Transactions of A. Razmadze
Mathematical Institute
Vol. **179** (2025), issue 2, 301–307

ON THE ABSTRACT SINGULAR OPERATORS IN THE SENSE OF B. BOJARSKI

GIORGI KHIMSHIASHVILI

Dedicated to the memory of Professor Elene Obolashvili

Abstract. We are concerned with certain topological aspects of the abstract singular operators introduced by B. Bojarski as a general framework for the classical Riemann–Hilbert problems. More concretely, we present a short review of the results dealing with Bojarski's index theorem and Bojarski's problem on the topological structure of invertible singular operators.

Introduction

First of all, it should be noted that the topics discussed in the present paper became accessible and instructive for the author to a large extent thanks to Professor Elene Obolashvili, who kindly shared her knowledge in the realm of singular integral equations and introduced him to the prominent researchers in this field, Professors Giorgi Manjavidze and Bogdan Bojarski. These contacts turned out to be very fruitful and, in particular, resulted in a long-term collaboration and several joint papers with Professor Bojarski. The concept of abstract singular operators and related problems became known to the author during a visit of Professor Bojarski to Tbilisi in 1981. In the present note, we give a short review of our results concerned with Bojarski's index theorem and Bojarski's problem on the topological structure of invertible singular operators. We also present several new observations and comments and with great pleasure and respect dedicate this note to the memory of Professor Elene Obolashvili.

To put our presentation in a proper perspective, we begin with recalling that B. Bojarski formulated in the mid-seventies a concept of abstract singular operators and an interesting topological problem that emerged from his investigation of the classical Riemann–Hilbert transmission and boundary value problems (see [2] and an extended survey [8]). This problem was later solved independently in [14] and [26] (cf. also [20]). Later on, these results and approach were used in studying several related topics of global analysis and operator theory [12,23,27]. An important feature of the geometric formulation of elliptic transmission problems in terms of Fredholm pairs of subspaces of a Hilbert space given in [2] was that it suggested several natural modifications and generalizations so that it became possible to study similar problems in more general contexts [5, 6, 8]. Along these lines, the author obtained a generalization of the approach and results of [2] in the context of Hilbert modules over C^*-algebras studied in [11, 17], which led to some progress in the theory of generalized transmission problems [12, 13].

This approach also enabled us to investigate the case of transmission problems over an arbitrary C^*-algebra. Clearly, it gives a wide generalization of the above-mentioned results, since they correspond to the case in which the considered algebra is taken to be the field of complex numbers C. Moreover, it allows one to study elliptic problems associated with abstract singular and bisingular operators over C.-algebras. It should be noted that our generalized setting may be regarded as the investigation of the homotopy classes of families of elliptic transmission problems parameterized by a (locally) compact topological space X. In fact, this corresponds to considering transmission problems over the algebra of continuous functions on the parameter space C(X), which is also a very special case of our results.

2020 *Mathematics Subject Classification.* 52C35, 32S40.

Key words and phrases. Fredholm pair of subspaces; Fredholm operator; Index; Compact operator; Abstract singular operator; Banach space; C^*-algebra; Fredholm module; Kuiper's theorem.

To make the presentation concise, we freely use the terms and constructions from several basic papers on related topics [2, 8, 20, 22]. An exhaustive description of the background and necessary topological notions are contained in [2, 19] and [5].

1. Fredholm Pairs and Abstract Singular Operators

We proceed to present some important details concerning the concept of an abstract singular operator introduced by B. Bojarski in [2]. Recall that a pair $W = (H_+, H_-)$ of subspaces in a Hilbert space is called a *Fredholm pair of subspaces* (FPS) if their sum $H_+ + H_-$ is closed and the numbers $a_W = \dim(H_+ \cap H_-)$ and $b_W = \operatorname{codim}(H_+ + H_-)$ are finite. To fit the original setting of B.Bojarski, here and in the sequel we assume that both H_+ and H_- are of infinite dimension and infinite codimension. The index of such a pair is defined as $\operatorname{ind} W = a_W - b_W$. These concepts were introduced by T. Kato, who, in particular, established that $\operatorname{ind} W$ is not changed by compact perturbations of a FPS. This concept is a natural generalization of the concept of a Fredholm operator. Namely, for a closed operator $A : H_1 \to H_2$ between Hilbert spaces the pair $W_A = (\operatorname{graph} A, \tilde{H}_1)$ is a Fredholm pair, where $\tilde{H}_1$ is a "coordinate subspace" $(H_1, 0)$ in $H_1 \oplus H_2$. Moreover, if A is a Fredholm operator then $\operatorname{ind} W_A = \operatorname{Ind} A$, where $\operatorname{Ind} A$ is the usual Fredholm index of operator A.

Now, an abstract singular operator T is defined as

$$T = \phi P_- + P_+,$$

where P_- and P_+ are the corresponding projectors on H_- and H_+.

Suppose that the Hilbert space is decomposed into a direct sum $H = H_- \oplus H_+$, where both summands are closed subspaces of infinite dimension. Let us say that this decomposition is given by the symmetry $S = P_+ - P_-$ satisfying $S^2 = 1$. The operator S is an abstract analogue of the basic one-dimensional singular operator with the Cauchy kernel [18]. In the classical theory of the linear conjugation problem, it is well known that for any continuous loop-matrix ψ on the unit circle, the commutator $[M_\psi, S]$ is a compact operator (the so-called Mikhlin lemma) [18]. By imposing an abstract analogue of the latter property one obtains the concept of an abstract singular operator formulated as follows.

Let $J \in L(H)$ be an ideal containing the ideal of finite-rank operators and contained in the ideal $K(H)$ of compact linear operators in H. In this situation, B. Bojarski introduced the group of all invertible linear automorphisms $GL(S, J)$ commuting with S up to the ideal J. In other words,

$$GL(S, J) = \phi \in GL(H) : [\phi, S] \in J\}.$$

These definitions allowed B. Bojarski to prove the following index theorem, which is an abstract analogue of the classical index theorem for singular integral operators [18].

Theorem 1.1 ([2])**.** *Let (H_+, H_-) be a Fredholm pair of subspaces such that H_- is given by a projector P_- satisfying $P_- - P_+ \in J$. Then there exists $\phi \in GL(S, J)$ such that $H_- = \phi(H_+)$. Moreover, the operator $L_\phi = \phi P_- + P_+$ is Fredholm and*

$$\operatorname{Ind} L_\phi = \operatorname{ind}(H_-, H_+).$$

Moreover, the map $\kappa : GL(S, J) \to \mathbf{Z}, \kappa(\phi) = \operatorname{Ind} L_\phi$ is a group homomorphism.

This important theorem has led to further developments and generalizations, of which we mention here only three. First, in [12], the author of the present paper obtained a generalization of Theorem 1.1 in the context of Hilbert modules over C^*-algebras (Theorem 2.1 below). Second, a generalization of Theorem 1.1 to the Fredholm pairs of operators was obtained by E.Boasso in 2010 [3] (Theorem 3.1). Both these results can be generalized to an index theorem for a regular Fredholm pair of operators in a Hilbert module over a C^*-algebra.

In the same seminal paper [2], B. Bojarski also formulated the problem of describing the homotopy structure of the group $GL(S, J)$, which led to the second line of developments. The original B. Bojarski's problem was solved by the present author in 1982 [14] and independently by W. Wojciechowski in 1985 [26]. Its solution also follows from the results on the so-called bounded Grassmanian of a polarized Hilbert space given in the book [20]. These results have been generalized by the present author in the context of abstract singular operators over C^*-algebra (Theorem 2.2 below).

2. Abstract Singular Operators over C^*-algebras

We pass now to the generalization of Bojarski's aproach in the context of Hilbert modules over C^*-algebras, to precise definitions needed to present B. Bojarski's geometric approach to transmission problems [2]. We mainly use the same concepts as in [2], but sometimes in a slightly different form adjusted to the case of Hilbert C^*-modules.

Let A be a unital C^*-algebra and H_A be the canonical Hilbert module over A (see, e.g., [17]). As is well known, there exists a natural A-valued scalar product on H_A possessing usual properties [17], and so one can introduce direct sum decompositions and consider various types of bounded linear operators on H_A. Denote by $B(H_A)$ the set of all A-bounded linear operators having A-bounded adjoints. This algebra is one of the most fundamental objects in the theory of Hilbert C^*-modules [11, 16, 17].

It is also well known that $B(H_A)$ is a Banach algebra. Let $GB = GB(H_A)$ denote its group of units, and let $U = U(H_A)$ denote the subgroup of unitaries in GB. For our purposes, it is important to have adjoints, which, as shown in [11], is not the case for an arbitrary bounded operator in the Hilbert A-module H_A. That is why we need to work with $B(H_A)$ and not with the algebra of all A-linear operators in H_A. Moreover, for the algebra BA, we have an analogue of the polar decomposition [11], which implies that $GB(H_A)$ is extendable to $U(H_A)$. Thus these two operator groups are homotopy equivalent, which is important for our considerations.

Compact linear operators on H_A are defined as the limits of the A-norm of linear operators of finite rank [11]. Their set is denoted by $K(H_A)$. Recall that the main object of B. Bojarski's approach in [2] is a special group of operators associated with a fixed direct sum decomposition (called a splitting in [2] and [8]) of a given complex Hilbert space. With this in mind, we fix a direct sum decomposition of the canonical Hilbert A-module of the form $H_A = H_- \oplus H_+$, where H_- and H_+ are both isomorphic to H_A as A-modules. As is well known, any operator in H_A can now be written as a (2×2)-matrix of operators with respect to this decomposition. We denote by π_- and π_+ the natural orthogonal projections defined by this decomposition.

Introduce now the subgroup $GB_r = GB_r(H_A)$ of $GB(H_A)$ consisting of operators whose off-diagonal terms belong to $K(H_A)$. Let $U_r = U_r(H_A)$ denote the subgroup of its unitary elements. To relate this to abstract singular operators, we use an analogue of the so-called bounded Grassmannian introduced in [20]. In fact, this is equivalent to working with Fredholm pairs of subspaces as in [2] but more convenient since we can use the arguments from [20].

Recall that there is a well-defined notion of finite rank of an A-submodule of a Hilbert A-module [17]. This enabled A. Mishchenko and A. Fomenko to introduce the notion of a Fredholm operator in a Hilbert A-module, requiring that its kernel and image be finite-rank A-submodules [17]. It turned out that many important properties of ordinary Fredholm operators remain valid in this context, as well. In particular, if the set of all Fredholm operators in H_A is denoted by $F(H_A)$, then there exists a canonical homomorphism

$$\mathrm{ind} = \mathrm{ind}_A : F(H_A) \to K_0(A),$$

where $K_0(A)$ is the usual topological K-homology group of the basic algebra A [16]. This means that Fredholm operators over C^*-algebras have indices obeying the usual additivity law. In the sequel, we freely refer to the detailed exposition of these results in [16].

Given the above technical details, we can now introduce a special Grassmannian $Gr_+ = Gr_+(H_A)$ associated with the given decomposition. It consists of all A-submodules V of H_A such that the projection π_+ restricted to V is a Fredholm operator while the projection π_- restricted on V is compact. Using the analogues of the local coordinate systems for $Gr_+(H_C)$ constructed in [20], one can verify that $Gr_+(H_A)$ is a Banach manifold modelled on the Banach space $K(H_A)$. For our purposes, it suffices to consider Gr_+ as a metrizable topological space with the topology induced by the standard one on the infinite Grassmannian $Gr(H_A)$ described in [16].

The problem we are interested in, is the study of the topology of $Gr_+(H_A)$ and $GB_r(H_A)$. Note that for $A = \mathbf{C}$ this is exactly the aforementioned problem suggested by B. Bojarski in [2]. We can now present the main results of [12] which, in particular, yield a solution to Bojarski's original problem.

Theorem 2.1. *The group $GB_r(H_A)$ acts transitively on $Gr_+(H_A)$ with the contractible isotropy subgroups.*

Theorem 2.2. *All even-dimensional homotopy groups of $Gr_(HA)$ are isomorphic to the index group $K_0(A)$, while its odd-dimensional homotopy groups are isomorphic to the Milnor group $K_1(A)$.*

Moreover, the same statements hold for the homotopy groups of $GBr(H_A)$, since in the proof of Theorem 2.1 given in [12] was established that these two spaces are homotopy equivalent. We formulate the result for $Gr_+(H_A)$ because for this set the proofs closely follow the argument given in [20] in the case $A = \mathbf{C}$.

In the proof of Theorem 2.1 given in [12], was also obtained precise description of the structure of isotropy subgroups. It should be noted that the contractibility of isotropy subgroups follows from a fundamental result on C^*-algebras called the generalization of Kuiper's theorem for Hilbert C^*-modules, which was independently proven by E. Troitsky [24] and J. Mingo [16]. For commutative C^*-algebras, a particular case of Theorem 2.2 yields a classifying space for K-theory.

The homotopy groups of $GB_r(H_A)$ were first computed by the author in 1987 [12] without mentioning the restricted grassmannian. Later, similar results were obtained by S. Zhang [27] in the framework of K-theory. The contractibility of isotropy subgroups involved in Theorem 2.1 was previously established only for $A = \mathbf{C}$ [20].

Corollary 2.1. *The even-dimensional homotopy groups of the set of classical Riemann–Hilbert problems are trivial, while the odd-dimensional ones are isomorphic to the additive group of integers* $\mathbf{Z}$.

Note that the non-triviality of these groups was interpreted in terms of the so-called spectral flow of order zero pseudo-differential operators by B. Booss and K. Wojciechowsky [9]. This problem is closely related to the Atiyah–Singer index formula in the odd-dimensional case [9].

Due to the aforementioned relation between the bounded Grassmannian and abstract singular operators, the solution of B. Bojarski's original problem is now straightforward.

Corollary 2.2 ([14, 26])**.** *Homotopy groups of invertible singular operators over a unital C.-algebra A are expressed by the relations* (*below n is an arbitrary natural number*)*:*

$$\pi_0 \tilde{=} K_0(A); \quad \pi_1 \tilde{=} \mathbf{Z} \oplus \mathbf{Z} \oplus K_1(A); \quad \pi_{2n} \tilde{=} K_0(A); \pi_{2n+1} \tilde{=} K_1(A).$$

Specifying this result for the algebras of continuous functions yields, in particular, homotopy classes of invertible classical singular integral operators on an arbitrary regular closed contour in the complex plane (see [2, 13] for the precise definitions).

Corollary 2.3. *If $K \subset \mathbf{C}$ is a smooth closed contour with k components, then the homotopy groups of invertible classical singular integral operators on K are expressed by the relations* (*where n is a natural and arbitrary number*)*:*

$$\pi_0 \tilde{=} Z; \quad \pi_1 \tilde{=} Z^{2k+1}; \quad \pi_{2n} \tilde{=} 0; \quad \pi_{2n+1} \tilde{=} Z.$$

As shown in [13], this information also provided homotopy classes and index formulas for the so-called bisingular operators. These operators can be defined in purely algebraic terms, starting from the algebra of abstract singular operators. This allows one to define bisingular operators over the C^*-algebra and to describe the homotopy classes of elliptic bisingular operators [13].

Corollary 2.4. *Abstract elliptic bisingular operators over a C^*-algebra A are homotopically classified by their indices which take values in $K_0(A)$. The index homomorphism is an epimorphism onto $K_0(A)$.*

These results have a number of other applications in global analysis and topology (see, e.g., [6–8]). Here, we only mention the construction and classification of the so-called Fredholm structures on loop groups of compact Lie groups presented in [6, 7], and applications to the bordism theory given in [8]. A description of these developments requires a separate presentation, so we conclude our review by presenting another line of generalization of Bojarski's approach in the context of several variable operator theory.

3. Fredholm Pairs of Operators in Banach Spaces

In this section we describe a generalization of Bojarski's approach to the context of Fredholm pairs of operators (FPO) between Banach spaces given in [3, 4]. We will freely use the concepts and basic results from [3,4,25]. It is known that a direct generalization of the Fredholm index to Fredholm pairs of operators between arbitrary Banach spaces meets technical problems related with the absence of adjoint operators in the general case. For this reason, it is necessary to introduce a more restricted notion of Fredholm pair of operators. As shown in [3], one of reasonable possibilities is to consider the so-called regular Fredholm pairs of operators. In this setting, there is a reasonable notion of index and an analogue of the Bojarski's index theorem which expresses the index of a regular Fredholm pair of operators as the usual index of a certain Fredholm operator. Below, we reproduce the relevant definitions and results from [3] and explain how they can be generalized to the context of Fredholm pairs of operators in the Hilbert module over a unital C^*-algebra.

Let X, Y be two Banach spaces, $L(X, Y)$ be the algebra of linear and continuous operators from X to Y, and $K(X, Y)$ be the closed ideal of all compact operators of $L(X, Y)$. For every $S \in L(X, Y)$, the null-space of S is denoted by $N(S)$ and the range is denoted by $R(S)$.

A pair of operators $S \in L(X, Y), T \in L(Y, X)$ is called a Fredholm pair of operators (FPO) if the following dimensions are finite:

$$a = \dim N(S)/(N(S)\Lambda R(T)), \quad b = \dim R(T)/(N(S)\Lambda R(T)),$$
$$c = \dim N(T)/(N(T)\Lambda R(S)), \quad d = \dim R(S)/(N(T)\Lambda R(S)).$$

Let $P(X, Y)$ denote the set of all Fredholm pairs. For $(S, T) \in P(X, Y)$, the index of (S, T) is defined as $\operatorname{ind}(S, T) = a - b - c + d$. If $b = d = 0$, then (S, T) is called a symmetrical Fredholm pair. Clearly, if S is a Fredholm operator, then the pair $(S, 0)$ is an FPO, so the concept of FPO is simultaneously a generalization of Fredholm operator and of the concept of Fredholm pair of subspaces. However,to guarantee the usual stability properties of the Fredholm index of a single operator and to obtain an analogue of Bojarski's index theorem, it is necessary to restrict the class of FPOs under consideration.

To this end, recall that an operator $T \in L(X, Y)$ is regular if there exists an operator $S \in L(Y, X)$ such that $TST = S$. Such an operator S is called a generalized inverse of T. A Fredholm pair of operators $(S, T) \in L(X, Y)$ is called a regular Fredholm pair of operators if both operators S and T are regular.

In this case, to formulate a proper analogue of Bojarski's index theorem, we need to perform some preparations of technical nature. Introduce $X' = X/R(TS)$, $Y' = Y/R(ST)$, and also the linear and bounded maps $S' \in L(X', Y')$ and $T' \in L(Y', X')$ as the factorizations of S and T through the corresponding invariant subspaces. Clearly, $S'T' = 0$ and $T'S' = 0$.

Since $R(ST)$ and $R(TS)$ are finite-dimensional subspaces of Y and X, respectively, therefore X' and Y' are isomorphic to finite codimensional closed subspaces of X and Y, respectively. Consequently, using isomorphisms of Banach spaces, S' and T' can be viewed as operators defined on and in finite-codimensional closed subspaces of X and Y. It can be easily verify that both S' and T' are regular operators. Now, a proper analogue of Bojarski's index theorem can be formulated as follows.

Theorem 3.1 ([4]). *With the above assumptions and notations, the following statements are equivalent:*

(i) (S, T) *is a regular Fredholm pair.*
(ii) $S + T' \in L(X, Y)$ *is a Fredholm operator.*
(iii) $T + S' \in L(Y, X)$ *is a Fredholm operator.*
Furthermore, in this case,

$$\operatorname{ind}(S, T) = \operatorname{Ind}(S + T') = -\operatorname{Ind}(T + S').$$

This means that, as in the original Bojarski's index theorem, the index of Fredholm pair is expressed as the usual Fredholm index of a certain explicitly constructible Fredholm operator. As shown in [4], if both operators act in the same Banach space, then the above index can also be expressed as the index of a certain Fredholm pair of subspaces. In the case if both operators S and T act in a Hilbert space, the assumption of regularity is automatically fulfilled and one can verify that Theorem 3.1 becomes equivalent to the original Bojarski's index theorem.

4. Concluding Remarks

In conclusion, we outline several possible research perspectives suggested by the results discussed in this note, in a hope that this may lead to further development of the approach initiated by B. Bojarski. First of all, one may try to obtain analogous results in the context of Fredholm fans of subspaces considered by B. Bojarski [5] and Fredholm complexes of Banach spaces considered in several variable operator theory [25].

Next, all these concepts have natural analogues for operators in Hilbert modules over C^*-algebras and one may wish to generalize the main results of [2, 8] and [7] in this context. In particular, the concept of regular Fredholm pair of operators and Theorem 3.1 should have direct analogues for Hilbert modules over C^*-algebras.

Further, given the well known index formulas in terms of commutator traces, it would be interesting to find their analogues for Redholm bi-Grassmannians satisfying certain trace-class conditions. Under the conditions of Theorem 1.1, one can consider the cases where J belongs to certain Shatten-von Neumann class. A Fredholm pair of trace-class operators in a Hilbert space yields a quadruple of subspaces for which there is a natural analogue of the cross-ratio, and it would be interesting to investigate its functorial properties.

Finally, it would be interesting to obtain analogues of Theorem 2.1 for the topological structure of Fredholm pairs of operators between Banach spaces of various classes. For operators in a Hilbert space, such results should follow directly from Theorem 2.1 but to the best of our knowledge they have not yet appeared in the literature.

References

1. B. Blackadar, *K-theory for Operator Algebras.* Mathematical Sciences Research Institute Publications, 5. Springer-Verlag, New York, 1986.
2. B. Bojarski, The abstract linear conjugation problem and Fredholm pairs of subspaces. (Russian) in: *Differential and integral equations. Boundary value problems. Collection of papers dedicated to the memory of I. N. Vekua,* 45–60. Tbilisi University Press, Tbilisi, 1979.
3. E. Boasso, Regular Fredholm pairs. *J. Operator Theory* **55** (2006), no. 2, 311–337.
4. E. Boasso, Further results on regular Fredholm pairs and chains. *Rev. Roumaine Math. Pures Appl.* **55** (2010), no. 3, 149–157.
5. B. Bojarski, Connections between complex and global analysis: some analytical and geometrical aspects of the Riemann-Hilbert transmission problem. In: *Complex analysis*, 97–110, Math. Lehrbücher Monogr. II. Abt. Math. Monogr., 61, Akademie-Verlag, Berlin, 1983.
6. B. Bojarski, G. Khimshiashvili, Global geometric aspects of Riemann-Hilbert problems. *Georgian Math. J.* **8** (2001), no. 4, 713–726.
7. B. Bojarski, G. Khimshiashvili, Geometry of Sato Grassmannians. *Open Math.* **3** (2005), no. 4, 511–521.
8. B. Bojarski, A. Weber, Generalize Riemann-Hilbert transmission and boundary value problems. Fredholm pairs and bordisms. *Bull. Pol. Acad. Sci.* **50** (2002), no. 4, 479–496.
9. B. Booss, K. Wojciechowsky, *Elliptic Boundary Problems for Dirac Operators.* Birkh¨auser, Boston, Basel, Berlin 1993.
10. J. Jr. Eells, Fredholm structures. In: *Nonlinear Functional Analysis (Proc. Sympos. Pure Math., Vol. XVIII, Part 1, Chicago, Ill., 1968)*, 62–85, Proc. Sympos. Pure Math., XVIII, Part 1, Amer. Math. Soc., Providence, RI, 1970.
11. G. G. Kasparov, Hilbert C^*-modules: theorems of Stinespring and Voiculescu. *J. Operator Theory* **4** (1980), no. 1, 133–150.
12. G. N. Khimshiashvili, The topology of invertible singular integral operators. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **108** (1982), no. 2, 273–276 (1983).
13. G. Khimshiashvili, Polysingular operators and the topology of invertible singular operators. *Z. Anal. Anwendungen* **5** (1986), no. 2, 139–145.
14. G. Khimshiashvili, On the homotopy structure of invertible singular operators. In: *Complex analysis and applications '87 (Varna, 1987)*, 230–234, Publ. House Bulgar. Acad. Sci. Sofia, 1989.
15. G. Khimshiashvili, Homotopy classes of elliptic transmission problems over C^*-algebras. *Georgian Math. J.* **5** (1998), no. 5, 453–468.
16. J. A. Mingo, K-theory and multipliers of stable C^*-algebras. *Trans. Amer. Math. Soc.* **299** (1987), no. 1, 397–411.
17. A. S. Miščenko, A. T. Fomenko, The index of elliptic operators over C^*-algebras. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **43** (1979), no. 4, 831–859, 967.

18. N. I. Muskhelishvili, *Singular Integral Equations.* (Russian) Third, corrected and augmented edition. With an appendix by B. Bojarski. Izdat. Nauka, Moscow, 1968.
19. R. Palais, On the homotopy type of certain groups of operators. *Topology* **3** (1965), 271–279.
20. A. Pressley, G. Segal, *Loop Groups.* Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1986.
21. Z. Prössdorf, *Einige Klassen Singulärer Gleichungen.* (German) Mathematische Reihe, Band 46. Birkhäuser Verlag, Basel-Stuttgart, 1974.
22. M. A. Rieffel, Dimension and stable rank in the K-theory of C^*-algebras. *Proc. London Math. Soc. (3)* **46** (1983), no. 2, 301–333.
23. K. Thomsen, Non-stable K-theory for operator algebras. *K-theory* **4** (1991), no. 3, 245–267.
24. E. Troitsky, On the contractibility of general linear groups of a Hilbert C^*-module. (Russian) *Funktsional. Anal. i Prilozhen.* **20** (1986), 58–64.
25. F.-H. Vasilescu, *Analytic Functional Calculus and Spectral Decompositions.* Translated from the Romanian. Mathematics and its Applications (East European Series), 1. D. Reidel Publishing Co., Dordrecht; Editura Academiei Republicii Socialiste România, Bucharest, 1982.
26. K. Wojciechowski, A note on the space of pseudodifferential projections with the same principal symbol. *J. Operator Theory* **15** (1986), no. 2, 207–216.
27. S. Zhang, Factorizations of invertible operators and K-theory of C^*-algebras. *Bull. Amer. Math. Soc. (N.S.)* **28** (1993), no. 1, 75–83.

(Received 10.02.2025)

Ilia State University, 3/5 K. Cholokashvili Avenue, Tbilisi, Georgia
Email address: giorgi.khimshiashvili@iliauni.edu.ge