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ON REAL-VALUED FUNCTIONS HAVING THIN GRAPHS IN THE PLANE

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Abstract. Two classes of functions acting from the real line $\mathbf{R}$ into itself are considered and compared: those whose graphs are totally imperfect in the plane $\mathbf{R}^2$ and those whose graphs are the absolute null sets in $\mathbf{R}^2$.

Throughout this short communication, the following notation will be used.

$\mathbf{R}$ is the set of all real numbers (the real line);

$\mathbf{c}$ is the cardinality of the continuum (i.e., $\mathbf{c} = \mathrm{card}(\mathbf{R})$);

ω is the least infinite cardinal number and ω_1 is the least uncountable cardinal number;

λ is the standard Lebesgue measure on $\mathbf{R}$;

$\mathbf{R}^2$ is the Euclidean plane (i.e., $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$);

λ_2 is the Lebesgue two-dimensional measure on $\mathbf{R}^2$ (i.e., λ_2 is the completion of the product measure $\lambda \otimes \lambda$).

$\mathcal{M}_0(\mathbf{R})$ is the class of the completions of all those nonzero σ-finite Borel measures on $\mathbf{R}$ which vanish at every singleton in $\mathbf{R}$.

$\mathcal{M}(\mathbf{R})$ is the class of all those nonzero σ-finite measures on $\mathbf{R}$ which vanish at every singleton in $\mathbf{R}$.

Clearly, the proper inclusion $\mathcal{M}_0(\mathbf{R}) \subset \mathcal{M}(\mathbf{R})$ holds true.

ZF & **DC** is an abbreviation of **ZF** set theory with the Axiom of Dependent Choices (see [2]).

We say that a function $f : \mathbf{R} \to \mathbf{R}$ is absolutely nonmeasurable with respect to the class $\mathcal{M}_0(\mathbf{R})$ ($\mathcal{M}(\mathbf{R})$) if f is nonmeasurable with respect to any measure belonging to $\mathcal{M}_0(\mathbf{R})$ (to $\mathcal{M}(\mathbf{R})$).

Recall that a subset X of a topological space E is totally imperfect if there exists no subspace of E homeomorphic to the Cantor space $\{0,1\}^\omega$ and entirely contained in X (cf. [1, 4–8]).

Observe that a subset X of a metric space E is totally imperfect if and only if X does not contain an uncountable compact subspace of E.

In the definition below, a topological space E is assumed to have the following property: each singleton in E is a Borel subspace of E.

A subset Y of a space E is called an absolute null if $\mu^*(Y) = 0$ for every nonzero σ-finite Borel measure μ on E vanishing at all singletons in E (here, μ^* denotes, as usual, the outer measure produced by μ).

Clearly, any absolute null subset of E is simultaneously totally imperfect in E.

As is known, absolute null sets and totally imperfect sets are the natural representatives of the so-called thin sets in topological spaces (see, e.g., [6]). There are many examples of totally imperfect subsets of $\mathbf{R}$ which are not absolute null. Moreover, the family of all absolute null sets in $\mathbf{R}$ forms a proper σ-ideal in the boolean of $\mathbf{R}$. On the other hand, if B is a Bernstein subset of $\mathbf{R}$ (see, e.g., [1, 4–8]), then both sets B and $\mathbf{R} \setminus B$ are totally imperfect, but their union coincides with the entire $\mathbf{R}$.

Lemma 1. *Let E_1 and E_2 be two metric spaces and let X be a totally imperfect subset of E_1. Suppose also that for every point $x \in X$, there is given a nonempty subset Y_x of E_2 that is totally imperfect in E_2.*

Then the set $Z = \cup\big\{\{x\} \times Y_x : x \in X\big\}$ is totally imperfect in the topological product space $E_1 \times E_2$.

Remark 1. The converse assertion does not hold, in general. Also, it may happen that a set Z is totally imperfect in the product space $E_1 \times E_2$ and all sets $(\{x\} \times E_2) \cap Z$ are one-element whenever $x \in E_1$, but $\mathrm{pr}_2(Z) = E_2$ (cf., Theorem 3 below).

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Lemma 2. *Let E_1 and E_2 be two topological spaces and let X be an absolute null subset of E_1. Suppose also that for every point $x \in X$, there is given a nonempty subset Y_x of E_2, that is absolute null in E_2.*

Then the set $Z = \cup\big\{\{x\} \times Y_x : x \in X\big\}$ is an absolute null in the topological product space $E_1 \times E_2$.

Remark 2. The converse assertion does not hold, in general. Also, it may happen that a set Z is an absolute null in the product space $E_1 \times E_2$ and all sets $\big(\{x\} \times E_2\big) \cap Z$ are one-element whenever $x \in E_1$, but $\mathrm{pr}_2(Z) = E_2$ (cf., Theorem 4 below).

Theorem 1. *In* **ZF** & **DC** *theory, the following three assertions are equivalent:*

(1) *there exists a totally imperfect subset of* $\mathbf{R}$ *whose cardinality is* $\mathbf{c}$*;*

(2) *there exists a function acting from* $\mathbf{R}$ *into* $\mathbf{R}$ *which is absolutely nonmeasurable with respect to the class* $\mathcal{M}_0(\mathbf{R})$*;*

(3) *there exists a partition of* $\mathbf{R}$ *into* $\mathbf{c}$ *many totally imperfect sets, all of which have cardinality* $\mathbf{c}$.

In the same theory, for a function $f : \mathbf{R} \to \mathbf{R}$, the following three assertions are also equivalent:

(a) *f is absolutely nonmeasurable with respect to $\mathcal{M}_0(\mathbf{R})$;*

(b) *the graph of f is totally imperfect in $\mathbf{R}^2$;*

(c) *the composition $f \circ \phi$ is λ-nonmeasurable whenever ϕ is a Borel bijection of* $\mathbf{R}$ *onto itself.*

Let us mention one consequence of Theorem 1. Let f and g be two functions from $\mathbf{R}$ to $\mathbf{R}$ such that $f \circ g$ is the identity mapping of $\mathbf{R}$. If f is absolutely nonmeasurable with respect to $\mathcal{M}_0(\mathbf{R})$, then g is also absolutely nonmeasurable with respect to $\mathcal{M}_0(\mathbf{R})$.

In particular, a bijection f of $\mathbf{R}$ onto itself is absolutely nonmeasurable with respect to $\mathcal{M}_0(\mathbf{R})$ if and only if its reverse f^{-1} is absolutely nonmeasurable with respect to $\mathcal{M}_0(\mathbf{R})$.

Theorem 2. *In* **ZF** & **DC** *theory, the following two assertions are equivalent:*

(1) *there exists an absolute null subset of* $\mathbf{R}$ *whose cardinality is* $\mathbf{c}$*;*

(2) *there exists a function $g : \mathbf{R} \to \mathbf{R}$ absolutely nonmeasurable with respect to the class* $\mathcal{M}(\mathbf{R})$.

In the same theory, a function $g : \mathbf{R} \to \mathbf{R}$ is absolutely nonmeasurable with respect to $\mathcal{M}(\mathbf{R})$ if and only if the range of g is an absolute null subset of $\mathbf{R}$ *and the sets $g^{-1}(r)$ are at most countable for all* $r \in \mathbf{R}$.

Remark 3. It follows from Lemma 2 and Theorem 2 that if a function $g : \mathbf{R} \to \mathbf{R}$ is absolutely nonmeasurable with respect to $\mathcal{M}(\mathbf{R})$, then the graph of g is an absolute null in $\mathbf{R}^2$ and hence is totally imperfect in $\mathbf{R}^2$.

Remark 4. If a function $h : \mathbf{R} \to \mathbf{R}$ is such that the graph of h is an absolute null subset of $\mathbf{R}^2$, then it may happen that h is not absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$.

Remark 5. By the definition, a Sierpiński–Zygmund function is any function $f : \mathbf{R} \to \mathbf{R}$ such that for each set $X \subset \mathbf{R}$ with $\mathrm{card}(X) = \mathbf{c}$, the restriction $f|X$ is not continuous on X (see [5, 9]). Every Sierpiński–Zygmund function is absolutely nonmeasurable with respect to the class $\mathcal{M}_0(\mathbf{R})$. At the same time, there exists a Sierpiński–Zygmund function g whose graph is a λ_2-thick set in $\mathbf{R}^2$ (this phrase means that the graph of g has common points with every Borel subset of $\mathbf{R}^2$ of strictly positive λ_2-measure). Therefore, g is not absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$. Moreover, the existence of absolutely nonmeasurable functions with respect to $\mathcal{M}(\mathbf{R})$ cannot be established within the **ZFC** set theory, because there are models of **ZFC** in which $\mathbf{c} > \omega_1$ and the cardinality of any absolute null set in $\mathbf{R}$ (or in $\mathbf{R}^2$) does not exceed ω_1 (for further details, see [2]).

Lemma 3. *Let X and Y be two infinite sets such that*

$$\mathrm{card}(X) = \mathrm{card}(Y) = \mathbf{a}$$

and let Z be a subset of $X \times Y$ satisfying the relations

$$(\forall x \in X)\big(\mathrm{card}((\{x\} \times Y) \cap Z) = \mathbf{a}\big),$$
$$(\forall y \in Y)\big(\mathrm{card}((X \times \{y\}) \cap Z) = \mathbf{a}\big).$$

Then there exists a bijection $h : X \to Y$ whose graph is contained in Z.

Using Lemma 3 and a Bernstein subset of the plane $\mathbf{R}^2$, one obtains the next statement.

Theorem 3. *There exists a function* $f : \mathbf{R} \to \mathbf{R}$ *such that:*

(1) f *is a bijection of* $\mathbf{R}$ *onto itself;*

(2) *the graph of* f *is totally imperfect in* $\mathbf{R}^2$ (*i.e.,* f *is absolutely nonmeasurable with respect to the class* $\mathcal{M}_0(\mathbf{R})$).

Remark 6. The existence of a function acting from $\mathbf{R}$ into $\mathbf{R}$ and absolutely nonmeasurable with respect to $\mathcal{M}_0(\mathbf{R})$ cannot be established within **ZF** & **DC** theory. It is easy to see that the characteristic function of a Bernstein set in $\mathbf{R}$ is absolutely nonmeasurable with respect to $\mathcal{M}_0(\mathbf{R})$. So, any representative of Bernstein sets in $\mathbf{R}$ turns out to be nonmeasurable with respect to each measure from the class $\mathcal{M}_0(\mathbf{R})$. The converse assertion is also true.

Theorem 4. *Assuming Martin's Axiom, there exists a function* $g : \mathbf{R} \to \mathbf{R}$ *such that:*

(1) g *is a bijection of* $\mathbf{R}$ *onto itself;*

(2) *the graph of* g *is an absolute null in* $\mathbf{R}^2$.

Remark 7. Theorem 4 cannot be proved without using additional set-theoretical assumptions (see Remark 5). In view of (1) of this theorem, the function g is not absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$.

Remark 8. If there exists a well-ordering $\preceq$ of $\mathbf{R}$ which is isomorphic to ω_1 and simultaneously is a projective subset of $\mathbf{R}^2$ (in the sense of Luzin and Sierpiński), then there exists a function $h : \mathbf{R} \to \mathbf{R}$ such that:

(a) the graph of h is a projective subset of $\mathbf{R}^2$;

(b) h is absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$.

It follows from (a) and (b) that the graph of h turns out to be an absolute null subset of $\mathbf{R}^2$ and so is also totally imperfect in $\mathbf{R}^2$. In addition, the existence of h implies that there is a countable family $\{X_i : i \in I\}$ of projective subsets of $\mathbf{R}$ which satisfy the following relations:

(c) the projective order of any set X_i ($i \in I$) does not exceed some fixed natural number n (where n depends only on the projective order of $\preceq$);

(d) for every measure $\mu \in \mathcal{M}(\mathbf{R})$, at least one set X_i is not measurable with respect to μ.

For more details about h and the family $\{X_i : i \in I\}$, see [3]. Also, it can be deduced from the result of [3] that the following statement is consistent with **ZFC** set theory:

(e) there exists a projective subset P of $\mathbf{R}^2$ such that all vertical sections of P are of cardinality $\mathbf{c}$ (hence $\mathrm{pr}_1(P) = \mathbf{R}$) and any uniformization of P is absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$.

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