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ON THE SOLVABILITY OF THE DIRICHLET PROBLEM FOR ONE CLASS OF FOURTH–ORDER NONLINEAR HYPERBOLIC SYSTEMS

TEONA BIBILASHVILI[1] AND SERGO KHARIBEGASHVILI[1,2]

Dedicated to the memory of Professor Elene Obolashvili

Abstract. The Dirichlet problem for one class of the fourth−order nonlinear hyperbolic systems in a characteristic rectangle is considered. The theorems on the existence, uniqueness and nonexistence of solutions of this problem are proved.

On a plane of variables x and t consider the following fourth−order hyperbolic system

$$\square^2 u + f\left(u_1, \dots, u_N\right) = F\left(x, t\right), \tag{1}$$

where $\square := \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$; $f = (f_1, \dots, f_N)$ and $F = (F_1, \dots, F_N)$ are the given vector functions, while $u = (u_1, \dots, u_N)$ is an unknown $N-$dimensional vector function, $N \geq 2$.

Denote by $D_T : |x| < t < T - |x|$, $|x| < \frac{1}{2}T$, the characteristic rectangle.

For system (1) in the characteristic rectangle D_T consider the Dirichlet problem: find in the domain D_T a solution $u = (u_1, \dots, u_N)$ to system (1) according to the boundary condition

$$u|_{\partial D_T} = 0. \tag{2}$$

Note that one multidimensional analogue of problem (1), (2) in the scalar case was considered in the work [2]. As shown in [1], if instead of the characteristic rectangle $D_T : |x| < t < T - |x|$, $|x| < \frac{1}{2}T$, we consider the non−characteristic rectangle $G_T : 0 < x, t < T$, then for the boundary value problem to be correct in G_T, the Dirichlet boundary condition alone is not enough, it is necessary to require additionally the fulfillment of the Neumann boundary condition $\frac{\partial u}{\partial v}|_{\partial G_T} = 0$.

Let

$$\overset{o}{C}{}^k\left(\overline{D}_T\right) := \left\{u \in C^k\left(\overline{D}_T\right) : u|_{\partial D_T} = 0\right\}, \quad k \geq 1,$$

and introduce the Hilbert space $\overset{o}{W}{}^1_\square\left(D_T\right)$ as a completion of the space $\overset{o}{C}{}^2\left(\overline{D}_T\right)$ with respect to the norm

$$\|u\|^2_{\overset{o}{W}{}^1_\square(D_T)} = \int\limits_{D_T} \left[u^2 + \left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 + (\square u)^2\right] dxdt. \tag{3}$$

It follows from (3) that if $u \in \overset{o}{W}{}^1_\square\left(D_T\right)$, then $u \in \overset{o}{W}{}^1_2\left(D_T\right)$ and $\square u \in L_2\left(D_T\right)$. Here, $W^1_2\left(D_T\right)$ is the well−known Sobolev space consisting of the elements of $L_2\left(D_T\right)$, having the first order generalized derivatives from $L_2\left(D_T\right)$, and $\overset{o}{W}{}^1_2\left(D_T\right) := \{u \in W^1_2\left(D_T\right) : u|_{\partial D_T} = 0\}$, where the equality $u|_{\partial D_T} = 0$ is understood in the sense of the trace theory.

Remark 1. It is easy to verify that if $u \in \overset{o}{C}{}^4\left(\overline{D}_T\right)$ is a classical solution of problem (1), (2), then multiplying scalarly both sides of system (1) by any test vector function $\varphi = (\varphi_1, \dots, \varphi_N) \in \overset{o}{C}{}^2\left(\overline{D}_T\right)$, after integration by parts, we obtain the equality

$$\int\limits_{D_T} \square u \square \varphi dxdt + \int\limits_{D_T} f\left(u\right) \varphi dxdt = \int\limits_{D_T} F \varphi dxdt. \tag{4}$$

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In deriving equality (4), we used the fact that $u|_{\partial D_T} = 0$, $\varphi|_{\partial D_T} = 0$ and the derivative with respect to the conormal $\frac{\partial}{\partial N} = v_t \frac{\partial}{\partial t} - v_x \frac{\partial}{\partial x}$ is an inner differential operator on the boundary ∂D_T of the characteristic rectangle D_T, where $v = (v_x, v_t)$ is the unit vector of the outer normal to ∂D_T.

We take equality (4) as a basis for the definition of a weak generalized solution of problem (1), (2).

Below, on the nonlinear vector function $f = (f_1, \ldots, f_N)$ from (1) we impose the following requirements:

$$f \in C\left(\mathbb{R}^{\mathbb{N}}\right), \quad |f(u)| \leq M_1 + M_2 |u|^{\alpha}, \quad \alpha = \text{const} > 1, \quad u \in \mathbb{R}^N, \tag{5}$$

where $|\cdot|$ is the norm of the space $\mathbb{R}^N$, $M_i = \text{const} \geq 0$, $i = 1, 2$.

Remark 2. As is known, since the dimension of the domain $D_T \subset \mathbb{R}^2$ equals 2, then the embedding operator $I : W_2^1(D_T) \to L_p(D_T)$ represents a linear continuous compact operator for any fixed $q = \text{const} > 1$ [4]. At the same time, the Nemitski operator $K : L_q(D_T) \to L_2(D_T)$, acting by the formula $Ku = f(u)$, where $u \in L_q(D_T)$ and the function f satisfies condition (5), is continuous and bounded for $q \geq 2\alpha$ [3]. Thus, for $q = 2\alpha$, the operator $K_0 = KI$; $W_2^1(D_T) \to L_2(D_T)$ will be continuous and compact. Whence, in particular, it follows that if $u \in W_2^1(D_T)$, then $f(u) \in L_2(D_T)$ and, if $u_m \to u$ in the space $W_2^1(D_T)$, then $f(u_m) \to f(u)$ in the space $L_2(D_T)$.

Definition. Let the vector function f satisfy condition (5) and $F \in L_2(D_T)$. The vector function $u \in \overset{o}{W}{}^1_{\square}(D_T)$ is said to be a weak generalized solution of problem (1), (2), if for any vector function $\varphi \in \overset{o}{W}{}^1_{\square}(D_T)$ the integral equality (4) is valid, i.e.,

$$\int_{D_T} \square u \square \varphi dx dt + \int_{D_T} f(u) \varphi dx dt = \int_{D_T} F \varphi dx dt \quad \forall \varphi \in \overset{o}{W}{}^1_{\square}(D_T). \tag{6}$$

Note that due to Remark 2, the integral $\int_{D_T} f(u) \varphi dx dt$ in the left−hand side of equality (6) is well defined, since $u \in \overset{o}{W}{}^1_{\square}(D_T)$ implies $f(u) \in L_2(D_T)$ and, therefore, $f(u)\varphi \in L_1(D_T)$, since $\varphi \in L_2(D_T)$.

It is not difficult to see that if the solution u of problem (1), (2) belongs to the class $\overset{o}{C}{}^4\left(\overline{D}_T\right)$ in the sense of the definition, then it will also be a classical solution of this problem.

Consider the following condition:

$$\lim_{|u| \to \infty} \inf \frac{uf(u)}{|u|^2} \geq 0, \tag{7}$$

which concerns the behavior of the vector function f in a neighborhood of infinity, where $uf(u) = \sum_{i=1}^{N} u_i f_i(u)$, $|u|^2 = \sum_{i=1}^{N} u_i^2$.

Let $F \in L_2(D_T)$ and conditions (5) and (7) be fulfilled. Then for a weak generalized solution $u \in \overset{o}{W}{}^1_{\square}(D_T)$ of the boundary value problem (1), (2) the following a priori estimate

$$\|u\|_{\overset{o}{W}{}^1_{\square}(D_T)} \leq c_1 \|F\|_{L_2(D_T)} + c_2$$

is valid, where the constants $c_1 > 0$ and $c_2 \geq 0$ are independent of u and F. From here, taking into account Remark 2 and the Leray−Schauder theorem, we have the following

Theorem 1. *Let conditions* (5) *and* (7) *be fulfilled. Then for any vector function* $F \in L_2(D_T)$ *the boundary value problem* (1), (2) *has at least one weak generalized solution in the space* $\overset{o}{W}{}^1_{\square}(D_T)$.

Consider the following condition imposed on the vector function f,

$$(f(u) - f(v))(u - v) \leq 0 \quad \forall u, v \in \mathbb{R}^N. \tag{8}$$

Theorem 2. *Let the vector function* f *satisfy conditions* (5) *and* (8). *Then for any vector function* $F \in L_2(D_T)$ *the boundary value problem* (1), (2) *cannot have more than one weak generalized solution* $u \in \overset{o}{W}{}^1_{\square}(D_T)$.

The following theorem follows from Theorems 1 and 2.

Theorem 3. *Let the vector function f satisfy conditions* (5), (7) *and* (8). *Then for any vector function $F \in L_2(D_T)$ the boundary value problem* (1), (2) *has a unique weak generalized solution in the space* $\overset{o}{W}{}^1_{\square}(D_T)$.

Note that if condition (7) is violated, a sufficiently wide class of vector functions $F \in L_2(D_T)$ can be represented when problem (1), (2) does not have a weak generalized solution from the space $\overset{o}{W}{}^1_{\square}(D_T)$. Indeed, let

$$f_i(u_1, \dots, u_N) = \sum_{j=1}^{N} a_{ij} |u_j|^{\beta_{ij}} + b_i, \quad i = 1, \dots, N,$$

where the constants a_{ij}, β_{ij} and b_i satisfy the following inequalities:

$$a_{ij} > 0, \quad \beta_{ij} = \text{const} > 1, \quad \sum_{j=1}^{N} b_i > 0, \quad i, j = 1, \dots, N.$$

Then, if $F^0 = (F_1^0, \dots, F_N^0) \in L_2(D_T)$, $G = \sum_{j=1}^{N} F_i^0 < 0$, and $F = \mu F^0$, $\mu = \text{const} > 0$, then there exists a number $\mu_0 = \mu_0(G, \beta_{ij}) > 0$ such that problem (1), (2) has no weak generalized solution $u \in \overset{o}{W}{}^1_{\square}(D_T)$, when $\mu > \mu_0$.

References

1. T. Bibilashvili, S. Kharibegashvili, On the solvability of the boundary value problem for one class of higher-order nonlinear hyperbolic systems. *Trans. A. Razmadze Math. Inst.* **178** (2024), no. 2, 317–319.
2. S. Kharibegashvili, B. Midodashvili, Solvability of characteristic boundary-value problems for nonlinear equations with iterated wave operator in the principal part. *Electron. J. Differential Equations* **2008**, no. 72, 12 pp.
3. A. Kufner, S. Fučík, *Nonlinear Differential Equations.* Studies in Applied Mechanics, 2. Elsevier Scientific Publishing Co., Amsterdam-New York, 1980.
4. O. A. Ladyzhenskaya, *The Boundary Value Problems of Mathematical Physics.* Translated from the Russian by Jack Lohwater [Arthur J. Lohwater]. Applied Mathematical Sciences, 49. Springer-Verlag, New York, 1985.

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[1]Department of Mathematics, Georgian Technical University, 77 Kostava Str., Tbilisi 0171, Georgia

[2]A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 2 Merab Aleksidze II Lane, Tbilisi 0193, Georgia

Email address: teonabibilashvili12@gmail.com
Email address: kharibegashvili@yahoo.com