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MULTILINEAR EXTRAPOLATION IN GRAND LEBESGUE SPACES

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Dedicated to the memory of Professor Elene Obolashvili

Abstract. In this note, we present multilinear Rubio de Francía's extrapolation theorem in grand Lebesgue spaces. The mapping properties of some operators of harmonic analysis are also discussed.

1. Preliminaries

The Rubio de Francía's extrapolation theorem (see [15]) is one of the powerful tools in harmonic analysis. It states that if a given operator T is bounded in a weighted Lebesgue space $L^{p_0}_w$ for some fixed p_0 and all weights w from A_{p_0}, then T is bounded in all L^p_w for all $1 < p < \infty$ and all $w \in A_p$. For the perfection of the extrapolation theory in the classical Lebesgue spaces as well as in various function spaces, we refer, e.g., to the monograph [5].

The grand Lebesgue spaces $L^{p)}(G)$, $1 < p < \infty$, for a constant exponent p and a bounded domain G, were introduced in 1992 by Iwaniec and Sbordone [10] in the context of finding a minimal condition for the integrability of the Jacobian, while the more general space $L^{p),\theta}(G)$ is related to the study of the inhomogeneous $n-$harmonic equation div $A(x, \nabla u) = \mu$ (see Greco, Iwaniec, and Sbordone [9]).

The theory of these spaces is currently one of the intensively developing directions in modern analysis (see, e.g., the recent monograph [12] and the references therein).

We are interested in generalized grand Lebesgue spaces $L^{p),\varphi(\cdot)}_\rho(G)$ with a weight function ρ on G (i.e., ρ is an integrable function on G). The norm in $L^{p),\varphi(\cdot)}_\rho(G)$ is defined as follows:

$$\|f\|_{L^{p),\varphi(\cdot)}_\rho(G)} := \sup_{0<\varepsilon<p_- -1} \left(\frac{\varphi(\varepsilon)}{|G|} \int_G |f(x)|^{p-\varepsilon} \rho(x)dx \right)^{1/(p-\varepsilon)},$$

where $\varphi(\cdot)$ is a measurable function on $(0, p-1)$ which is non-decreasing on some small interval $(0, p-1)$ such that $\lim\limits_{t\to 0} \varphi(x) = 0$. We denote the class of such functions by the symbol $\mathcal{A}^{(p)}$.

If $\varphi(x) = x^\theta$ and $\rho =$ const, where θ is a positive number, then we have the class $L^{p),\theta}(G)$ of Greco, Iwaniec and Sbordone [9]. In particular, if $\theta = 1$ in $L^{p),\theta}(G)$, then we have the space $L^{p)}(G)$ of Iwaniec and Sbordone.

Recall that (see, e.g., [12]) the space $L^{p),\varphi(\cdot)}_\rho(G)$ is a Banach space and that the following continuous imebeddings:

$$L^p_\rho(G) \hookrightarrow L^{p),\varphi(\cdot)}_\rho(G) \hookrightarrow L^{p-\varepsilon}_\rho(G), \ \ 0 < \varepsilon \le p-1,$$

hold.

For the history and properties of grand function spaces we refer, e.g., to [12].

To define the product of a grand Lebesgue space, we first note that for a grand Lebesgue space, the following holds (see also [11]): let $1 < r < \infty$ and let ρ be a weight function on G. Then

$$\|f\|_{L^{r),\varphi}_\rho(G)} = \sup_{0<\varepsilon\le r-1} \varphi(\varepsilon)^{\frac{1}{r-\varepsilon}} \|f\|_{L^{r-\varepsilon}_\rho(G)} = \sup_{1<q<r} (\varphi(r/q'))^{\frac{q}{r}} \|f\|_{L^{r/q}_\rho(G)},$$

where

$$q' = \frac{q}{q-1}\,.$$

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Taking this note into account, we define the class $\mathcal{L}\prod_{j=1}^{m}\mathcal{L}_{\rho_j}^{p_j),\varphi(\cdot)}(G)$ of vector-functions $\overrightarrow{f} = (f_1,\dots,f_m)$ and vector weight $\overrightarrow{\rho} := (\rho_1,\dots,\rho_m)$ as follows: $\overrightarrow{f} \in \mathcal{L}\prod_{j=1}^{m}\mathcal{L}_{\rho_j}^{p_j),\varphi(\cdot)}(G)$, if

$$\|\overrightarrow{f}\|_{\mathcal{L}\prod_{j=1}^{m}\mathcal{L}_{\rho_j}^{p_j),\varphi(\cdot)}(G)} = \sup_{1<r<p}\left\{\varphi\Big(\frac{p}{r'}\Big)^{\frac{r}{p}}\prod_{j=1}^{m}\|f_j\|_{L_{\rho_j}^{p_j/r}(G)}\right\}$$
$$= \sup_{1<r<p}\left\{\prod_{j=1}^{m}\varphi\big(\frac{p}{r'}\big)^{\frac{r}{p_j}}\|f_j\|_{L_{\rho_j}^{p_j/r}(G)}\right\} < \infty,$$

where p is defined by

$$\frac{1}{p} = \sum_{j=1}^{m}\frac{1}{p_j}. \tag{1.1}$$

The expression $\|\overrightarrow{f}\|_{\mathcal{L}\prod_{j=1}^{m}\mathcal{L}_{\rho_j}^{p_j),\varphi(\cdot)}(X,\mu)}$ can be rewritten in the form

$$\|\overrightarrow{f}\|_{\mathcal{L}\prod_{j=1}^{m}\mathcal{L}_{\rho_j}^{p_j),\varphi(\cdot)}(G)}$$
$$= \sup_{0<\eta\le p-1}\left\{\prod_{j=1}^{m}\varphi\Big(\eta\Big)^{\frac{1}{p_j-\eta_j}}\|f_j\|_{L_{\rho_j}^{p_j-\eta_j}(G)} : \frac{p}{p-\eta} = \frac{p_j}{p_j-\eta_j},\ \ j=1,\dots,m\right\},$$

where p is defined by (1.1), and

$$\frac{1}{\eta} = \sum_{j=1}^{m}\frac{1}{\eta_j}. \tag{1.2}$$

It is easy to see that (1.1) and (1.2) imply the identities

$$\frac{1}{p-\eta} = \sum_{j=1}^{m}\frac{1}{p_j-\eta_j}; \quad \frac{\eta}{\eta_j} = \frac{p}{p_j},$$

appearing in the definition of $\|\overrightarrow{f}\|_{\mathcal{L}\prod_{j=1}^{m}\mathcal{L}_{\rho_j}^{p_j),\varphi(\cdot)}(G)}$.

Further, let $\widetilde{p} := \min\{p_1,\dots,p_m\}$. It is easy to check that if $\varphi \in \mathcal{A}^{(\widetilde{p})}$, then

$$\|\overrightarrow{f}\|_{\prod_{j=1}^{m}\mathcal{L}_{\rho_j}^{p_j),\varphi(\cdot)}(G)} \le \|\overrightarrow{f}\|_{\mathcal{L}\prod_{j=1}^{m}L_{\rho_j}^{p_j),\varphi(\cdot)}(G)},$$

where

$$\|\overrightarrow{f}\|_{\prod_{j=1}^{m}L_{\rho_j}^{p_j),\varphi(\cdot)}(G)} := \prod_{j=1}^{m}\|f_j\|_{L_{\rho_j}^{p_j),\varphi(\cdot)}(G)}.$$

This follows from the fact that $\eta \le \eta_j$, $j=1,\dots,m$ (see condition (1.2)).

If $\varphi(x) = x^\theta$, where θ is a positive number, then we denote $\mathcal{L}\prod_{j=1}^{m}\mathcal{L}_{\rho_j}^{p_j),\varphi(\cdot)}(G)$ by $\mathcal{L}\prod_{j=1}^{m}\mathcal{L}_{\rho_j}^{p_j),\theta}(G)$. In particular, obviously,

$$\|\overrightarrow{f}\|_{\mathcal{L}\prod_{j=1}^{m}\mathcal{L}_{\rho_j}^{p_j,\theta}(G)} \le \|\overrightarrow{f}\|_{\prod_{j=1}^{m}L_{\rho_j}^{p_j),\theta}(G)}.$$

If $m=1$, then the grand product space $\prod_{j=1}^{m}\mathcal{L}_{\rho_j}^{p_j),\varphi(\cdot)}(G)$ coincides with the grand Lebesgue space $L_{\rho_j}^{p),\varphi(\cdot)}(G)$.

To formulate the main statement of this work, we need to recall the definition of the multilinear Muckenhoupt class $A_{\overrightarrow{p}}$. Let $1 \le p_j < \infty$ for each $1 \le j \le m$ and $0 < p < \infty$, where p is defined by (1.1). We say that $\overrightarrow{w} := (w_1,\dots,w_m)$ belongs to the class $A_{\overrightarrow{p}}(X)$ if

$$\|\overrightarrow{w}\|_{A_{\overrightarrow{p}}} := \sup_{B\subset\mathbb{R}^n}\left(\frac{1}{|B|}\int_B \nu_{\overrightarrow{w}}(x)dx\right)\prod_{j=1}^{m}\left(\frac{1}{|B|}\int_B w_j^{1-p'}(x)\,dx\right)^{p/p_j'} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. For $p_j = 1$, the expression $\left(\frac{1}{\mu(B)}\int\limits_B w_j^{1-p'}(x)dx\right)^{1/p'_j}$ is understood as $(\inf_B w_j)^{-1}$.

If $m = 1$, then this class is the well-known Muckenhoupt A_p class.

It is known that (see [13]) the multilinear Hardy–Littlewood maximal operator

$$\mathcal{M}(f_1, \dots, f_m)(x) = \sup_{B \ni x} \prod_{j=1}^m \frac{1}{|B|} \int\limits_B |f_j(y)|dy,$$

is bounded from $\prod_{j=1}^m L^{p_j}_{w_j}(\mathbb{R}^n)$ to $L^p_{\nu_{\vec{w}}}(\mathbb{R}^n)$, where p is defined by (1.1), if and only if $\vec{w} \in A_{\vec{p}}$.

The multilinear version of Rubio de Francía's extrapolation theorem in the classical Lebesgue spaces is written as follows:

Theorem 1.1 ([14])**.** *Let $\mathcal{F}$ be a collection of $m+1$-tuple of non-negative functions on $\mathbb{R}^n$. Assume that we have exponents $\vec{p} = (p_1, \dots, p_m)$, with $1 \le p_1, \dots, p_m < \infty$ such that for any $\vec{w} = (w_1, \dots, w_m) \in A_{\vec{p}}$, the inequality*

$$\|f\|_{L^p_w} \le C\big([w]_{A_{\vec{p}}}\big) \prod_{i=1}^m \|f_i\|_{L^{p_i}_{w_i}}$$

holds for every $(f, f_1, \dots, f_m) \in \mathcal{F}$, where p is defined by (1.1), and

$$w := \prod_{j=1}^m w_j^{p/p_j}. \tag{1.3}$$

Then for all exponents $\vec{q} := (q_1, \dots, q_m)$, with $1 < q_1, \dots, q_m < \infty$, and for all weights $\vec{v} \in A_{\vec{q}}$, where

$$\vec{v} := (v_1, \dots, v_m), \tag{1.4}$$

the inequality

$$\|f\|_{L^p_v} \le C\big([\vec{v}]_{A_{\vec{q}}}\big) \prod_{i=1}^m \|f_i\|_{L^{q_i}_{v_i}}$$

holds for every $(f, f_1, \dots, f_m) \in \mathcal{F}$, where

$$\frac{1}{q} := \sum_{j=1}^m \frac{1}{q_j}, \quad v := \prod_{j=1}^m v_j^{q/q_j}. \tag{1.5}$$

Moreover, for the same family of exponents and weights, and for all exponents $\vec{s} = (s_1, \dots, s_m)$ with $1 < s_1, \dots, s_m < \infty$,

$$\left\|\Big(\sum_j (f^j)^s\Big)^{1/s}\right\|_{L^q_v} \le C\big([\vec{v}]_{A_{\vec{q}}}\big) \prod_{i=1}^m \left\|\Big(\sum_j (f_i^j)^{s_i}\Big)^{1/s_i}\right\|_{L^{q_i}_{v_i}}$$

for all $\{(f^j, f_1^j, \dots f_m^j)\}_j \subset \mathcal{F}$, where

$$\frac{1}{s} := \sum_{j=1}^m \frac{1}{s_j}\,. \tag{1.6}$$

2. Main Results

To formulate our main results, we need to introduce the following notation:

$$\overrightarrow{F^{(s)}} := \left(\Big(\sum_j (f_1^j)^{s_1}\Big)^{1/s_1}, \dots, \Big(\sum_j (f_m^j)^{s_m}\Big)^{1/s_m}\right), \tag{2.1}$$

where s is defined by (1.6).

$$\overrightarrow{F^{(r)}} := \bigg(\Big(\sum_j (f_1^j)^r\Big)^{1/r}, \dots, \Big(\sum_j (f_m^j)^r\Big)^{1/r}\bigg), \tag{2.2}$$

where s is defined by (1.6), and r is a constant such that $1 \le r < \infty$.

The symbol $A \lesssim B$ means that there is a positive constant C such that $A \le CB$. Our main result is the following statement:

Theorem 2.1. *Let $\mathcal{F}$ be a collection of $m+1$-tuple of non-negative functions defined on $\mathbb{R}^n$. Assume that we have the exponents $\overrightarrow{p} = (p_1, \dots, p_m)$, with $1 \le p_1, \dots, p_m < \infty$ such that for any $\overrightarrow{w} = (w_1, \dots, w_m) \in A_{\overrightarrow{p}}$, the inequality*

$$\|f\|_{L^p_w(\mathbb{R}^n)} \le C\big([\overrightarrow{w}]_{A_{\overrightarrow{p}}}\big) \prod_{i=1}^m \|f_i\|_{L^{p_i}_{w_i}(\mathbb{R}^n)}$$

holds for every $(f, f_1, \dots, f_m) \in \mathcal{F}$, where the mapping $\cdot \mapsto C(\cdot)$ is non-decreasing, and w is defined by (1.3). Then for all bounded domains $G \subset \mathbb{R}^n$, for all exponents $\overrightarrow{q} := (q_1, \dots, q_m)$, with $1 < q_1, \dots, q_m < \infty$, for all weights $\overrightarrow{v} \in A_{\overrightarrow{q}}(\mathbb{R}^n)$ and for any $\varphi(\cdot) \in \mathcal{A}^{(q)}$, the inequality

$$\|f\|_{L_v^{q),\varphi(\cdot)}(G)} \lesssim \|(f_1, \dots, f_m)\|_{\mathcal{L}\prod_{i=1}^m \mathcal{L}_{v_i}^{q_i),\varphi(\cdot)}(G)}$$

holds for every $(f, f_1, \dots, f_m) \in \mathcal{F}$, where q is defined by (1.5) and v is defined by (1.4).

Moreover, for the same family of exponents and weights, and for all exponents $\overrightarrow{s} = (s_1, \dots, s_m)$ with $1 < s_1, \dots, s_m < \infty$,

$$\bigg\|\Big(\sum_j (f^j)^s\Big)^{1/s}\bigg\|_{L_v^{q),\varphi(\cdot)}(G)} \lesssim \Big\|\overrightarrow{F^{(s)}}\Big\|_{\mathcal{L}\prod_{i=1}^m \mathcal{L}_{v_i}^{q_i),\varphi(\cdot)}(G)}$$

for all $\{(f^j, f_1^j, \dots, f_m^j)\}_j \subset \mathcal{F}$, where s is defined by (1.6) and $\overrightarrow{F^{(s)}}$ is defined by (2.1).

Now, we apply this statement, for example, to derive appropriate weighted estimates for the bilinear operators (for the classical Lebesgue spaces, see [14] and references therein).

Let

$$\mathcal{M}(f,g)(x) = \sup_{Q \ni x} \bigg(\frac{1}{|Q|}\int_Q |f(y)|dy\bigg)\bigg(\frac{1}{|Q|}\int_Q |g(y)dy\bigg),$$

where f and g are the locally integrable functions on $\mathbb{R}^n$, and the supremum is taken over all cubes with sides, parallel to the coordinate axes.

Further, given a bilinear operator T defined a priori from $S \times S$ into S' of the form

$$T(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y,z) f(y) g(z)\, dy dz,$$

we say that T is a Calderón-Zygmund bilinear operator if it can be extended as a bounded operator from $L^{p_1} \times L^{p_2}$ to L^p for some $1 < p_1, p_2 < \infty$ with $1/p_1 + 1/p_2 = 1/p$, and its distributive kernel K coincides, away from the diagonal $\big\{(x,y,z) \in \mathbb{R}^{3n} :\ x = y = z\big\}$, with a locally integrable function $K(x,y,z)$ satisfying the estimates of the form

$$|\partial^\alpha K(x,y,z)| \lesssim \big(|x-y| + |x-z| + |y-z|\big)^{-2n-|\alpha|}, \quad |\alpha| \le 1.$$

The Calderón–Zygmund bilinear operators and $\mathcal{M}$ are known to satisfy the weighted norm inequalities of the classes $A_{\overrightarrow{p}}$ (see [13]). As a consequence of Theorem 2.1, we easily obtain the following vector-valued inequalities.

Theorem 2.2. *Let $m = 2$ and T be a Calderón–Zygmund bilinear operator on $\mathbb{R}^{n \times 2}$. Let G be a bounded domain in $\mathbb{R}^n$. Then for every $\overrightarrow{p} = (p_1, p_2)$, $\overrightarrow{s} = (s_1, s_2)$ with $1 < p_1, p_2, s_1, s_2 < \infty$, for* *every* $\overrightarrow{w}=(w_1,w_2)\in A_{\overrightarrow{p}}$, *for every* $\varphi(\cdot)\in\mathcal{A}^{(p)}$,

$$\Big\|\Big(\sum_j \mathcal{M}(f_1^j,f_2^j)^s\Big)^{\frac{1}{s}}\Big\|_{L_w^{p),\varphi(\cdot)}(G)} \lesssim \Big\|\overrightarrow{F^{(s)}}\Big\|_{\mathcal{L}\prod_{i=1}^2 \mathcal{L}_{w_i}^{p_1),\varphi(\cdot)}(G)}$$

for every non-negative f_i^j *with compact supports* G, *and*

$$\Big\|\Big(\sum_j \Big|T(f_j^1,f_j^2)\Big|^s\Big)^{\frac{1}{s}}\Big\|_{L_w^{p),\varphi(\cdot)}(G)} \lesssim \Big\|\overrightarrow{F^{(s)}}\Big\|_{\mathcal{L}\prod_{i=1}^2 \mathcal{L}_{w_i}^{p_i),\varphi(\cdot)}(G)}$$

for every $f_i^j\in C_0^\infty(G)$, *where* $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, $\frac{1}{s}=\frac{1}{s_1}+\frac{1}{s_2}$, $w=w_1^{\frac{p}{p_1}}w_2^{\frac{p}{p_2}}$, *and* $\overrightarrow{F^{(s)}}$ *is defined by* (2.1).

We can also formulate the Marcinkiewicz–Zygmund inequalities for the Calderón–Zygmund multilinear operators in the frame of grand Lebesgue spaces (see [2,3,8] for the case of classical Lebesgue spaces).

Corollary 2.1. *Let* $m=2$, T *be a Calderón–Zygmund bilinear operator. Let* $1<r\le 2$ *and* $1<q_1,q_2<\infty$. *Then for* $\overrightarrow{w}=(w_1,w_2)\in A_{\overrightarrow{q}}$ *and* $\varphi\in\mathcal{A}^{(q)}$, *the inequality*

$$\Big\|\Big(\sum_j \Big|T(f_j^1,f_j^2)\Big|^r\Big)^{\frac{1}{r}}\Big\|_{L_w^{q),\varphi(\cdot)}(G)} \lesssim \Big\|\overrightarrow{F^{(r)}}\Big\|_{\mathcal{L}\prod_{i=1}^2 \mathcal{L}_{w_i}^{q_i),\varphi(\cdot)}(G)}$$

holds for every $f_i^j\in C_0^\infty(G)$, *where* $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$, $w=w_1^{\frac{q}{q_1}}w_2^{\frac{q}{q_2}}$, *and* $\overrightarrow{F}^{(r)}$ *is defined by* (2.2).

Using extrapolation, we can remove the restriction $q_1,q_2<r$ for $1<r<2$ (for a version of the following result in the context of the Muckenhoupt product classes, we refer the reader to [4]).

Now we give an example for a class of rough bilinear singular integrals given by

$$T_\Omega(f_1,f_2)(x)=\text{p.v.}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f_1(x-y_1)f_2(x-y_2)\frac{\Omega\left((y_1,y_2)/|(y_1,y_2)|\right)}{|(y_1,y_2)|^{2n}}\,dy_1dy_2,$$

where $\Omega\in L^\infty\left(\mathbb{S}^{2n-1}\right)$ has a vanishing integral. All this was introduced by Coifman and Meyer and further studied by Grafakos, He and Honzík [7].

We now discuss the weighted norm inequalities for the bilinear rough singular integral operators T_Ω introduced above. The unweighted and weighted cases in the classical Lebesgue spaces were studied in [1,6,7,14].

We have

Corollary 2.2. *Let* $m=2$, G *be a bounded domain in* $\mathbb{R}^n$, T_Ω *be a bilinear rough singular integral operator with* $\Omega\in L^\infty\left(\mathbb{S}^{2n-1}\right)$ *and* $\int_{\mathbb{S}^{2n-1}}\Omega d\sigma=0$. *Further, suppose that for every* $\overrightarrow{p}=(p_1,p_2)$, $\overrightarrow{s}=(s_1,s_2)$ *with* $1<p_1,p_2,s_1,s_2<\infty$, *for every* $\overrightarrow{w}=(w_1,w_2)\in A_{\overrightarrow{p}}$ *and for every* $\varphi(\cdot)\in\mathcal{A}^{(p)}$, *one has*

$$\Big\|\Big(\sum_j \Big|T(f_j^1,f_j^2)\Big|^s\Big)^{\frac{1}{s}}\Big\|_{L_w^{p),\varphi(\cdot)}(G)} \lesssim \Big\|\overrightarrow{F^{(s)}}\Big\|_{\mathcal{L}\prod_{i=1}^2 \mathcal{L}_{w_i}^{q_i),\varphi(\cdot)}(G)}$$

that holds for every $f_j^i\in C_0^\infty(G)$, *where* $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, $\frac{1}{s}=\frac{1}{s_1}+\frac{1}{s_2}$, $w=w_1^{\frac{p}{p_1}}w_2^{\frac{p}{p_2}}$, *and* $\overrightarrow{F^{(s)}}$ *is defined by* (2.1).

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