Transactions of A. Razmadze
Mathematical Institute
Vol. **179** (2025), issue 1, 95–105

THE BOUNDS FOR THE ZEROS OF POLYNOMIALS

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Abstract. In this paper we introduce new bounds on the location of zeros of a polynomial with complex coefficients. The upper bound of our result is the best possible. In case where all coefficients, except the leading one, tend to zero, this reflects the fact that all zeros of the polynomial tend to the origin $z = 0$. In some cases, using this, we can get better bounds of polynomial zeros than those obtained by others known results, which is also illustrated by an example. In addition, an attempt has been made to investigate the result obtained by A. Joyal et al. (1967) and to generalize and refine this result for the class of lacunary type polynomials under consideration.

1. Introduction

Estimating the moduli of zeros of a polynomial has a long history. In recent years, many papers (see [2–4, 8, 15–18, 24, 26, 27]) and comprehensive books (see [19, 23]) have been published to determine the bounds of the zeros of polynomial with real or complex coefficients. Historically speaking, the study of the zero region of a polynomial began since the time when the geometric representation of complex numbers was introduced into mathematics. The first researches in this subject were Lagrange [18], Gauss [9] and Cauchy [5], respectively. Let

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

be a polynomial of degree n with real or complex coefficients. In 1816, Gauss proved the following result.

Theorem 1.1. *All zeros of $p(z)$ with real coefficients lie in the circular region*

$$|z| \le M,$$

where

$$M = \max_{0\le k\le n-1} \left\{ \left(2^{\frac{1}{2}} n \left| \frac{a_k}{a_n} \right| \right)^{\frac{1}{n-k}} \right\}.$$

In 1829, Cauchy [5] (see also [19]) produced more simple bound of polynomial zeros which can be stated as follows.

Theorem 1.2. *All zeros of $p(z)$ lie in an open circular region*

$$|z| < 1 + A,$$

where

$$A = \max_{0\le k\le n-1} \left| \frac{a_k}{a_n} \right|.$$

In 1971, Simeon Reich proposed and among others, O. P. Lossers [24] obtained the following

Theorem 1.3. *All zeros of $p(z)$ with $a_{n-1} = 0$ lie in the region*

$$|z| \le A + A^2 + \cdots + A^{n-1},$$

where

$$A = \left(\max_{0\le k\le n-2} \left| \frac{a_k}{a_n} \right| \right)^{\frac{1}{n}} > 1.$$

2020 *Mathematics Subject Classification.* 30C15, 30C10, 26C10.
Key words and phrases. Zeros; Cauchy bound; Circular region; Lacunary type polynomials.

Theorem 1.4. *All zeros of $p(z)$ with $a_{n-1}=0$ lie in the region*

$$|z|<\frac{1}{2}+\sqrt{\frac{1}{4}+A^n}.$$

Moreover, if $\left|\frac{a_j}{a_n}\right|=A^n$ *for some* $j;\ 0\le j\le n-2$, *and* $|a_i|\le\alpha=\sqrt{2}-1;\ \forall\, i\,(\ne j)\in\{0,1,2,\dots,n-2\}$, *then zeros of* $p(z)$ *lie in the disc*

$$|z|<\sqrt{1+A^n},$$

where

$$A=\left(\max_{0\le k\le n-2}\left|\frac{a_k}{a_n}\right|\right)^{\frac{1}{n}}>1.$$

Aziz [1] (see also [27, Theorem B]) generalized Theorem 1.3 by using a lacunary type polynomial [19, Ch. VIII, Sect. 34, pp. 156]

$$P(z)=z^n+a_\lambda z^\lambda+\dots+a_1z+a_0,\quad a_\lambda\ne 0,\ \ 0\le\lambda<n,$$

and obtained the following result.

Theorem 1.5. *All zeros of $P(z)$ lie in the disc*

$$|z|\le A+A^2+\dots+A^{\lambda+1},$$

where

$$A=\left(\max_{0\le k\le\lambda}\left|\frac{a_k}{a_n}\right|\right)^{\frac{1}{n}}.$$

In 2013, Aziz and Rather [2] obtained the following improvement of Theorem 1.3 and Theorem 1.5.

Theorem 1.6. *All zeros of $P(z)$ lie in the disc*

$$|z|\le\left(A^n+A^{n-1}+\dots+A^{n-\lambda}\right)^{\frac{1}{n-\lambda}},$$

where

$$A=\left(\max_{0\le k\le\lambda}\left|\frac{a_k}{a_n}\right|\right)^{\frac{1}{n}}.$$

Applying Holder's inequality, in 1967, Joyal et. al. [16] proved the following

Theorem 1.7. *For any real $p>1$ with $\frac{1}{p}+\frac{1}{q}=1$, all zeros of $p(z)$ lie in*

$$|z|\le k,$$

where $k\ge\max\left\{1,\frac{|a_{n-1}|}{|a_n|}\right\}$ *is a positive root of the equation*

$$\left(|z|-\frac{|a_{n-1}|}{|a_n|}\right)^q(|z|^q-1)-B_{n-1}^q=0;\quad B_{n-1}=\left(\sum_{j=0}^{n-2}\left|\frac{a_j}{a_n}\right|^p\right)^{\frac{1}{p}}.$$

In this paper, we obtain new explicit bounds of polynomial zeros which do not require any numerical method. Also, the second result produces an important corollary which gives a generalization of Theorem 1.7. Moreover, our third and forth results are the refinement and extension of Theorem 1.7. More precisely, we consider the polynomial of the form

$$T(z)=a_nz^n+a_{\lambda_1}z^{\lambda_1}+a_{\lambda_2}z^{\lambda_2}+\dots+a_{\lambda_j}z^{\lambda_j}+\dots+a_{\lambda_{k-2}}z^{\lambda_{k-2}}+a_{\lambda_{k-1}}z^{\lambda_{k-1}}+a_{\lambda_k}z^{\lambda_k},$$

where

$$\lambda_j\in\{0,1,2,\dots,n-1\}\ \text{ for }\ j=1,2,\dots,k$$

with

$$\lambda_1>\lambda_2>\dots>\lambda_j>\dots>\lambda_{k-2}>\lambda_{k-1}>\lambda_k\ge 0$$

and a_{λ_j}'s $j=1,2,\dots,k$ are the non-zero complex numbers.

In particular, when $\lambda_k=0$, $\lambda_1=n-1$ and λ_j's are consecutive, $T(z)$ reduces to $p(z)$. Also, for $\lambda_k=0$, $\lambda_1=\lambda$ and λ_j's consecutive, then $T(z)$ is converted to $P(z)$.

2. Main Results

In this section, we present first our new result.

Theorem 2.1. *All zeros of $T(z)$ lie in the closed circular region*

$$|z| \leq \max_{j=1}^{k} \left\{ M^{\frac{1}{n-\lambda_j}} \right\}.$$

Moreover, all zeros of $T(z)$, different from 0, lie in the closed annular region

$$\left(\max \left\{ \max_{j=1}^{k-1} \left\{ N^{\frac{1}{\lambda_j - \lambda_k}} \right\}, N^{\frac{1}{n-\lambda_k}} \right\} \right)^{-1} \leq |z| \leq \max_{j=1}^{k} \left\{ M^{\frac{1}{n-\lambda_j}} \right\},$$

where

$$M = \frac{\sum_{j=1}^{k} |a_{\lambda_j}|}{|a_n|} \quad \text{and} \quad N = \frac{|a_n| + \sum_{j=1}^{k-1} |a_{\lambda_j}|}{|a_{\lambda_k}|},$$

respectively.

Remark 2.1. As $a_{\lambda_j} \to 0$; $j = 1, 2, \ldots, k$, all zeros of $T(z)$ approach the origin $z = 0$.

Remark 2.2. The upper bound in Theorem 2.1 is attained by the polynomial

$$p(z) = -nz^n + z^{n-1} + z^{n-2} + \cdots + z + 1.$$

Remark 2.3. In some cases, our result in Theorem 2.1 gives better bound as compared to the bounds of the other known results. To illustrate this, we consider the polynomial

$$p(z) = z^{12} + 24z^7 + 18z^5 + 8z^3 + 20.$$

All zeros of $p(z)$ lie in the following regions:

(i) $|z| \leq 3.32645$, by Theorem 1.1,
(ii) $|z| \leq 25$, by Theorem 1.2,
(iii) $|z| \leq 74.85262$, by Theorem 1.3,
(iv) $|z| < 5.42443$, by Theorem 1.4,
(v) $|z| \leq 31.4624$, by Theorem 1.5,
(vi) $|z| \leq 2.46361$, by Theorem 1.6,
(vii) $|z| < 2.37483$, by Rahman [22, Theorem 1],
(viii) $|z| < 24.99999$, Boese and Luther [4, Theorem 1],
(ix) $|z| < 24.99999$, by Datt and Govil [7, Theorem 2],
(x) $|z| \leq 24.04159$, by Jain and Tewary [15, Theorem],
(xi) $|z| \leq 31.46236$, by Zargar [27, Theorem 1 for $t = 1$],
(xii) $|z| \leq 9.62355$, by Zargar [27, Theorem 2],
(xiii) $|z| \leq 36.94591$, by Fujiwara [8, p. 83],
(xiv) $|z| \leq 5.42443$, by Joyal, Labelle and Rahman [16, Theorem 1],
(xv) $|z| \leq 23.470588$, by Jain [14, Theorem 1],
(xvi) $|z| \leq 3.39938$, by Lagrange [18] (see also [3, Theorem 1.1]),
(xvii) $|z| \leq 3.22261$, by Batra, Mignotte and Stefanescu [3, Theorem 3.1],
(xviii) $|z| < 5.42443$, by Mohammad [20, Theorem 2],
(xix) $|z| \leq 127.93748$, by Mohammad [21, Theorem 1 for $p = q = 2$],
(xx) $|z| \leq 5.42443$, by Jain [11, Theorem 1],
(xxi) $|z| < 2.51275$, by Jain [12, Theorem 1],

(xxii) $|z| < 3.41095$, by Jain [13, Theorem 2],

(xxiii) $|z| \le 3.25928$, by Sun and Hsieh [25, Theorem 1],

(xxiv) $|z| < 5.94287$, by Walsh [26, p. 286],

(xxv) $|z| < 3.38138$, by Jain [10, Therem for $p = q = 2$],

(xxvi) $0.454545 \le |z| \le 5.42443$, by Das [6, Theorem 4],

(xxvii) $0.33333 \le |z| \le 3.77635$, by Kojima [17, eq. (8), p. 121],

(xxviii) $0.73196 \le |z| \le 2.33894$, by Theorem 2.1.

If $\lambda_1 = n-1$ and λ_j's are consecutive, then $k = n$,
i.e.,

$$a_{\lambda_1} = a_{n-1},\ \ a_{\lambda_2} = a_{n-2},\ \ldots,\ a_{\lambda_j} = a_{n-j},\ \ldots,\ a_{\lambda_{n-1}} = a_1,\ \ a_{\lambda_n} = a_0.$$

So, $T(z)$ reduces to $p(z)$ and we can produce the following

Corollary 2.1. *All zeros of $p(z)$ with non-zero coefficients lie in the closed annular region*

$$\left(\max_{j=1}^{n}\left\{N^{\frac{1}{j}}\right\}\right)^{-1} \le |z| \le \max_{j=1}^{n}\left\{M^{\frac{1}{j}}\right\},$$

where

$$M = \frac{\sum\limits_{j=0}^{n-1}|a_j|}{|a_n|} \ \ \text{and} \ \ N = \frac{\sum\limits_{j=1}^{n}|a_j|}{|a_0|},$$

respectively.

Theorem 2.2. *For any real $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, all zeros of $T(z)$ lie in*

$$|z| \le r_0,$$

where $r_0 \ge \max\left\{1, \left|\frac{a_{\lambda_1}}{a_n}\right|^{\frac{1}{n-\lambda_1}}\right\}$ is a positive root of the equation

$$r^{q(\lambda_1-\lambda_2-1)}\left(r^{n-\lambda_1} - \frac{|a_{\lambda_1}|}{|a_n|}\right)^q (r^q - 1) - B^q_{\lambda_1,k} = 0;\ \ B_{\lambda_1,k} = \left(\sum_{j=2}^{k}\left|\frac{a_{\lambda_j}}{a_n}\right|^p\right)^{\frac{1}{p}}.$$

Taking $\lambda_1 = \lambda\,(< n)$ and λ_j's are consecutive, then $k = \lambda + 1$,
i.e.,

$$a_{\lambda_1} = a_\lambda,\ \ a_{\lambda_2} = a_{\lambda-1},\ \ldots\ , a_{\lambda_j} = a_{\lambda-j+1},\ \ldots,\ a_{\lambda_{\lambda+1}=}a_0$$

and $T(z)$ reduces to $P(z)$. Also,

$$B_{\lambda,\lambda+1} = \left(\sum_{j=2}^{\lambda+1}\left|\frac{a_{\lambda_j}}{a_n}\right|^p\right)^{\frac{1}{p}} = \left(\sum_{j=0}^{\lambda-1}\left|\frac{a_j}{a_n}\right|^p\right)^{\frac{1}{p}} = B_\lambda \text{ (say)}$$

and we obtain the following

Corollary 2.2. *For any real $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, all zeros of $P(z)$ lie in*

$$|z| \le r_0',$$

where $r_0' \ge \max\left\{1, \left|\frac{a_\lambda}{a_n}\right|^{\frac{1}{n-\lambda}}\right\}$ is a positive root of the equation

$$\left(r^{n-\lambda} - \frac{|a_\lambda|}{|a_n|}\right)^q (r^q - 1) - B^q_\lambda = 0;\ \ \ B_\lambda = \left(\sum_{j=0}^{\lambda-1}\left|\frac{a_j}{a_n}\right|^p\right)^{\frac{1}{p}}.$$

Observing that for $\lambda = n-1, B_\lambda = B_{n-1}$. In this case, Corollary 2.2 reduces to Theorem 1.7. So, Corollary 2.2 is a generalization of Theorem 1.7. The next result is a refinement of Theorem 1.7 as follows:

Theorem 2.3. *For any non-zero real or complex t and a real $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, all zeros of $p(z)$ lie in*

$$|z| < r_0'',$$

where $r_0'' \geq \max\left\{1, \frac{|t-a_{n-1}|}{|a_n|}\right\}$ is a positive root of the equation

$$\left(r - \frac{|t-a_{n-1}|}{|a_n|}\right)^q (r^q - 1) - Q_t^q = 0; \quad Q_t = \left(\sum_{j=0}^{n-1} \left|\frac{ta_j - a_n a_{j-1}}{a_n^2}\right|^p\right)^{\frac{1}{p}}, \quad a_{-1} = 0.$$

Putting $t = a_{n-1}$ in Theorem 2.3, we have the following

Corollary 2.3. *For any real $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, all zeros of $p(z)$ lie in*

$$|z| < \left(\frac{1 + \sqrt{1 + 4Q_{a_{n-1}}^q}}{2}\right)^{\frac{1}{q}},$$

where

$$Q_{a_{n-1}} = \left(\sum_{j=0}^{n-1} \left|\frac{a_{n-1}a_j - a_n a_{j-1}}{a_n^2}\right|^p\right)^{\frac{1}{p}}, \quad a_{-1} = 0.$$

From Corollary 2.3, we can easily obtain the following corollary on the polynomial with non-zero real coefficients having some restriction.

Corollary 2.4. *For any real $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, all zeros of $p(z)$ with non-zero real coefficients satisfying the condition*

$$\frac{a_n}{a_{n-1}} = \frac{a_{n-1}}{a_{n-2}} = \cdots = \frac{a_2}{a_1} = \frac{a_1}{a_0} = k,$$

lie in

$$|z| < \left(\frac{1 + \sqrt{1 + \frac{4}{|k|^{q(n+1)}}}}{2}\right)^{\frac{1}{q}}.$$

Theorem 2.4. *For any real $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and for some real or complex t, all zeros of $P(z)$ lie in*

$$|z| < r_0''',$$

where $r_0''' > \max\left\{1, \frac{|t|}{|a_n|}\right\}$ is a positive root of the equation

$$r^{q(n-\lambda-2)} \left(r - \frac{|t|}{|a_n|}\right)^q (r^q - 1) - \Omega_{\lambda,t}^q = 0; \quad \Omega_{\lambda,t} = \left(\sum_{j=0}^{\lambda+1} \left|\frac{a_j t - a_{j-1} a_n}{a_n^2}\right|^p\right)^{\frac{1}{p}}$$

with $a_{\lambda+1} = a_{-1} = 0$.

3. Proof of Theorems

Proof of Theorem 2.1. For some $|z| > 0$,

$$|T(z)| \geq |a_n| |z|^n - \left(|a_{\lambda_1}| |z|^{\lambda_1} + |a_{\lambda_2}| |z|^{\lambda_2} + \cdots + |a_{\lambda_j}| |z|^{\lambda_j} + \cdots + |a_{\lambda_k}| |z|^{\lambda_k}\right).$$

Now, we express $|a_n|$, a sum of k numbers non-zero positive real in the form

$$|a_n| = \mu_{\lambda_1} + \mu_{\lambda_2} + \cdots + \mu_{\lambda_j} + \cdots + \mu_{\lambda_{k-1}} + \mu_{\lambda_k}$$

satisfying the property

$$\frac{\mu_{\lambda_1}}{|a_{\lambda_1}|} = \frac{\mu_{\lambda_2}}{|a_{\lambda_2}|} = \cdots = \frac{\mu_{\lambda_j}}{|a_{\lambda_j}|} = \cdots = \frac{\mu_{\lambda_{k-1}}}{|a_{\lambda_{k-1}}|} = \frac{\mu_{\lambda_k}}{|a_{\lambda_k}|}.$$

From the above we see that each ratio $\frac{\mu_{\lambda_j}}{|a_{\lambda_j}|}$; $j = 1, 2, \ldots, k$ is equal to

$$\frac{\sum_{j=1}^{k} \mu_{\lambda_j}}{\sum_{j=1}^{k} |a_{\lambda_j}|} = \frac{|a_n|}{\sum_{j=1}^{k} |a_{\lambda_j}|} = \frac{1}{M} \text{ (say).}$$

Now,

$$\begin{aligned} |T(z)| &\geq \left(\sum_{j=1}^{k} \mu_{\lambda_j} \right) |z|^n - \sum_{j=1}^{k} |a_{\lambda_j}| \, |z|^{\lambda_j} \\ &= \sum_{j=1}^{k} \left(\mu_{\lambda_j} |z|^n - |a_{\lambda_j}| \, |z|^{\lambda_j} \right) \\ &= \sum_{j=1}^{k} \mu_{\lambda_j} \left(|z|^{n-\lambda_j} - \frac{|a_{\lambda_j}|}{\mu_{\lambda_j}} \right) |z|^{\lambda_j} \\ &= \sum_{j=1}^{k} \mu_{\lambda_j} \left(|z|^{n-\lambda_j} - M \right) |z|^{\lambda_j} . \end{aligned}$$

This gives $|T(z)| > 0$ if $|z| \geq M^{\frac{1}{n-\lambda_j}}$ for all $j = 1, 2, \ldots, k$ with at least $|z| > M^{\frac{1}{n-\lambda_j}}$ for some $j \in \{1, 2, \ldots, k\}$, i.e., $|T(z)| > 0$ if

$$|z| > \max_{j=1}^{k} \left\{ M^{\frac{1}{n-\lambda_j}} \right\}.$$

So, all zeros of $T(z)$ lie in the closed circular region

$$|z| \leq \max_{j=1}^{k} \left\{ M^{\frac{1}{n-\lambda_j}} \right\}.$$

In the later part of our paper, we will see that 0 is the zero of $T(z)$ of multiplicity λ_k and the remaining $(n - \lambda_k)$ zeros of $T(z)$ different from 0 coincide with the zeros of

$$T'(z) = a_n z^{n-\lambda_k} + a_{\lambda_1} z^{\lambda_1 - \lambda_k} + \cdots + a_{\lambda_j} z^{\lambda_j - \lambda_k} + \cdots + a_{\lambda_{k-1}} z^{\lambda_{k-1} - \lambda_k} + a_{\lambda_k}.$$

We introduce the polynomial $Q(z)$ defined by

$$\begin{aligned} Q(z) &= z^{n-\lambda_k} T'\left(\frac{1}{z}\right) \\ &= z^{n-\lambda_k} \left(\frac{a_n}{z^{n-\lambda_k}} + \frac{a_{\lambda_1}}{z^{\lambda_1 - \lambda_k}} + \cdots + \frac{a_{\lambda_j}}{z^{\lambda_j - \lambda_k}} + \cdots + \frac{a_{\lambda_{k-1}}}{z^{\lambda_{k-1} - \lambda_k}} + a_{\lambda_k} \right) \\ &= a_{\lambda_k} z^{n-\lambda_k} + a_{\lambda_{k-1}} z^{n-\lambda_{k-1}} + \cdots + a_{\lambda_j} z^{n-\lambda_j} + \cdots + a_{\lambda_1} z^{n-\lambda_1} + a_n. \end{aligned}$$

Now, we are looking for k numbers of non-zero positive real $\mu'_{\lambda_{k-1}}, \mu'_{\lambda_{k-2}}, \ldots, \mu'_{\lambda_j}, \ldots, \mu'_{\lambda_2}, \mu'_{\lambda_1}, \mu'_n$ satisfying the conditions

$$\frac{\mu'_{\lambda_{k-1}}}{|a_{\lambda_{k-1}}|} = \frac{\mu'_{\lambda_{k-2}}}{|a_{\lambda_{k-2}}|} = \cdots = \frac{\mu'_{\lambda_j}}{|a_{\lambda_j}|} = \cdots = \frac{\mu'_{\lambda_2}}{|a_{\lambda_2}|} = \frac{\mu'_{\lambda_1}}{|a_{\lambda_1}|} = \frac{\mu'_n}{|a_n|}$$

and

$$\mu'_{\lambda_{k-1}} + \mu'_{\lambda_{k-2}} + \cdots + \mu'_{\lambda_j} + \cdots + \mu'_{\lambda_2} + \mu'_{\lambda_1} + \mu'_n = |a_{\lambda_k}|,$$

respectively. Clearly, each ration of the above is equal to

$$\frac{\sum_{j=1}^{k-1} \mu'_{\lambda_j} + \mu'_n}{\sum_{j=1}^{k-1} |a_{\lambda_j}| + |a_n|} = \frac{|a_{\lambda_k}|}{|a_n| + \sum_{j=1}^{k-1} |a_{\lambda_j}|} = \frac{1}{N} \text{ (say).}$$

Again, for some $|z| > 0$,

$$\begin{aligned} |Q(z)| &\geq |a_{\lambda_k}| \, |z|^{n-\lambda_k} - \left(\sum_{j=1}^{k-1} \left|a_{\lambda_j}\right| |z|^{n-\lambda_j} \right) - |a_n| \\ &= \left(\sum_{j=1}^{k-1} \mu'_{\lambda_j} + \mu'_n \right) |z|^{n-\lambda_k} - \left(\sum_{j=1}^{k-1} \left|a_{\lambda_j}\right| |z|^{n-\lambda_j} \right) - |a_n| \\ &= \sum_{j=1}^{k-1} \left(\mu'_{\lambda_j} |z|^{n-\lambda_k} - \left|a_{\lambda_j}\right| |z|^{n-\lambda_j} \right) + \left(\mu'_n |z|^{n-\lambda_k} - |a_n| \right) \\ &= \sum_{j=1}^{k-1} \left(|z|^{\lambda_j-\lambda_k} - \frac{\left|a_{\lambda_j}\right|}{\mu'_{\lambda_j}} \right) \mu'_{\lambda_j} |z|^{n-\lambda_j} + \mu'_n \left(|z|^{n-\lambda_k} - \frac{|a_n|}{\mu'_n} \right) \\ &= \sum_{j=1}^{k-1} \left(|z|^{\lambda_j-\lambda_k} - N \right) \mu'_{\lambda_j} |z|^{n-\lambda_j} + \mu'_n \left(|z|^{n-\lambda_k} - N \right) \end{aligned}$$

which shows that $|Q(z)| > 0$ if

$$|z| > \max \left\{ \max_{j=1}^{k-1} \left\{ N^{\frac{1}{\lambda_j - \lambda_k}} \right\}, N^{\frac{1}{n-\lambda_k}} \right\}.$$

So, all zeros of $Q(z)$ lie in the region

$$|z| \leq \max \left\{ \max_{j=1}^{k-1} \left\{ N^{\frac{1}{\lambda_j - \lambda_k}} \right\}, N^{\frac{1}{n-\lambda_k}} \right\}.$$

Let $z_0 \, (\neq 0)$ be any zero of $T(z)$. So, it is a zero of $T'(z)$ and, consequently, $\frac{1}{z_0}$ is a zero of $Q(z)$. Therefore,

$$\left| \frac{1}{z_0} \right| \leq \max \left\{ \max_{j=1}^{k-1} \left\{ N^{\frac{1}{\lambda_j - \lambda_k}} \right\}, N^{\frac{1}{n-\lambda_k}} \right\},$$

i.e.,

$$|z_0| \geq \left(\max \left\{ \max_{j=1}^{k-1} \left\{ N^{\frac{1}{\lambda_j - \lambda_k}} \right\}, N^{\frac{1}{n-\lambda_k}} \right\} \right)^{-1}$$

and this leads us to the desired result. □

Proof of Theorem 2.2. Clearly, for some $|z| > 0$,

$$\begin{aligned} |T(z)| &\geq |a_n| \, |z|^n - |a_{\lambda_1}| \, |z|^{\lambda_1} - \sum_{j=2}^{k} \left|a_{\lambda_j}\right| |z|^{\lambda_j} \\ &= |a_n| \, |z|^{\lambda_1} \left[\left(|z|^{n-\lambda_1} - \frac{|a_{\lambda_1}|}{|a_n|} \right) - \sum_{j=2}^{k} \frac{\left|a_{\lambda_j}\right|}{|a_n|} \frac{1}{|z|^{\lambda_1-\lambda_j}} \right]. \end{aligned}$$

Applying Holder's inequality, we have

$$|T(z)| \geq |a_n| \, |z|^{\lambda_1} \left[\left(|z|^{n-\lambda_1} - \left| \frac{a_{\lambda_1}}{a_n} \right| \right) - \left(\sum_{j=2}^{k} \left| \frac{a_{\lambda_j}}{a_n} \right|^p \right)^{\frac{1}{p}} \left(\sum_{j=2}^{k} \left(\frac{1}{|z|^{\lambda_1-\lambda_j}} \right)^q \right)^{\frac{1}{q}} \right]$$

for some positive reals p, q with $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

For $|z| > 1$, $\sum_{j=2}^{k} \left(\frac{1}{|z|^{\lambda_1-\lambda_j}} \right)^q$

$$= \sum_{j=2}^{k} \left(\frac{1}{|z|^q} \right)^{\lambda_1-\lambda_j}$$

$$= \left(\frac{1}{|z|^q}\right)^{\lambda_1-\lambda_2} + \left(\frac{1}{|z|^q}\right)^{\lambda_1-\lambda_3} + \left(\frac{1}{|z|^q}\right)^{\lambda_1-\lambda_4} + \cdots + \left(\frac{1}{|z|^q}\right)^{\lambda_1-\lambda_k}$$

$$= \left(\frac{1}{|z|^q}\right)^{\lambda_1-\lambda_2} \left[1 + \left(\frac{1}{|z|^q}\right)^{\lambda_2-\lambda_3} + \left(\frac{1}{|z|^q}\right)^{\lambda_2-\lambda_4} + \cdots + \left(\frac{1}{|z|^q}\right)^{\lambda_2-\lambda_k}\right]$$

$$\leq \frac{1}{\left(|z|^q\right)^{\lambda_1-\lambda_2}} \left[\frac{1 - \left(\frac{1}{|z|^q}\right)^{\lambda_2-\lambda_k+1}}{1 - \frac{1}{|z|^q}}\right]$$

$$= \frac{1}{\left(|z|^q\right)^{\lambda_1-\lambda_2}} \left[\frac{|z|^q \left\{\left(|z|^q\right)^{\lambda_2-\lambda_k+1} - 1\right\}}{\left(|z|^q\right)^{\lambda_2-\lambda_k+1} \left(|z|^q - 1\right)}\right]$$

$$= \frac{1}{\left(|z|^q\right)^{\lambda_1-\lambda_k}} \left[\frac{\left(|z|^q\right)^{\lambda_2-\lambda_k+1} - 1}{|z|^q - 1}\right]$$

$$< \frac{1}{\left(|z|^q\right)^{\lambda_1-\lambda_k}} \left[\frac{\left(|z|^q\right)^{\lambda_2-\lambda_k+1}}{|z|^q - 1}\right]$$

$$= \frac{1}{\left(|z|^q\right)^{\lambda_1-\lambda_2-1} \left(|z|^q - 1\right)}$$

which implies

$$|T(z)| > |a_n| \, |z|^{\lambda_1} \left[\left(|z|^{n-\lambda_1} - \left|\frac{a_{\lambda_1}}{a_n}\right|\right) - \frac{B_{\lambda_1,k}}{|z|^{\lambda_1-\lambda_2-1} \left(|z|^q - 1\right)^{\frac{1}{q}}}\right].$$

So, $|T(z)| > 0$ if

$$|z|^{q(\lambda_1-\lambda_2-1)} \left(|z|^{n-\lambda_1} - \left|\frac{a_{\lambda_1}}{a_n}\right|\right)^q \left(|z|^q - 1\right) - B^q_{\lambda_1,k} \geq 0 \text{ for } \ |z| > 1.$$

We introduce a function

$$f(r) = r^{q(\lambda_1-\lambda_2-1)} \left(r^{n-\lambda_1} - \left|\frac{a_{\lambda_1}}{a_n}\right|\right)^q \left(r^q - 1\right) - B^q_{\lambda_1,k}.$$

Let $r_0 > \max\left\{1, \left|\frac{a_{\lambda_1}}{a_n}\right|^{\frac{1}{n-\lambda_1}}\right\}$ be a positive root of the equation $f(r) = 0$, then $|T(z)| > 0$ if

$$|z| \geq r_0.$$

In particular, when $a_{\lambda_j} = 0$, $j = 2, 3, \ldots, k$,

$$|T(z)| \geq |a_n| \, |z|^{\lambda_1} \left(|z|^{n-\lambda_1} - \frac{|a_{\lambda_1}|}{|a_n|}\right) > 0$$

if

$$|z| > \left(\frac{|a_{\lambda_1}|}{|a_n|}\right)^{\frac{1}{n-\lambda_1}}.$$

Combining all the possibilities, we obtain the desired result. □

Proof of Theorem 2.3. For some real or complex t, we construct the polynomial $Q(z)$ defined by

$$Q(z) = (t - a_n z) \, p(z)$$

which can also be written as

$$Q(z) = -a_n^2 z^{n+1} + (ta_n - a_{n-1}a_n) z^n + \sum_{j=0}^{n-1} (ta_j - a_{j-1}a_n) z^j$$

with $a_{-1} = 0$. Now, for $|z| > 1$, we have

$$\begin{aligned} |Q(z)| &\geq |a_n|^2 |z|^{n+1} - |a_n| |t - a_{n-1}| |z|^n - \sum_{j=0}^{n-1} \left| \frac{ta_j - a_{j-1}a_n}{a_n^2} \right| |z|^j \\ &= |a_n|^2 |z|^n \left[\left(|z| - \frac{|t - a_{n-1}|}{|a_n|} \right) - \sum_{j=0}^{n-1} \left| \frac{ta_j - a_{j-1}a_n}{a_n^2} \right| \frac{1}{|z|^{n-j}} \right]. \end{aligned}$$

Applying Holder's inequality, we have

$$|Q(z)| \geq |a_n|^2 |z|^n \left[\left(|z| - \frac{|t - a_{n-1}|}{|a_n|} \right) - \left(\sum_{j=0}^{n-1} \left| \frac{ta_j - a_{j-1}a_n}{a_n^2} \right|^p \right)^{\frac{1}{p}} \left(\sum_{j=0}^{n-1} \left(\frac{1}{|z|^{n-j}} \right)^q \right)^{\frac{1}{q}} \right].$$

Since for $|z| > 1$,

$$\begin{aligned} \sum_{j=0}^{n-1} \left(\frac{1}{|z|^{n-j}} \right)^q &= \sum_{j=0}^{n-1} \left(\frac{1}{|z|^q} \right)^{n-j} \\ &= \frac{1}{|z|^q} \sum_{j=0}^{n-1} \left(\frac{1}{|z|^q} \right)^j \\ &= \frac{1}{|z|^{qn}} . \frac{|z|^{qn} - 1}{|z|^q - 1} \\ &< \frac{1}{|z|^{qn}} . \frac{|z|^{qn}}{|z|^q - 1} \\ &= \frac{1}{|z|^q - 1} \end{aligned}$$

which implies that

$$\begin{aligned} |Q(z)| &> |a_n|^2 |z|^n \left[\left(|z| - \frac{|t - a_{n-1}|}{|a_n|} \right) - \left(\sum_{j=0}^{n-1} \left| \frac{ta_j - a_{j-1}a_n}{a_n^2} \right|^p \right)^{\frac{1}{p}} \cdot \frac{1}{(|z|^q - 1)^{\frac{1}{q}}} \right] \\ &= |a_n|^2 |z|^n \left[\left(|z| - \frac{|t - a_{n-1}|}{|a_n|} \right) - \frac{Q_t}{(|z|^q - 1)^{\frac{1}{q}}} \right]. \end{aligned}$$

So, $|Q(z)| > 0$ if

$$(|z|^q - 1) \left(|z| - \frac{|t - a_{n-1}|}{|a_n|} \right)^q - Q_t^q \geq 0 \;\text{ provided }\; |z| > 1.$$

Now, we consider a function $g(r)$ defined by

$$g(r) = (r^q - 1) \left(r - \frac{|t - a_{n-1}|}{|a_n|} \right)^q - Q_t^q.$$

Let $r_0'' \geq \max\left\{1, \frac{|t - a_{n-1}|}{|a_n|}\right\}$ be a positive root of the equation $g(r) = 0$. Then

$$|Q(z)| > 0, \;\text{ if }\; |z| \geq r_0''$$

and this leads us to the desired result. □

Proof of Theorm 2.4. For some real or complex t, we consider the polynomial $F(z)$ defined by

$$\begin{aligned} F(z) &= (t - a_n z) P(z) \\ &= -a_n^2 z^{n+1} + t a_n z^n + \sum_{j=0}^{\lambda+1} (t a_j - a_{j-1} a_n) z^j \end{aligned}$$

with $a_{\lambda+1} = a_{-1} = 0$.
Clearly, for some $|z| > 1$,

$$|F(z)| \geq |a_n|^2 |z|^n \left[\left(|z| - \frac{|t|}{|a_n|} \right) - \sum_{j=0}^{\lambda+1} \left| \frac{t a_j - a_{j-1} a_n}{a_n^2} \right| \frac{1}{|z|^{n-j}} \right].$$

Applying Holder's inequality, we have

$$\begin{aligned} |F(z)| &\geq |a_n|^2 |z|^n \left[\left(|z| - \frac{|t|}{|a_n|} \right) - \left(\sum_{j=0}^{\lambda+1} \left| \frac{t a_j - a_{j-1}}{a_n} \right|^p \right)^{\frac{1}{p}} \left(\sum_{j=0}^{\lambda+1} \left(\frac{1}{|z|^{n-j}} \right)^q \right)^{\frac{1}{q}} \right] \\ &= |a_n|^2 |z|^n \left[\left(|z| - \frac{|t|}{|a_n|} \right) - \Omega_{\lambda,t} \left(\sum_{j=n-(\lambda+1)}^{n} \left(\frac{1}{|z|^q} \right)^j \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Now, for $|z| > 1$,

$$\begin{aligned} \sum_{j=n-(\lambda+1)}^{n} \left(\frac{1}{|z|^q} \right)^j &= \frac{1}{|z|^{q(n-\lambda-1)}} \left[1 + \frac{1}{|z|^q} + \left(\frac{1}{|z|^q} \right)^2 + \cdots + \left(\frac{1}{|z|^q} \right)^{\lambda+1} \right] \\ &= \frac{1}{|z|^{q(n-\lambda-1)}} \left[\frac{1 - \left(\frac{1}{|z|^q} \right)^{\lambda+2}}{1 - \frac{1}{|z|^q}} \right] \\ &= \frac{1}{|z|^{q(n-\lambda-1)}} \cdot \frac{1}{|z|^{q(\lambda+1)}} \left[\frac{|z|^{q(\lambda+2)} - 1}{|z|^q - 1} \right] \\ &< \frac{1}{|z|^{q(n-\lambda-2)} (|z|^q - 1)}. \end{aligned}$$

Using the above, we get

$$|F(z)| > |a_n|^2 |z|^n \left[\left(|z| - \frac{|t|}{|a_n|} \right) - \frac{\Omega_{\lambda,t}}{|z|^{(n-\lambda-2)} (|z|^q - 1)^{\frac{1}{q}}} \right].$$

So, $|F(z)| > 0$ if

$$|z|^{q(n-\lambda-2)} \left(|z|^q - 1 \right) \left(|z| - \frac{|t|}{|a_n|} \right)^q - \Omega_{\lambda,t}^q \geq 0 \text{ provided } |z| > 1.$$

Now, we construct a function $h(r)$ defined by

$$h(r) = r^{q(n-\lambda-2)} (r^q - 1) \left(r - \frac{|t|}{|a_n|} \right)^q - \Omega_{\lambda,t}^q.$$

Let $r_0''' > \max\left\{1, \frac{|t|}{|a_n|}\right\}$ be a positive root of the equation $h(r) = 0$. Then $|F(z)| > 0$ if

$$|F(z)| > 0 \text{ if } |z| \geq r_0'''$$

and this completes the proof. □

References

1. A. Aziz, *Studies in Zeros and Extremal Properties of Polynomials*. Ph. D. Thesis, submitted to Kashmir University, India, 1981.
2. A. Aziz, N. A. Rather, Bounds for the zeros of a class of lacunary-type polynomials. *J. Math. Inequal.* **7** (2013), no. 3, 445–452.
3. P. Batra, M. Mignotte, D. Stefănescu, Improvements of Lagrange's bound for polynomial roots. *J. Symbolic Comput.* **82** (2017), 19–25.
4. F. G. Boese, W. J. Luther, A note on a classical bound for the moduli of all zeros of a polynomial. *IEEE Trans. Automat. Control* **34** (1989), no. 9, 998–1001.
5. A. L. Cauchy, *Exercise de Mathématiques*. IV. Anné de Bure Fréres, Paris, 1829.
6. S. Das, Annular bounds for the zeros of polynomials. *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* **63(111**) (2020), no. 4, 335–347.
7. B. Datt, N. K. Govil, On the location of the zeros of a polynomial. *J. Approx. Theory* **24** (1978), no. 1, 78–82.
8. M. Fujiwara, Über die Wurzeln der algebraischen Gleichungen. *Tohoku Math. J.* **8** (1915), 78–85.
9. K. F. Gauss, Beiträge zur Theorie der algebraischen Gleichungen. *Abh. Ges. Wiss. Gottingen* **4** (1950), p. 3.
10. V. K. Jain, On the zeros of a polynomial. *Publ. Inst. Math. (Beograd) (N.S.)* **41(55)** (1987), 75–77.
11. V. K. Jain, On Cauchy's bound for zeros of a polynomial. *Approx. Theory Appl.* **6** (1990), no. 4, 18–24.
12. V. K. Jain, On Cauchy's bound for zeros of a polynomial. *Turkish J. Math.* **30** (2006), no. 1, 95–100.
13. V. K. Jain, On Cauchy's type bounds for zeros of a polynomial. *Ann. Univ. Mariae Curie-Skłodowska Sect. A* **49** (1995), 109–116.
14. V. K. Jain, On the location of zeros of a polynomial. *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* **54(102)** (2011), no. 4, 337–352.
15. V. K. Jain, V. Tewary, A refinement of Cauchy's bound for the moduli of zeros of a polynomial. *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* **61(109)** (2018), no. 2, 173–185.
16. A. Joyal, G. Labelle, Q. I. Rahman, On the location of zeros of polynomials. *Canad. Math. Bull.* **10** (1967), 53–63.
17. T. Kojima, On the limits of the roots of an algebraic equation. *Tohoku Math. J.* **11** (1917), 119–127.
18. J. L. Lagrange, Sur la résolution des équations numériques. *Mémoires de l'Académie royale des Sciences et Belles-lettres de Berlin* **XXIII** (1769), 539–578.
19. M. Marden, *The Geometry of Polynomials*. Amer. Math. Soc. Math. Surveys 3, Amer. Math. Soc. Providence R.I., 1966.
20. Q. G. Mohammad, Mathematical Notes: On the Zeros of Polynomials. *Amer. Math. Monthly* **69** (1962), no. 9, 901–904.
21. Q. G. Mohammad, Location of the zeros of polynomials. *Amer. Math. Monthly* **74** (1967), 290–292.
22. Q. I. Rahman, A bound for the moduli of the zeros of polynomials. *Canad. Math. Bull.* **13** (1970), 541–542.
23. Q. I. Rahman, G. Schmeisser, *Analytic Theory of Polynomials*. London Mathematical Society Monographs. New Series, 26. The Clarendon Press, Oxford University Press, Oxford, 2002.
24. S. Reich, O. P. Lossers, Problems and Solutions: Solutions of Advanced Problems: 5737. *Amer. Math. Monthly* **78** (1971), no. 6, 681–683.
25. Y. J. Sun, J. G. Hsieh, A note on the circular bound of polynomial zeros. *IEEE Trans. Circuits Systems I Fund. Theory Appl.* **43** (1996), no. 6, 476–478.
26. J. L. Walsh, An inequality for the roots of an algebraic equation. *Ann. of Math. (2)* **25** (1924), no. 3, 285–286.
27. B. A. Zargar, Zeros of Lacunary type of polynomials. *Mathematical Journal of Interdisciplinary Sciences* **5** (2017), no. 2, 93–99.

(Received 30.11.2022)

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