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CHARACTERIZATION OF G-LIPSCHITZ SPACES VIA COMMUTATORS OF THE G-MAXIMAL FUNCTION

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Abstract. Let M_G be the Gegenbauer maximal (G-maximal) function and b be a locally integrable function. Denote M_G^b the G-maximal commutator of M_G with b. In this paper, we consider the boundedness M_G^b on the Lebesgue and G-Morrey spaces when b belongs to the Gegenbauer–Lipschitz (G-Lipschitz) space, by which some characterizations of the Gegenbauer–Lipschitz spaces are given.

1. Introduction, Definition, Notation and Results

The boundedness of the fractional maximal operators, fractional integral and its commutators plays an important role in harmonic analysis and their applications. In recent decades, many authors have proved the boundedness of the commutators with BMO functions of fractional maximal and fractional integral operators on some function spaces. The reader can find detailed information in papers [3, 23, 25, 26, 30].

The fractional integral operator I_α and fractional maximal operator M_α are defined as follows:

$$I_\alpha f(x) := \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-a}} dy, \quad n \geq 1, \;\; 0 < \alpha < n,$$

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int\limits_Q |f(y)|\, dy, \tag{1.1}$$

where the supremum is taken over the cubes $Q \subset \mathbb{R}^n$ containing x.

Let $b \in L_{\mathrm{loc}}(\mathbb{R}^n)$, then the commutator generated by the function I_α and a suitable function b, and also b and M_α are defined as follows:

$$[b, I_\alpha]\, f(x) = b(x) I_\alpha f(x) - I_\alpha(bf)(x) = \int\limits_{\mathbb{R}^n} \frac{[b(x)-b(y)]}{|x-y|^{n-\alpha}} f(y) dy,$$

$$M_{b,\alpha} f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int\limits_Q |b(x)-b(y)|\, |f(y)|\, dy. \tag{1.2}$$

The commutators $[b, I_\alpha]$ were introduced by Chanillo [2].

Definition 1.1 (John–Nirenberg space)**.** $BMO(\mathbb{R}^n)$ is the John–Nirenberg space. That $BMO(\mathbb{R}^n)$ is a Banach space, modulo constants, with the norm $\|\cdot\|_*$ defined by

$$\|b\|_* := \sup_Q \frac{1}{|Q|} \int\limits_Q |b(x) - b_Q|\, dx,$$

where

$$b_Q := \frac{1}{|Q|} \int\limits_Q |b(y)|\, dy$$

and the supremum is taken over all cubes Q in $\mathbb{R}^n$.

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Theorem 1.2 (Chanillo [2])**.** *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $b \in BMO(\mathbb{R}^n)$. Then the operator $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, i.e.,*

$$\|[b, I_\alpha]f\|_{L^q(\mathbb{R}^n)} \le C\, \|b\|_* \, \|f\|_{L^p(\mathbb{R}^n)} .$$

Conversely, if $n - \alpha$ is an even integer and $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, then $b \in BMO(\mathbb{R}^n)$.

The results obtained in [27] (see Theorem 3.5.1 and Theorem 3.5.2) strengthen Theorem 1.2.

Theorem 1.3 ([27])**.** *Let $0 < \alpha < n$, $1 < p < n/\alpha$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. Then $[b, I_\alpha]$ and $[b, M_\alpha]$ are bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $b \in BMO(\mathbb{R}^n)$.*

In fact, the statement of Theorem 1.3 remains valid for $M_{b,\alpha}$ and for the operator

$$|b, I_\alpha| := \int\limits_{R^n} \frac{|b(x) - b(y)|}{|x - y|^{n-\alpha}} f(y) dy,$$

(see [5, Theorem 2.2]).

Further, in works [25] and [31], the results of Theorem 1.3 were obtained in the Morrey spaces.

Morrey spaces were originally introduced by Morrey in [28] to study the local behavior of solutions to the second-order elliptic partial differential equations. Many classical operators of harmonic analysis were studied in Morrey-type spaces during the last decades. We refer the readers to Adams [1] and references therein.

Definition 1.4. Let $1 \le p < \infty$ and $0 \le \theta \le n$. The classical Morrey spaces are defined by

$$L^{p,\theta}(\mathbb{R}^n) = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\theta}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{L^{p,\theta}(\mathbb{R}^n)} := \sup_Q \left(\frac{1}{|Q|^{\theta/n}} \int\limits_Q |f(x)|^p \, dx \right)^{1/p}.$$

The weak Morrey space is defined by

$$WL^{p,\theta}(\mathbb{R}^n) = \{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{WL^{p,\theta}(\mathbb{R}^n)} < \infty \},$$

where

$$\|f\|_{WL^{p,\theta}(\mathbb{R}_+)} := \sup_{r>0} r \sup_{x \in \mathbb{R}^n, t>0} \left(t^{-\frac{\theta}{n}} \left| \{ x \in Q : |f(x)| > r \} \right| \right)^{\frac{1}{p}}.$$

It is well known that if $1 \le p < \infty$, then $L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{p,n}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$.

The next theorem is a generalization of Theorem 1.2 to the Morrey space.

Theorem 1.5 (Di Fazio and Ragusa [4])**.** *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $0 < \theta < n - \alpha p$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\theta}$.*

If $b \in BMO(\mathbb{R}^n)$, then the commutator $[b, I_\alpha]$ is bounded from $L^{p,\theta}(\mathbb{R}^n)$ to $L^{q,\theta}(\mathbb{R}^n)$.

Conversely, if $n - \alpha$ is an even integer and $[b, I_\alpha]$ is bounded from $L^{p,\theta}(\mathbb{R}^n)$ to $L^{q,\theta}(\mathbb{R}^n)$, then $b \in BMO(\mathbb{R}^n)$.

The $L^{p,\theta}$ theory about the fractional integral operator I_α and its commutator $[b, I_\alpha]$ is based of the following theorems:

Theorem 1.6 (Adams [1])**.** *Let $0 < \alpha < n, 0 \le \theta < n$ and $1 \le p < \frac{n-\theta}{\alpha}$.*

(i) *If $1 < p < \frac{n-\theta}{\alpha}$, then $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\theta}$ is a necessary and sufficient condition for the boundedness of I_α from $L^{p,\theta}(\mathbb{R}^n)$ to $L^{q,\theta}(\mathbb{R}^n)$.*

(ii) *If $p = 1$, then $1 - \frac{1}{q} = \frac{\alpha}{n-\theta}$ is a necessary and sufficient condition for the boundedness of I_α from $L^{1,\theta}(\mathbb{R}^n)$ to $WL^{q,\theta}(\mathbb{R}^n)$.*

Theorem 1.7 (Spanne, but published by Peetre [29]). *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \theta < n - \alpha p$. Moreover, let $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $\mu = n\theta/(n - \alpha p)$ (i.e., $\frac{\theta}{p} = \frac{\mu}{q}$).*

(a) *For $p > 1$, I_α is bounded from $L^{p,\theta}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$ if and only if $\frac{\theta}{p} = \frac{\mu}{q}$.*

(b) *For $p = 1$, I_α is bounded from $L^{1,\theta}(\mathbb{R}^n)$ to $WL^{q,\mu}(\mathbb{R}^n)$ if and only if $\mu = \theta q$.*

Theorem 1.8 (Komori and Mizuhara [26]). *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $0 < \theta < n - \alpha p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\theta}$.*

Then the following conditions are equivalent:

(a) *$b \in BMO(\mathbb{R}^n)$.*

(b) *$[b, I_\alpha]$ is bounded from $L^{p,\theta}(\mathbb{R}^n)$ to $L^{q,\theta}(\mathbb{R}^n)$.*

Note that Theorem 1.8 is a strengthening of Theorem 1.5.

Theorem 1.9 (Satori Shirai [31]). *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $0 < \theta < n - \alpha p$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $\mu = n\theta/(n - \alpha p)$ (i.e., $\theta/p = \mu/q$).*

Then the following conditions are equivalent:

(a) *$b \in BMO(\mathbb{R}^n)$.*

(b) *$[b, I_\alpha]$ is bounded from $L^{p,\theta}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$.*

In [31], analogues results are obtained in a Lipschitz space. In the classical analysis, the Lipschitz space is defined as follows.

Definition 1.10. Let $0 < \beta \le 1$. The Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ of order β is defined by

$$\dot{\Lambda}_\beta(\mathbb{R}^n) = \left\{ f : |f(x + y) - f(x)| \le C\,|y|^\beta \right\},$$

and the smallest such constant C is called the $\dot{\Lambda}_\beta(\mathbb{R}^n)$ norm of f and denoted $\|f\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}$.

The above definition implies that if $f \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, then by $\beta > 1$, $f(x) \equiv$ const. Therefore, it makes sense to consider a case for $0 < \beta \le 1$.

The following results were obtained in [31].

Theorem 1.11 ([31]). *Let $1 < p < q < \infty$, $0 < \alpha < n(1/p - 1/q)$, $0 < \beta < 1$, $0 < \alpha + \beta = n(1/p - 1/q) < n, 0 < \theta < n - (\alpha + \beta)$ and $\mu/q = \theta/p$.*

Then the following conditions are equivalent:

(a) *$b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.*

(b) *$[b, I_\alpha]$ is bounded from $L^{p,\theta}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$.*

Theorem 1.12 ([31]). *Let $1 < p < q < \infty$, $0 < \alpha < n(1/p - 1/q)$, $0 < \beta < 1$, and $0 < \alpha + \beta = n(1/p - 1/q) < n$.*

Then the following conditions are equivalent:

(a) *$b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.*

(b) *$[b, I_\alpha]$ is bounded from $L^{p,\theta}(\mathbb{R}^n)$ to $L^{q,\theta}(\mathbb{R}^n)$.*

Further research in this direction can be found in work [27]. According to (1.1) and (1.2), for a locally integrable function f, the Hardy–Littlewood maximal function M is given by

$$M_0 f(x) = Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

the maximal commutator of M with a locally integrable function b is defined by

$$[b, M] f(x) = M_b f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(x) - b(y)|\, |f(y)|\, dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x.

The following theorems were proved in [32].

Theorem 1.13 ([32])**.** *Let b be a locally integrable function and $0 < \beta < 1$, then the following statement are equivalent:*

(1) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$;

(2) M_b *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$ *for* $1 < p < n/\beta$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{n}$;

(3) M_b *satisfies the weak-type* $(1, n/(n-\beta))$ *estimates, namely, there exists a positive constant C such that for all $\theta > 0$,*

$$\left|\{x \in \mathbb{R}^n : M_b f(x) > \theta\}\right| \le C\left(\theta^{-1}\|f\|_{L^1(\mathbb{R}^n)}\right)^{n/(n-\beta)}.$$

Theorem 1.14 ([32] Adams type result)**.** *Let b be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < n/\beta$, $0 < \theta < n - \beta p$, $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{n-\theta}$. Then $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ if and only if M_b is bounded from $L^{p,\theta}(\mathbb{R}^n)$ to $L^{q,\theta}(\mathbb{R}^n)$.*

Theorem 1.15 ([32] Spanne type result)**.** *Let b be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < n/\beta$, $0 < \theta < n - \beta p$, $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{n}$ and $\frac{\theta}{p} = \frac{\mu}{q}$.*

Then $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ if and only if M_b is bounded from $L^{p,\theta}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$.

The purpose of this paper to receive the results of the type of Theorems 1.13, 1.14 and 1.15 in the G-Lipschitz spaces.

2. Some Information from the Gegenbauer Harmonic Analysis

In 2011, in the paper [15] new integral transformation that formed the basis of theory of Harmonic analysis of the Gegenbauer differential operator were constructed. Later, this theory was intensively developed in various directions: approximation theory, embedding theory, transformation theory, theory of maximal functions and potential theory (see [9–14, 16, 18–22, 24]). The basis of this theory was the Gegenbauer differential operator G,

$$G \equiv G_\lambda = (x^2-1)\frac{d^2}{dx^2} + (2\lambda+1)x\frac{d}{dx}, \quad x \in [1,\infty), \quad \lambda \in \left(0, \frac{1}{2}\right)$$

introduced in [7].

The generalized shift operator associated with the operator G is of the form (see [15])

$$A^\lambda_{chy} f(chx) = \frac{\Gamma\left(\lambda + \frac{1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)} \int_0^\pi f\left(chxchy - shxshy\cos\varphi\right)(\sin\varphi)^{2\lambda-1} d\varphi.$$

The properties of this operator are described in [9].

One of the important directions of the Gegenbauer harmonic analysis is the boundedness of maximal operator and potential, generated by the Gegenbauer differential operator G.

In what follows, the expression $A \lesssim B$ will mean that there exists a constant C such that $0 < A \le CB$, where C may depend on some inessential parameters. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$ and say that A and B are equivalent.

For a locally integrable function f, the Gegenbauer maximal function M_G (G-maximal function) is given by (see [16])

$$M_G f(chx) = \sup_{r>0} \frac{1}{|H_r|_\lambda} \int_{H_r} A^\lambda_{chy} |f(chx)|\, d\mu_\lambda(y),$$

where $H_r = (0, r)$ and (see [19])

$$|H_r|_\lambda = \int_0^r sh^{2\lambda} x dx \approx \left(sh\frac{r}{2}\right)^\gamma,$$

where

$$\gamma = \gamma_\lambda(r) = \begin{cases} 2\lambda + 1, & \text{if } r \in (0,2), \\ 4\lambda, & \text{if } r \in [2,\infty). \end{cases}$$

The Gegenbauer potential (G-potential) is defined in [16] as follows:

$$I_G^{\alpha} f(chx) = \frac{1}{\Gamma(\alpha/2)} \int_0^{\infty} \Big(\int_0^{\infty} r^{\alpha/2-1} h_r(cht) dr \Big) A_{cht}^{\lambda}(chx) d\mu_\lambda(t),$$

where

$$h_r(cht) = \int_1^{\infty} e^{-u(u+2\lambda)r} P_u^{\lambda}(cht)(u^2-1)^{\lambda-\frac{1}{2}} du, \quad 0 < \alpha < 2\lambda+1,$$

and

$$P_u^{\lambda}(cht) = \frac{\Gamma(u+2\lambda)\cos\pi\lambda}{\Gamma(\lambda)\Gamma(\lambda+1)} (2cht)^{-u-2\lambda} {}_2F_1\Big(\frac{u}{2}+\lambda, \frac{u}{2}+\lambda+\frac{1}{2}; u+\lambda+1; (cht)^{-2}\Big)$$

is an eigenfunction of the operator G, ${}_2F_1(\alpha,\beta;\gamma;x)$ is a Gaussian hypergeometric function.

The Hardy–Littlewood–Sobolev type theorem for the I_G^{α} holds (see [14, Theorem 3]).

Let $L_p(\mathbb{R}_+, G) = L_{p,\lambda}(\mathbb{R}_+)$, $\mathbb{R}_+ = [0,\infty)$ be the space of a $\mu(x) = sh^{2\lambda}x-$ measurable function on $\mathbb{R}_+$ with the finite norm

$$\|f\|_{L_{p,\lambda}(\mathbb{R}_+)} = \Big(\int_{\mathbb{R}_+} |f(chx)|^p \, d\mu_\lambda(x) \Big)^{\frac{1}{p}}, \quad 1 \le p < \infty, \quad d\mu_\lambda(x) = sh^{2\lambda}x dx,$$

$$\|f\|_{L_{\infty,\lambda}(\mathbb{R}_+)} = \|f\|_{\infty} = \operatorname*{ess\,sup}_{x\in\mathbb{R}_+} |f(chx)|, \quad p = \infty.$$

For $f \in L_{1,\lambda}^{\rm loc}(\mathbb{R}_+)$, the G-fractional maximal operator $\mathfrak{M}_G^{\alpha}$ and the G-fractional integral operator J_G^{α} are defined in [18] as follows:

$$\mathfrak{M}_G^{\alpha} f(chx) = \sup_{r>0} \frac{1}{|H_r|_\lambda^{1-\frac{\alpha}{\gamma}}} \int_{H_r} A_{chy}^{\lambda} |f(chx)| \, d\mu_\lambda(y), \quad 0 < \alpha < \gamma,$$

$\mathfrak{M}_G^{0} \equiv M_G$, and

$$J_G^{\alpha} f(chx) = \int_0^{\infty} \frac{A_{chy}^{\lambda} f(chx)}{\big(sh\frac{y}{2}\big)^{\gamma-\alpha}} d\mu_\lambda(y), \quad 0 < \alpha < \gamma,$$

respectively.

Let $b \in L_{1,\lambda}^{\rm loc}(\mathbb{R}_+)$, then the commutator generated by the function b and the $\mathfrak{M}_G^{\alpha}$, as well as J_G^{α} are defined as follows:

$$\mathfrak{M}_G^{b,\alpha} f(chx) = \sup_{r>0} \frac{1}{|H_r|_\lambda^{1-\frac{\alpha}{\gamma}}} \int_{H_r} \big| A_{chy}^{\lambda} b(chx) - b_{H_r}(chx) \big| \, A_{chy}^{\lambda} |f(chx)| \, d\mu_\lambda(y), \quad 0 < \alpha < \gamma,$$

moreover, $\mathfrak{M}_G^{b,0} \equiv M_G^b$, and also

$$J_G^{b,\alpha} f(chx) = \int_0^{\infty} \frac{[A_{chy}^{\lambda} b(chx) - b_{H_r}(chx)]}{\big(sh\frac{y}{2}\big)^{\gamma-\alpha}} \, A_{chy}^{\lambda} |f(chx)| \, d\mu_\lambda(y), \quad 0 < \alpha < \gamma,$$

where

$$b_{H_r}(chx) = \frac{1}{|H_r|_\lambda} \int_{H_r} A_{chy}^{\lambda} f(chx) d\mu_\lambda(y).$$

By the definition (see [18]), the Gegenbauer –BMO space (G-BMO space) is denoted as

$$BMO_G(\mathbb{R}_+) := \Big\{ f \in L_{1,\lambda}^{\rm loc}(\mathbb{R}_+) : \|f\|_{BMO_G(\mathbb{R}_+)} < \infty \Big\},$$

where

$$\|f\|_{BMO_G(\mathbb{R}_+)} = \sup_{\substack{x\in\mathbb{R}_+\\ r>0}} \frac{1}{|H_r|_\lambda} \int_{H_r} |A_{chy}^{\lambda} f(chx) - f_{H_r}(chx)| d\mu_\lambda(y).$$

The next theorem is analogous of Theorem 1.3.

Theorem 2.1 ([18])**.** *Let $0<\alpha<\gamma$, $1<p<\frac{\gamma}{\alpha}$ and $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{\gamma}$. Then $\mathfrak{M}_G^{b,\alpha}$ and $J_G^{b,\alpha}$ are bounded from $L_{p,\lambda}(\mathbb{R}^n)$ to $L_{q,\lambda}(\mathbb{R}^n)$, if and only if $b\in BMO_G(\mathbb{R}_+)$.*

The space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ was introduced by the German mathematician Rudolf Lipschitz in 1864 in connection with the study of the question of the convergence of Fourier series to its sum. More precisely, the Lipschitz conditions turned out to be sufficient for the convergence of Fourier series in a trigonometric system to their sum. Currently, the Lipschitz conditions are used in direct and inverse theorems (Jackson and Bernstein) in a system of functions, as well as in various questions of Fourier harmonic analysis. This indicates the relevance of studying the classes of functions satisfying Lipschitz conditions.

It is known that every continuously differentiable function on a compact subset of Euclidean space is Lipschitz. On the other hand, any Lipschitz function defined on an open set of Euclidean space is differentiable almost everywhere on this set. It is also known that if a function $f(x)$ has a bounded derivative, then $f\in\dot{\Lambda}_1(\mathbb{R}^n)$.

The above allows us to suggest a close connection between the functions from $\dot{\Lambda}_\beta(\mathbb{R}^n)$ and their differentiability.

It was this circumstance that gave us the idea to introduce a space of functions associated with the Gegenbauer differential operator. We called this space the Gegenbauer–Lipschitz space in honor of two remarkable German mathematicians Gegenbauer and Lipschitz.

By analogy with the classical case, we introduce the G-Lipschitz space as follows.

Definition 2.2. Let $0<\beta\le 1$, we say a function f belongs to the G-Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}_+,G)$ if there exists a constant C such that for all $x,y\in\mathbb{R}_+$,

$$\dot{\Lambda}_\beta(\mathbb{R}_+,G)=\left\{f:|A_{chy}^\lambda f(chx)-f(chx)|<C(chy-1)^\beta\right\}, \tag{2.1}$$

and the smallest such constant C we call the $\dot{\Lambda}_\beta(\mathbb{R}_+,G)$ norm of f and denote it by $\|f\|_{\dot{\Lambda}_\beta(\mathbb{R}_+,G)}$.

The naturalness of the G-Lipschitz space introduced by us is substantiated as follows. For the equality (see [8], Lemma 1)

$$\lim_{y\to 0}\frac{A_{chy}^\lambda f(chx)-f(chx)}{chy-1}=\frac{sh^2x}{2\lambda+1}f''(chx)+(chx)f'(chx)=\frac{Gf(chx)}{2\lambda+1},$$

(2.1) implies that for $\beta>1$,

$$\lim_{y\to 0}\left|\frac{A_{chy}^\lambda f(chx)-f(chx)}{chy-1}\right|\le C\lim_{y\to 0}(chy-1)^{\beta-1}=0$$

$$\Leftrightarrow \lim_{y\to 0}\left|\frac{A_{chy}^\lambda f(chx)-f(chx)}{chy-1}\right|=|Gf(chx)|=0,$$

which means that this G-Lipschitz space for $\beta>1$ consists of $f(x)\equiv$ const. If $|Gf(chx)|\le M-$const, then $f\in\dot{\Lambda}_1(\mathbb{R}^n)$. Therefore, we consider the cases where $0<\beta\le 1$.

3. Theorems

Our first result can be stated as follows.

Theorem 3.1. *Let b be a locally integrable function and $0<\beta<1$. Then the following statements are equivalent:*

(1) $b\in\dot{\Lambda}_\beta(\mathbb{R}_+,G)$,
(2) M_G^b *is bounded from* $L_{p,\lambda}(\mathbb{R}_+,G)$ *to* $L_{q,\lambda}(\mathbb{R}_+)$ *for* $1<p<\frac{\gamma}{\beta}$ *and* $\frac{1}{p}-\frac{1}{q}=\frac{\beta}{\gamma}$,

(3) *M_G^b satisfies the weak-type $\left(1, \frac{\gamma}{\gamma-\beta}\right)$ estimates, namely, there exists a positive constant C such that for all $\nu > 0$,*

$$\left|\left\{x \in \mathbb{R}_+ : M_G^b f(chx) > \nu\right\}\right|_\lambda \le C\left(\nu^{-1}\|f\|_{L_{1,\lambda}(\mathbb{R}_+)}\right)^{\frac{\gamma}{\gamma-\beta}}. \tag{3.1}$$

The following definition was introduced in [19].

Definition 3.2. Let $1 \le p < \infty$ and $0 \le \nu \le \gamma$. We denote by $L_{p,\lambda,\nu}(\mathbb{R}_+)$ the Gegenbauer–Morrey (G-Morrey) space associated with the Gegenbauer differential operator G as the set of locally integrable functions $f(chx)$, $x \in \mathbb{R}_+$ with the finite norm

$$\|f\|_{L_{p,\lambda,\nu}(\mathbb{R}_+)} = \sup_{\substack{x\in\mathbb{R}_+\\ r>0}}\left(|H_r|_\lambda^{-\frac{\nu}{\gamma}}\int_0^r A_{chy}^\lambda |f(chx)|^p\, d\mu_\lambda(y)\right)^{\frac{1}{p}}.$$

Thus by the definition,

$$L_{p,\lambda,\nu}(\mathbb{R}_+) = \left\{f \in L_{1,\lambda}^{\rm loc}(\mathbb{R}_+) : \|f\|_{L_{p,\lambda,\nu}(\mathbb{R}_+)} < \infty\right\}.$$

Let $f \in L_{p,\lambda}(\mathbb{R}_+), 1 \le p \le \infty$, then for any $y \in \mathbb{R}_+$, the following inequality (see [23], Lemma 2)

$$\left\|A_{chy}^\lambda f\right\|_{L_{p,\lambda}(\mathbb{R}_+)} \le \|f\|_{L_{p,\lambda}(\mathbb{R}_+)} \tag{3.2}$$

holds.

Note that $L_{p,\lambda,0}(\mathbb{R}_+) = L_{p,\lambda}(\mathbb{R}_+)$. This follows from inequality (3.2), since $A_1 f(chx) = f(chx)$.

Theorem 3.3 (Adams type result)**.** *Let b be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < \frac{\gamma}{\beta}$, $0 < \nu < \gamma - \beta p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{\gamma-\nu}$. Then $b \in \dot{\Lambda}_\beta(\mathbb{R}_+, G)$ if and only if M_G^b is bounded from $L_{p,\lambda,\nu}(\mathbb{R}_+)$ to $L_{q,\lambda,\nu}(\mathbb{R}_+)$.*

Theorem 3.4 (Spanne type result)**.** *Let b be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < \frac{\gamma}{\beta}$, $0 < \nu < \gamma - \beta p$, $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{\gamma}$ and $\frac{\nu}{p} = \frac{\mu}{q}$. Then $b \in \dot{\Lambda}_\beta(\mathbb{R}_+, G)$ if and only if M_G^b is bounded from $L_{p,\lambda,\nu}(\mathbb{R}_+)$ to $L_{q,\lambda,\mu}(\mathbb{R}_+)$.*

This paper is organized as follows. In Section 1, we recall some basic definitions, known results, and also some auxiliary statements. In Section 2, we give some information from the Gegenbauer harmonic analysis. In Section 3, we give the wording proven theorems. In Section 4, we provide some technical lemmas to prove our theorems. In Section 5, we give proofs of the theorems.

4. Preliminaries and Lemmas

For a measurable set E, by $|E|_\lambda$, we denote $E\,|_\lambda = \int_E sh^{2\lambda}x dx$, absolutely continuous with respect to the Lebesgue measure and by χ_E the characteristic function of E . For $p \in [1, \infty]$, we denote by p' the conjugate index of p, namely, $p' + p = pp\prime$. To prove the theorems, we need some auxiliary results.

To characterize the G-Lipschitz space, by analogy with [4], we introduce the following operator:

$$F_G^\beta f(chx) = \sup_{r>0}|H_r|_\lambda^{-(1+\beta/\gamma)}\int_{H_r}\left|A_{chy}^\lambda f(chx) - f_{H_r}(chx)\right| d\mu_\lambda(y)$$

for all locally integrable functions on $\mathbb{R}_+$.

If $1 \le p < \infty$ and $\beta > 0$, let

$$C_p^\beta = \left\{f \in L_{p,\lambda}(\mathbb{R}_+), F_G^\beta \in L_{p,\lambda}(\mathbb{R}_+)\right\}.$$

If $f \in C_p^\beta$, we define the seminorm

$$|f|_{C_p^\beta} := \left\|F_G^\beta f\right\|_{L_{p,\lambda}(\mathbb{R}_+)}$$

and the norm

$$\|f\|_{C_p^\beta} := \|f\|_{L_{p,\lambda}(\mathbb{R}_+)} + |f|_{C_p^\beta}\,.$$

Introducing the Gegenbauer–Lipschits space, we consider it important to prove that it is a Banach space.

The following statement is analogous to Lemma 6.1 in [6].

Proposition 4.1. *For $1 \le p < \infty$ and $\beta > 0$, C_p^β is a Banach space.*

Proof. We prove that C_p^β is complete.

Suppose $\{f_m\}$ is a Cauchy sequence in C_p^β. Since $L_{p,\lambda}(\mathbb{R}_+)$ is complete (see [11], Proposition 5.1), there exists an $f \in L_{p,\lambda}(\mathbb{R}_+)$ such that $f_m \to f$ in $L_{p,\lambda}(\mathbb{R}_+)$. Let $H_r = (0,r) \subset \mathbb{R}_+$ and $f(chx) = A^\lambda_{ch\frac{1}{\nu}} h_m(chx)$, where $h_m \to h$ in $L_{p,\lambda}(\mathbb{R}_+)$ and $\nu > 0$. Then $f_{H_r}(chx) = \left(A^\lambda_{ch\frac{1}{\nu}} h_m\right)_{H_r}(chx)$. Using the commutativity of the operator A^λ_{chy} (see [12]) $A^\lambda_{chx} A^\lambda_{chy} = A^\lambda_{chy} A^\lambda_{chx}$ and also inequality (3.2), we obtain

$$\begin{aligned}
&\frac{1}{|H_r|_\lambda^{1+\frac{\beta}{\gamma}}} \int\limits_{H_r} \left|A^\lambda_{chy} f(chx) - f_{H_r}(chx)\right| d\mu_\lambda(y) \\
&= \lim_{m\to\infty} \frac{1}{|H_r|_\lambda^{1+\frac{\beta}{\gamma}}} \int\limits_{H_r} \left|A^\lambda_{chy}\left(A^\lambda_{ch\frac{1}{\nu}} h_m\right)(chx) - \left(A^\lambda_{ch\frac{1}{\nu}} h_m\right)_{H_r}(chx)\right| d\mu_\lambda(y) \\
&= \lim_{m\to\infty} \frac{1}{|H_r|_\lambda^{1+\frac{\beta}{\gamma}}} \int\limits_{H_r} \left|A^\lambda_{ch\frac{1}{\nu}}\left(A^\lambda_{chy} h_m\right)(chx) - \left(A^\lambda_{ch\frac{1}{\nu}} h_m\right)_{H_r}(chx)\right| d\mu_\lambda(y) \\
&\le \lim_{m\to\infty} \frac{1}{|H_r|_\lambda^{1+\frac{\beta}{\gamma}}} \int\limits_{H_r} \left|A^\lambda_{chy} h_m(chx) - (h_m)_{H_r}(chx)\right| d\mu_\lambda(y) = \lim_{m\to\infty} F_G^\beta h_m(chx), \quad x \in \mathbb{R}_+.
\end{aligned}$$

Taking a supremum over all interval H_r containing x, we get

$$F_G^\beta h(chx) \le \lim_{m\to\infty} F_G^\beta h_m(chx), \quad x \in \mathbb{R}_+. \tag{4.1}$$

Applying this inequality to the sequence $\{f_m\}$, taking p-th povers, and applying Fatou's lemma, we deduce

$$\begin{aligned}
\left\|F_G^\beta f\right\|_{L_{p,\lambda}(\mathbb{R}_+)} &\le \left(\int\limits_{H_r} \lim_{m\to\infty} \left|F_G^\beta f_m(chx)\right|^p d\mu_\lambda(y)\right)^{\frac{1}{p}} \\
&\le \lim_{m\to\infty} \|f_m\|_{C_p^\beta}
\end{aligned}$$

and so, $f \in C_p^\beta$. Similar reasoning shows that inequality (4.1) applied to the sequence $\{f_m - f_n\}_{m=1}^\infty$ gives

$$\left\|F_G^\beta (f - f_n)\right\|_{L_{p,\lambda}(\mathbb{R}_+)} \le \lim_{m\to\infty} \left\|F_G^\beta (f_m - f_n)\right\|_{L_{p,\lambda}(\mathbb{R}_+)}.$$

But the right-hand side converges to zero as $n \to \infty$ since $\{f_m\}$ is a Cauchy sequence in C_p^β. Since $f_m \to f$ in $L_{p,\lambda}(\mathbb{R}_+)$ has already been established, $f_m \to f$ in C_p^β. $\square$

From the direct and inverse Hölder's inequality, we get the following statement (see [26]).

Lemma 4.2 ([26])**.** *For all $1 < q < \infty$, the relation*

$$\begin{aligned}
&\left(|H_r|_\lambda^{-1} \int\limits_{H_r} \left|A^\lambda_{chy} f(chx) - f_{H_r}(chx)\right|^q d\mu_\lambda|(y)\right)^{\frac{1}{q}} \\
&\approx \frac{1}{|H_r|_\lambda} \int\limits_{H_r} \left|A^\lambda_{chy} f(chx) - f_{H_r}(chx)\right| d\mu_\lambda|(y)
\end{aligned}$$

is valid.

Lemma 4.3. *Let $0<\beta<1$ and $1\le q<\infty$. Define*

$$\dot{\Lambda}_{\beta,q}(\mathbb{R}_+,G):=\Big\{f\in L^{\rm loc}_{1,\lambda}(\mathbb{R}_+):\|f\|_{\dot{\Lambda}_{\beta,q}(\mathbb{R}_+,G)}$$
$$=\sup_{\substack{x\in\mathbb{R}_+\\ r>0}}\frac{1}{|H_r|_\lambda^{\frac{\beta}{\gamma}}}\bigg(\frac{1}{|H_r|_\lambda}\int\limits_{H_r}\big|A^\lambda_{chy}f(chx)-f_{H_r}(chx)\big|^q d\mu_\lambda(y)\bigg)^{\frac1q}<\infty\Big\}.$$

Then for all $0<\beta<1$ and $1\le q<\infty$, $\dot{\Lambda}_\beta(\mathbb{R}_+,G)=\dot{\Lambda}_{\beta,q}(\mathbb{R}_+,G)$ with equivalent norms.

Proof. Using Lemma 4.2, we have

$$\sup_{\substack{x\in\mathbb{R}_+\\ r>0}}\frac{1}{|H_r|_\lambda^{1+\frac{\beta}{\gamma}}}\int\limits_{H_r}\big|A^\lambda_{chy}f(chx)-f_{H_r}(chx)\big|\,d\mu_\lambda(y)$$
$$\approx\sup_{\substack{x\in\mathbb{R}_+\\ r>0}}\frac{1}{|H_r|_\lambda^{\frac{\beta}{\gamma}}}\bigg(\frac{1}{|H_r|_\lambda}\int\limits_{H_r}\big|A^\lambda_{chy}f(chx)-f_{H_r}(chx)\big|^q d\mu_\lambda(y)\bigg)^{\frac1q}$$
$$\Leftrightarrow\|f\|_{\dot{\Lambda}_\beta(\mathbb{R}_+,G)}\approx\|f\|_{\dot{\Lambda}_{\beta,q}(\mathbb{R}_+,G)}.$$
□

Lemma 4.4 ([18], Corollary 3.3)**.** *Let $0<\alpha<\gamma$, $0<\nu<\gamma-\alpha\beta$ and $1\le p<\frac{\gamma-\nu}{\alpha}$.*

(i) *If $1<p<\frac{\gamma-\nu}{\alpha}$, then the condition $\frac1p-\frac1q=\frac{\alpha}{\gamma-\nu}$ is necessary and sufficient for the boundedness of $\mathfrak{M}^\alpha_G$ from $L_{p,\lambda,\nu}(\mathbb{R}_+)$ to $L_{q,\lambda,\nu}(\mathbb{R}_+)$.*

(ii) *If $p=1<\frac{\gamma-\nu}{\alpha}$, the condition $1-\frac1q=\frac{\alpha}{\gamma-\nu}$ is necessary and sufficient for the boundedness of $\mathfrak{M}^\alpha_G$ from $L_{1,\lambda,\nu}(\mathbb{R}_+)$ to $WL_{q,\lambda,\nu}(\mathbb{R}_+)$.*

Lemma 4.5. *Let $1\le p<\infty$ and $0<\nu<\gamma$, then*

$$\|\chi_{H_r}\|_{L_{p,\lambda,\nu}(\mathbb{R}_+)}\lesssim|H_r|_\lambda^{\frac{\gamma-\nu}{p\gamma}}.$$

Proof. Let $H_r=(0,r)\subset\mathbb{R}_+$. Then we have

$$\|\chi_{H_r}\|_{L_{p,\lambda,\nu}(\mathbb{R}_+)}\lesssim\bigg(|H_r|_\lambda^{-\nu/\gamma}\int\limits_{H_r}A^\lambda_{chy}\chi_{H_r}(chx)^p d\mu_\lambda(y)\bigg)^{\frac1p}$$
$$\lesssim\bigg(|H_r|_\lambda^{-\nu/\gamma}\int\limits_{H_r}d\mu_\lambda(y)\bigg)^{\frac1p}=\Big(|H_r|_\lambda^{1-\nu/\gamma}\Big)^{\frac1p}=|H_r|_\lambda^{\frac{\gamma-\nu}{p\gamma}}.$$

The following strong– and weak-type boundedness of M^α_G are well known (see [9, Corollary 5.6]). □

Lemma 4.6 ([9])**.** *Let $0<\alpha<\gamma$. Then:*

(1) *If $1<p<\frac{\alpha}{\gamma}$, then the condition $\frac1p-\frac1q=\frac{\alpha}{\gamma}$ is necessary and sufficient for the boundedness of $\mathfrak{M}^\alpha_G$ from $L_{p,\lambda}(\mathbb{R}_+)$ to $L_{q,\lambda}(\mathbb{R}_+)$.*

(2) *If $p=1$, then the condition $1-\frac1q=\frac{\alpha}{\gamma}$ is necessary and sufficient for the boundedness $\mathfrak{M}^\alpha_G$ from $L_{1,\lambda}(\mathbb{R}_+)$ to $WL_{q,\lambda}(\mathbb{R}_+)$.*

Here, $WL_{q,\lambda}(\mathbb{R}_+)$ is the weak space $L_{p,\lambda}(\mathbb{R}_+)$ defined as the locally integrable functions $f(chx)$, $x\in\mathbb{R}_+$ with the finite norm

$$\|f\|_{WL_{p,\lambda}(\mathbb{R}_+)}=\sup_{r>0}r\,|\{x\in\mathbb{R}_+:|f(chx)|>r\}|_\lambda^{\frac1p}$$
$$=\sup_{r>0}r\bigg(\int\limits_{\{x\in\mathbb{R}_+:|f(chx)|>r\}}d\mu_\lambda(y)\bigg)^{\frac1p}.$$

5. Proof of Theorems

Proof of Theorem 3.1. If $b \in \dot{\Lambda}_\beta(\mathbb{R}_+, G)$, then

$$M_G^b f(chx) = \sup_{r>0} \frac{1}{|H_r|_\lambda} \int_{H_r} \left| A_{chy}^\lambda b(chx) - b_{H_r}(chx) \right| A_{chy}^\lambda |f(chx)| \, d\mu_\lambda(y)$$

$$\lesssim \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}_+,G)} \sup_{r>0} \frac{1}{|H_r|_\lambda^{1-\frac{\beta}{\gamma}}} \int_{H_r} A_{chy}^\lambda |f(chx)| \, d\mu_\lambda(y)$$

$$= \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}_+,G)} \mathfrak{M}_G^\beta f(chx). \tag{5.1}$$

Obviously, (2) and (3) follow from Lemma 4.6 and (5.1)

(2) $\Longrightarrow$ (1) : Assume M_G^b is bounded from $L_{p,\lambda}(\mathbb{R}_+)$, $1 < p < \frac{\gamma}{\beta}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{\gamma}$.

For any interval $H_r \subset \mathbb{R}_+$, by Hölder's inequality and noting $\frac{1}{p} + \frac{1}{q'} = 1 + \frac{\beta}{\gamma}$, one gets

$$\frac{1}{|H_r|_\lambda^{1+\frac{\beta}{\gamma}}} \int_{H_r} \left| A_{chy}^\lambda b(chx) - b_{H_r}(chx) \right| d\mu_\lambda(y)$$

$$\approx \frac{1}{|H_r|_\lambda^{1+\frac{\beta}{\gamma}}} \int_{H_r} \left| A_{chy}^\lambda b(chx) - b_{H_r}(chx) \right| A_{chy}^\lambda \chi_{H_r}(chx) d\mu_\lambda(y)$$

$$\approx \frac{1}{|H_r|_\lambda^{1+\frac{\beta}{\gamma}}} \int_{H_r} \left| A_{chy}^\lambda b(chx) - b_{H_r}(chx) \right| \left(\frac{1}{|H_r|_\lambda} \int_{H_r} A_{chy}^\lambda \chi_{H_r}(chx) d\mu_\lambda(x) \right) d\mu_\lambda(y)$$

$$= \frac{1}{|H_r|_\lambda^{1+\frac{\beta}{\gamma}}} \int_{H_r} \left(\frac{1}{|H_r|_\lambda} \int_{H_r} \left| A_{chy}^\lambda b(chx) - b_{H_r}(chx) \right| A_{chy}^\lambda \chi_{H_r}(chx) d\mu_\lambda(y) \right) d\mu_\lambda(x)$$

$$\leq \frac{1}{|H_r|_\lambda^{1+\frac{\beta}{\gamma}}} \int_{H_r} M_G^b \chi_{H_r}(chx) d\mu_\lambda(x) \leq \frac{1}{|H_r|_\lambda^{1+\frac{\beta}{\gamma}}} \left(\int_{H_r} \left[M_G^b \chi_{H_r}(chx) \right]^q d\mu_\lambda(x) \right)^{\frac{1}{q}}$$

$$\times \left(\int_{H_r} \chi_{H_r}(chx) d\mu_\lambda(x) \right)^{\frac{1}{q'}} \lesssim \frac{1}{|H_r|_\lambda^{1+\frac{\beta}{\gamma}}} \left\| M_G^b \chi_{H_r} \right\|_{L_{q,\lambda}(\mathbb{R}_+)} \|\chi_{H_r}\|_{L_{q',\lambda}(\mathbb{R}_+)}$$

$$\lesssim \frac{1}{|H_r|_\lambda^{1+\frac{\beta}{\gamma}}} \|\chi_{H_r}\|_{L_{p,\lambda}(\mathbb{R}_+)} \|\chi_{H_r}\|_{L_{q',\lambda}(\mathbb{R}_+)} \lesssim |H_r|_\lambda^{\frac{1}{p}+\frac{1}{q'}-1-\frac{\beta}{\gamma}} \lesssim 1.$$

This, together with Lemma 4.3, gives $b \in \dot{\Lambda}_\beta(\mathbb{R}_+, G)$.

(3) $\Longrightarrow$ (1). We assume that (3.1) is true and verify $b \in \dot{\Lambda}_\beta(\mathbb{R}_+, G)$. For any fixed $H_0 = (0, r_0) \subset \mathbb{R}_+$, since for any $x \in H_0$,

$$\left| A_{chy}^\lambda b(chx) - b_{H_0}(chx) \right| \leq \frac{1}{|H_0|_\lambda} \int_{H_0} \left| A_{chy}^\lambda b(chx) - b_{H_0}(chx) \right| d\mu_\lambda(y),$$

then for all $x \in H_0$,

$$M_G^b \chi_{H_0}(chx) = \sup_{x \in H_r} \frac{1}{|H_r|_\lambda} \int_{H_r} \left| A_{chy}^\lambda b(chx) - b_{H_r}(chx) \right| A_{chy}^\lambda \chi_{H_0}(chx) d\mu_\lambda(y)$$

$$\gtrsim \frac{1}{|H_r|_\lambda} \int_{H_0} \left| A_{chy}^\lambda b(chx) - b_{H_r}(chx) \right| d\mu_\lambda(y)$$

$$\gtrsim \frac{1}{|H_0|_\lambda} \int_{H_0} \left| A_{chy}^\lambda b(chx) - b_{H_r}(chx) \right| d\mu_\lambda(y)$$

$$\geq \left|A^{\lambda}_{chy}b(chx) - b_{H_0}(chx)\right|.$$

This, together with (3.1), gives

$$\left|\left\{x \in H_0 : \left|A^{\lambda}_{chy}b(chx) - b_{H_0}(chx)\right| > \nu\right\}\right|_{\lambda}$$
$$\leq \left|\left\{x \in H_0 : M^b_G \chi_{H_0}(chx) > \nu\right\}\right|_{\lambda}$$
$$\leq C\left(\nu^{-1}\left\|\chi_{H_0}\right\|_{L_{1,\lambda}(\mathbb{R}_+)}\right)^{\frac{\gamma}{\gamma-\beta}} = C\left(\nu^{-1}\left|H_0\right|_{\lambda}\right)^{\frac{\gamma}{\gamma-\beta}}.$$

Let $t > 0$ be a constant to be determined later, then

$$\int_{H_0} \left|A^{\lambda}_{chy}b(chx) - b_{H_0}(chx)\right| d\mu_{\lambda}(y)$$
$$= \int_0^{\infty} \left|\left\{x \in H_0 : \left|A^{\lambda}_{chy}b(chx) - b_{H_0}(chx)\right| > \nu\right\}\right|_{\lambda} d\nu$$
$$= \int_0^{t} \left|\left\{x \in H_0 : \left|A^{\lambda}_{chy}b(chx) - b_{H_0}(chx)\right| > \nu\right\}\right|_{\lambda} d\nu$$
$$+ \int_t^{\infty} \left|\left\{x \in H_0 : \left|A^{\lambda}_{chy}b(chx) - b_{H_0}(chx)\right| > \nu\right\}\right|_{\lambda} d\nu$$
$$\leq t\left|H_0\right|_{\lambda} + C\int_t^{\infty}\left(\left(\nu^{-1}\left|H_0\right|_{\lambda}\right)^{\frac{\gamma}{\gamma-\beta}}\right) d\nu$$
$$\leq t\left|H_0\right|_{\lambda} + C\left|H_0\right|_{\lambda}^{\frac{\gamma}{\gamma-\beta}} \int_t^{\infty} \nu^{\frac{\gamma}{\gamma-\beta}} d\nu \lesssim \left(t\left|H_0\right|_{\lambda} + \left|H_0\right|_{\lambda}^{\frac{\gamma}{\gamma-\beta}} t^{1-\frac{\gamma}{\gamma-\beta}}\right).$$

Set $t = \left|H_0\right|_{\lambda}^{\frac{\beta}{\gamma}}$ in the above estimate, we have

$$\int_{H_0} \left|A^{\lambda}_{chy}b(chx) - b_{H_0}(chx)\right| d\mu_{\lambda}(y) \lesssim \left|H_0\right|_{\lambda}^{1+\frac{\beta}{\gamma}},$$

it follows from Lemma 4.3 that $b \in \dot{\Lambda}_{\beta}(\mathbb{R}_+, G)$, since H_0 is an arbitrary interval in $\mathbb{R}_+$.

The proof of Theorem 3.1 is completed, since (2) $\Longrightarrow$ (1) follows from (3) $\Longrightarrow$ (1).

Proof of Theorem 3.3. Assume $b \in \dot{\Lambda}_{\beta}(\mathbb{R}_+, G)$.

By (5.1) and Lemma 4.4 (i), we have

$$\left\|M^b_G f\right\|_{L_{q,\lambda,\nu}} \lesssim \|b\|_{\dot{\Lambda}_{\beta}(\mathbb{R}_+,G)} \left\|\mathfrak{M}^{\beta}_G f\right\|_{L_{q,\lambda,\nu}} \lesssim \|f\|_{\dot{\Lambda}_{\beta}(\mathbb{R}_+,G)} \|f\|_{L_{p,\lambda,\nu}}.$$

Conversely, if M^b_G is bounded from $L_{p,\lambda,\nu}(\mathbb{R}_+)$ to $L_{q,\lambda,\nu}(\mathbb{R}_+)$, then for any interval $H_r \subset \mathbb{R}_+$,

$$\frac{1}{\left|H_r\right|_{\lambda}^{\frac{\beta}{\gamma}}} \int_{H_r} \left(\frac{1}{\left|H_r\right|_{\lambda}} \int_{H_r} \left|A^{\lambda}_{chy}b(chx) - b_{H_r}(chx)\right|^q d\mu_{\lambda}(y)\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{\left|H_r\right|_{\lambda}^{\frac{\beta}{\gamma}}} \left(\frac{1}{\left|H_r\right|_{\lambda}} \int_{H_r} \left[\frac{1}{\left|H_r\right|_{\lambda}} \left|A^{\lambda}_{chy}b(chx) - b_{H_r}(chx)\right| A^{\lambda}_{chy}\chi_{H_r}(chx) d\mu_{\lambda}(y)\right]^q d\mu_{\lambda}(x)\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{\left|H_r\right|_{\lambda}^{\frac{\beta}{\gamma}}} \left(\frac{1}{\left|H_r\right|_{\lambda}} \int_{H_r} \left[M^b_G \chi_{H_r}(chx)\right]^q d\mu_{\lambda}(x)\right)^{\frac{1}{q}}$$

$$= \frac{1}{|H_r|_\lambda^{\frac{\beta}{\gamma}}} \left(\frac{|H_r|_\lambda^{\frac{\nu}{\gamma}}}{|H_r|_\lambda} \right)^{\frac{1}{q}} \left(\frac{1}{|H_r|_\lambda^{\frac{\nu}{\gamma}}} \int_{H_r} \left[M_G^b \chi_{H_r}(chx) \right]^q d\mu_\lambda(x) \right)^{\frac{1}{q}}$$

$$\leq |H_r|_\lambda^{-\frac{\beta}{\gamma} - \frac{1}{q} + \frac{\nu}{\gamma q}} \left\| M_G^b \chi_{H_r} \right\|_{L_{q,\lambda,\nu}(\mathbb{R}_+)} \lesssim |H_r|_\lambda^{-\frac{\beta}{\gamma} - \frac{1}{q} + \frac{\nu}{\gamma q}} \left\| H_r \right\|_{L_{p,\lambda,\nu}(\mathbb{R}_+)}$$

$$\lesssim |H_r|_\lambda^{-\frac{\beta}{\gamma} - \frac{1}{q} + \frac{\nu}{\gamma q} + \frac{\gamma - \nu}{p \gamma}} \lesssim 1,$$

where on the last step we have used $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{\gamma - \nu}$ and Lemma 4.5.

It follows from Lemma 4.3 that $b \in \dot{\Lambda}_\beta(\mathbb{R}_+, G)$ which completes the proof.

Proof of Theorem 3.4. Assume $b \in \dot{\Lambda}_\beta(\mathbb{R}_+, G)$. By (5.1) and Lemma 4.5, we have

$$\left\| M_G^b f \right\|_{L_{q,\lambda,\mu}} \lesssim \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}_+,G)} \left\| \mathfrak{M}_G^\beta f \right\|_{L_{q,\lambda,\mu}} \lesssim \|f\|_{\dot{\Lambda}_\beta(\mathbb{R}_+,G)} \|f\|_{L_{p,\lambda,\nu}}.$$

Conversely, M_G^b is bounded from $L_{p,\lambda,\nu}(\mathbb{R}_+)$ to $L_{q,\lambda,\mu}(\mathbb{R}_+)$, then for any interval $H_r \subset \mathbb{R}_+$,

$$\frac{1}{|H_r|_\lambda^{\frac{\beta}{\gamma}}} \left(\frac{1}{|H_r|_\lambda} \int_{H_r} \left| A_{chy}^\lambda b(chx) - b_{H_r}(chx) \right|^q sh^{2\lambda} d\mu_\lambda(y) \right)^{\frac{1}{q}}$$

$$\leq \frac{1}{|H_r|_\lambda^{\frac{\beta}{\gamma}}} \left(\frac{|H_r|_\lambda^{\frac{\nu}{\gamma}}}{|H_r|_\lambda} \right)^{\frac{1}{q}} \left(\frac{1}{|H_r|_\lambda^{\frac{\mu}{\gamma}}} \int_{H_r} \left[M_G^b \chi_{H_r}(chx) \right]^q sh^{2\lambda} d\mu_\lambda(x) \right)^{\frac{1}{q}}$$

$$\lesssim |H_r|_\lambda^{-\frac{\beta}{\gamma} - \frac{1}{q} + \frac{\mu}{\gamma q} + \frac{\gamma - \nu}{p\gamma}} \lesssim 1,$$

where on the last step we have used Lemma 4.5, $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{\gamma}$ and $\frac{\nu}{p} = \frac{\mu}{q}$.

This completes the proof.

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