughes, W. H. Reid, The effect of a transverse pressure gradient on the stability of couette flow. *Z. Angew. Math. Phys.* **15** (1964), 573–581.
4. D. C. Raney, T. S. Chang, Oscillatory modes of instability for flow between rotating cylinders with a transverse pressure gradient. *Z. Angew. Math. Phys.* **22** (1971), no. 4, 680–690.
5. L. Shapakidze, On the bifurcation of viscous flows between two permeable cylinders in the presence of a transverse pressure gradient. *Proc. A. Razmadze Math. Inst.* **130** (2002), 125–132.
6. L. Shapakidze, On the stability of Couette flow between two rotating cylinders in the presence of a transverse pressure gradient. *Proc. A. Razmadze Math. Inst.* **136** (2004), 115–126.
7. L. Shapakidze, On the bifurcations of Dean flow between porous horizontal cylinders with a radial flow and a radial temperature gradient. *Journal of Applied Mathematics and Physics* **5** (2017), no. 9, 1725–1738.
8. L. Shapakidze, On the complex regimes of the Taylor-Dean flow between two rotating porous cylinders. *Trans. A. Razmadze Math. Inst.* **175** (2021), no. 3, 461–466.
9. L. Shapakidze, Effect of Prandtl numbers on the transitions of diverging and converging heat-conducting flows in an annulus. *Trans. A. Razmadze Math. Inst.* **177** (2023), no. 3, 509–515.
10. V. Yudovich, Rotationally symmetric flows of incompressible liquid through a circular ring. (Russian) *Dep. VINITI, Part 1*, 1801–1862, 2001.

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