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# SOLUTION OF THE BOUNDARY VALUE PROBLEM OF THE COUPLED THEORY OF ELASTICITY FOR A CIRCULAR RING WITH DOUBLE-POROSITY

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**Abstract.** This paper concerns the coupled linear static theory of elasticity for materials with double porosity. In these materials, a coupled phenomenon of the extended Darcy law and the volume fraction concepts, is proposed. A two-dimensional system of equations of the plane deformation is written in a complex form and its general solution is represented by means of three analytic functions of a complex variable and three solutions of the Helmholtz equations. The specific boundary value problem for a circular ring is solved.

## Introduction

The proposed applications of the theories of elasticity for double- and multi-porosity media are related to geological materials such as rocks and soils, manufactured porous materials such as ceramics and pressed powders, and biomaterials such as bone (for details, see the books by Straugan [19] and Svanadze [20] and references therein).

The first theory of poroelasticity based on Darcy's law was proposed by Biot in paper [1] in which a coupling effect between fluid pressure and mechanical stress is introduced. Later, Biot's classical model was generalized and the mathematical model of double porosity materials based on Darcy's law was developed by Wilson and Aifantis [27]

In [5,18], on the basis of the volume fraction concept, Cowin and Nunziato introduced an alternative linear and nonlinear coupled theory of elasticity for deformable porous materials, respectively. Using the mechanics of materials with voids, Ieşan and Quintanilla [10] presented the theories of elasticity and thermoelasticity for materials with a double-porosity structure.

Recently, the coupled linear theories of elasticity and thermoelasticity for materials with single and double porosity have been presented by Svanadze [21–23, 25], in which the coupled effect of Darcy's law and the concept of the volume fraction are developed. This coupled phenomenon is extended to double-porosity elastic and viscoelastic materials [24]. Moreover, the basic BVPs of the coupled quasi-static theories of elasticity and thermoelasticity for solids with single porosity are studied by Mikelashvili [13, 14]. More recently, in papers [15, 16], the same author has investigated the BVPs of steady vibrations of the coupled quasi-static theory of elasticity for double porosity materials by using the potential method. The basic boundary value problems are studied by Bitsadze [2, 3] and Tsagareli [26] in the series of papers (see [6–9, 11, 12]). The Dirichlet type quasi-static boundary value problem of the simple coupled theory of elasticity for a porous circular ring is solved by Bitsadze [4].

In the present paper, the linear mathematical model of double-porosity materials is introduced in which the coupled phenomenon of the Darcy law and volume fractions concepts of two levels of (macro-pores and micro-pores) is proposed [21, 22]. The governing system of plane strain equations is rewritten in a complex form, and its general solution is represented by using three analytic functions of a complex variable and three solutions of the Helmholtz equations. The constructed general solution allows us to solve analytically the problem for a circular ring.

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## 1. Basic Equations for Materials with Double Voids

Let $x = (x_1; x_2; x_3)$ be a point in the Euclidean three-dimensional space $\mathbb{R}^3$. In what follows, we consider an isotropic, homogeneous elastic solid body with double voids occupying a region of $\Omega \in \mathbb{R}^3$. We assume that the subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, repeated indices are summed over the range $(1; 2; 3)$. The governing equations of the theory of elastic materials with double voids can be expressed in the following form [23, 25]:

• Equations of equilibrium

$$\begin{aligned} t_{ji,j} + \rho_0 f_i = 0, \ \ i, j = 1, 2, 3, \\ \sigma_{j,j} + \xi + \rho_0 g = 0, \\ \tau_{j,j} + \zeta + \rho_0 l = 0, \end{aligned} \tag{1.1}$$

where $t_{ij}$ is the symmetric stress tensor, $f_i$ is the body force per unit mass, $\rho_0$ is the mass density, $\sigma_i$ and $\tau_i$ are the equilibrated stress vectors, $\xi$ and $\zeta$ are the intrinsic equilibrated body forces, $g$ is the extrinsic equilibrated body force per unit mass associated to macro pores, $l$ is the extrinsic equilibrated body force per unit mass associated to fissures.

• Constitutive equations

$$\begin{aligned} t_{ij} &= \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + (b_1\varphi_1 + b_2\varphi_2)\delta_{ij} - (\beta_1 p_1 + \beta_2 p_2)\delta_{ij}, \\ \sigma_i &= a_1\varphi_{1,i} + a_3\varphi_{2,i}, \\ \tau_i &= a_3\varphi_{1,i} + a_2\varphi_{2,i}, \\ \xi &= -b_1 e_{kk} - \alpha_1\varphi_1 - \alpha_3\varphi_2 + m_1 p_1 + m_3 p_2, \\ \zeta &= -b_2 e_{kk} - \alpha_3\varphi_1 - \alpha_2\varphi_2 + m_3 p_1 + m_2 p_2, \end{aligned} \tag{1.2}$$

where $\lambda$ and $\mu$ are the Lamé constants, $b_1$, $b_2$, $\beta_1$, $\beta_2$, $a_1$, $a_2$, $a_3$, $\alpha_1$, $\alpha_2$, $\alpha_3$, $m_1$, $m_2$ and $m_3$ are the constants characterizing the body porosity, $\delta_{ij}$ is the Kronecker delta, $\varphi_1$ is a change of volume fraction corresponding to pores (macro pores), $\varphi_2$ is a change of volume fraction corresponding to fissures (micro pores), $p_1$ and $p_2$ are the changes of the fluid pressure in macro- and micro-pore networks, respectively, $e_{ij}$ is the strain tensor and

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{1.3}$$

where $u_i$ are the components of the displacement vector.

In the stationary case, the values $p_1$ and $p_2$ satisfy the following system:

$$\begin{aligned} k_1\tilde{\Delta}p_1 + k_3\tilde{\Delta}p_2 - \gamma_0(p_1 - p_2) = 0, \\ k_3\tilde{\Delta}p_1 + k_2\tilde{\Delta}p_2 + \gamma_0(p_1 - p_2) = 0, \end{aligned}$$

where $\mu'$ is the fluid viscosity, $k_1 = \frac{\kappa_1}{\mu'}$, $k_2 = \frac{\kappa_2}{\mu'}$, $k_3 = \frac{\kappa_3}{\mu'}$, $\kappa_1$ and $\kappa_2$ are the macroscopic intrinsic permeabilities associated with matrix and fissure porosity, $\kappa_3$ is the cross-coupling permeability for fluid flow at the interface between the matrix and fissure phases, $\gamma_0$ is the internal transport coefficient corresponding to a fluid transfer rate respecting the intensity of the flow between macro and micro pores, $\gamma_0 \geq 0$, $\tilde{\Delta} \equiv \partial_{11} + \partial_{22} + \partial_{33}$ is the three-dimensional Laplace operator, $\partial_i = \frac{\partial}{\partial x_i}$.

The constitutive equations also meet some other conditions following from physical considerations:

$$\begin{aligned} \mu > 0, \quad 3\lambda + 2\mu > 0, \quad a_1 > 0, \quad a_1 a_2 - a_3^2 > 0, \\ k_1 > 0, \quad k_1 k_2 - k_3^2 > 0. \end{aligned}$$

## 2. Basic (Governing) Equations of the Plane Strain

From the basic three-dimensional equations, we obtain the basic equations for the case of plane strain. Let $\Omega$ be a sufficiently long cylindrical body with generatrix parallel to the $Ox_3$-axis. Denote by $V$ the cross-section of this cylindrical body, thus $V \subset \mathbb{R}^2$. In the case of plane deformation, $u_3 = 0$, while the functions $u_1$, $u_2$, $\varphi_1$, $\varphi_2$, $p_1$ and $p_2$ do not depend on the coordinate $x_3$.

As follows from formulas (1.2) and (1.3), in the case of plane strain

$$t_{k3} = t_{3k} = 0, \quad \sigma_3 = 0, \quad \tau_3 = 0, \quad k = 1, 2.$$

Therefore the homogeneous system of equilibrium equations (1.1) takes the form

$$\begin{gathered} \partial_1 t_{11} + \partial_2 t_{21} = 0, \quad \partial_1 t_{12} + \partial_2 t_{22} = 0, \\ \partial_k \sigma_k + \xi = 0, \quad \partial_k \tau_k + \zeta = 0. \end{gathered} \tag{2.1}$$

Now, relations (1.2) are rewritten as

$$\begin{gathered} t_{11} = \lambda\theta + 2\mu\partial_1 u_1 + b_1\varphi_1 + b_2\varphi_2 - \beta_1 p_1 - \beta_2 p_2, \\ t_{22} = \lambda\theta + 2\mu\partial_2 u_2 + +b_1\varphi_1 + b_2\varphi_2 - \beta_1 p_1 - \beta_2 p_2, \\ t_{12} = t_{21} = \mu(\partial_1 u_2 + \partial_2 u_1), \\ t_{33} = \sigma(t_{11} + t_{22}) + \frac{\mu}{\lambda + \mu}(b_1\varphi_1 + b_2\varphi_2 - \beta_1 p_1 - \beta_2 p_2), \\ \sigma_k = a_1\partial_k\varphi_1 + a_3\partial_k\varphi_2, \\ \tau_k = a_3\partial_k\varphi_1 + a_2\partial_k\varphi_2, \\ \xi = -b_1 e_{kk} - \alpha_1\varphi_1 - \alpha_3\varphi_2 + m_1 p_1 + m_3 p_2, \\ \zeta = -b_2 e_{kk} - \alpha_3\varphi_1 - \alpha_2\varphi_2 + m_3 p_1 + m_2 p_2, \end{gathered} \tag{2.2}$$

where $\sigma = \frac{\lambda}{2(\lambda+\mu)}$ is the Poisson ratio, $\theta = \partial_1 u_1 + \partial_2 u_2$.

Substituted relations (2.2) into system (2.1), we obtain the following system of governing equations of statics with respect to the functions $u_1$, $u_2$, $\varphi_1$, $\varphi_2$, $p_1$ and $p_2$

$$\begin{gathered} \mu\Delta u_k + (\lambda + \mu)\partial_k\theta + b_1\partial_k\varphi_1 + b_2\partial_k\varphi_2 - \beta_1\partial_k p_1 - \beta_2\partial_k p_2 = 0, \quad k = 1, 2, \\ (a_1\Delta - \alpha_1)\varphi_1 + (a_3\Delta - \alpha_3)\varphi_2 - b_1\theta + m_1 p_1 + m_3 p_2 = 0, \\ (a_3\Delta - \alpha_3)\varphi_1 + (a_2\Delta - \alpha_2)\varphi_2 - b_2\theta + m_3 p_1 + m_2 p_2 = 0, \\ k_1\Delta p_1 + k_3\Delta p_2 - \gamma_0(p_1 - p_2) = 0, \\ k_3\Delta p_1 + k_2\Delta p_2 + \gamma_0(p_1 - p_2) = 0. \end{gathered} \tag{2.3}$$

Note that $\Delta \equiv \partial_{11} + \partial_{22}$ is the two-dimensional Laplace operator.

On the plane $Ox_1x_2$, we introduce the complex variable $z = x_1 + ix_2 = re^{i\vartheta}$, $(i^2 = -1)$ and the operators $\partial_z = 0.5(\partial_1 - i\partial_2)$, $\partial_{\bar{z}} = 0.5(\partial_1 + i\partial_2)$, $\bar{z} = x_1 - ix_2$, and $\Delta = 4\partial_z\partial_{\bar{z}}$.

To write system (2.1) in the complex form, we multiply the second equation of this system by $i$ and sum up with the first equation

$$\begin{gathered} \partial_z(t_{11} - t_{22} + 2it_{12}) + \partial_{\bar{z}}(t_{11} + t_{22}) = 0, \\ \partial_z\sigma_+ + \partial_{\bar{z}}\bar{\sigma}_+ + \xi = 0, \\ \partial_z\tau_+ + \partial_{\bar{z}}\bar{\tau}_+ + \zeta = 0, \end{gathered} \tag{2.4}$$

where $\sigma_+ = \sigma_1 + i\sigma_2$, $\tau_+ = \tau_1 + i\tau_2$, we rewrite formulas (2.2) as follows:

$$\begin{gathered}
t_{11} - t_{22} + 2it_{12} = 4\mu\partial_{\bar z} u_+,\\
t_{11} + t_{22} = 2(\lambda+\mu)\theta + 2b_1\varphi + 2b_2\varphi - 2\beta_1 p_1 - 2\beta_2 p_2,\\
\sigma_+ = 2a_1\partial_{\bar z}\varphi_1 + 2a_3\partial_{\bar z}\varphi_2,\\
\tau_+ = 2a_3\partial_{\bar z}\varphi_1 + 2a_2\partial_{\bar z}\varphi_2,\\
\xi = -b_1\theta - \alpha_1\varphi_1 - \alpha_3\varphi_2 + m_1p_1 + m_3p_2,\\
\zeta = -b_2\theta - \alpha_3\varphi_1 - \alpha_2\varphi_2 + m_3p_1 + m_2p_2,
\end{gathered} \tag{2.5}$$

$$\theta = \partial_z u_+ + \partial_{\bar z}\bar u_+, \quad u_+ = u_1 + iu_2.$$

Substituting relations (2.5) into system (2.4), we rewrite system (2.3) in the complex form

$$\begin{gathered}
2\mu\partial_{\bar z}\partial_z u_+ + (\lambda+\mu)\partial_{\bar z}\theta + b_1\partial_{\bar z}\varphi_1 + b_2\partial_{\bar z}\varphi_2 - \beta_1\partial_{\bar z}p_1 - \beta_2\partial_{\bar z}p_2 = 0,\\
(a_1\Delta - \alpha_1)\varphi_1 + (a_3\Delta - \alpha_3)\varphi_2 - b_1\theta + m_1p_1 + m_3p_2 = 0,\\
(a_3\Delta - \alpha_3)\varphi_1 + (a_2\Delta - \alpha_2)\varphi_2 - b_2\theta + m_3p_1 + m_2p_2 = 0,\\
k_1\Delta p_1 + k_3\Delta p_2 - \gamma_0(p_1 - p_2) = 0,\\
k_3\Delta p_1 + k_2\Delta p_2 + \gamma_0(p_1 - p_2) = 0.
\end{gathered} \tag{2.6}$$

## 3. Kolosov–Muskhelishvili's Analogues of Formulas for System (2.6)

**Theorem.** *The general solution of system* (2.6) *is represented as follows* [7, 11, 17]*:*

$$\begin{gathered}
2\mu u_+ = \varkappa f(z) - z\overline{f'(z)} - \overline{h(z)} + q_1(g(z) + z\overline{g'(z)}) - q_2\partial_{\bar z}\chi_1(z,\bar z) - q_3\partial_{\bar z}\chi_2(z,\bar z) + q_4\partial_{\bar z}\eta(z,\bar z),\\
\varphi_1 = l_{11}\chi_1(z,\bar z) + l_{12}\chi_2(z,\bar z) - e_1(f'(z) + \overline{f'(z)}) - e_3(g'(z) + \overline{g'(z)}) - e_5\eta(z,\bar z),\\
\varphi_2 = l_{21}\chi_1(z,\bar z) + l_{22}\chi_2(z,\bar z) - e_2(f'(z) + \overline{f'(z)}) - e_4(g'(z) + \overline{g'(z)}) - e_6\eta(z,\bar z),\\
p_1 = g'(z) + \overline{g'(z)} + (k_2 + k_3)\eta(z,\bar z),\\
p_2 = g'(z) + \overline{g'(z)} - (k_1 + k_3)\eta(z,\bar z),
\end{gathered} \tag{3.1}$$

*where $f(z)$, $h(z)$ and $g(z)$ are the arbitrary analytic functions of a complex variable $z$, $\eta(z,\bar z)$ is an arbitrary solution of the Helmholtz equation*

$$\triangle\eta - \nu^2\eta = 0,$$

$$\nu^2 = \frac{k_1 + k_2 + k_3^2}{k_1k_2 - k_3^2}\,\gamma_0,$$

*$\chi_1(z,\bar z)$ and $\chi_2(z,\bar z)$ are general solutions of the Helmholtz equations*

$$\Delta\chi_1(z,\bar z) - \varkappa_1\chi_1(z,\bar z) = 0, \quad \Delta\chi_2(z,\bar z) - \varkappa_2\chi_2(z,\bar z) = 0.$$

$\varkappa_\alpha$ are eigenvalues and $(l_{11},\ l_{21})$, $(l_{12},\ l_{22})$ are eigenvectors of the matrix $C$ and, owing to (2.2), they are positive numbers:

$$C=\begin{pmatrix} a_1 & a_3\\ a_3 & a_2\end{pmatrix}^{-1}\cdot\begin{pmatrix}\alpha_1-\dfrac{b_1^2}{\lambda+2\mu} & \alpha_3-\dfrac{b_1b_2}{\lambda+2\mu}\\ \alpha_3-\dfrac{b_1b_2}{\lambda+2\mu} & \alpha_2-\dfrac{b_2^2}{\lambda+2\mu}\end{pmatrix}$$

$\varkappa$, $q_1$, $q_2$, $q_3$ and $q_4$ equal

$$\varkappa=\frac{\lambda+3\mu+2\mu(b_1e_1+b_2e_2)}{\lambda+\mu-2\mu(b_1e_1+b_2e_2)},\quad q_1=\frac{\mu(b_1e_3+b_2e_4+\beta_1+\beta_2)}{\lambda+2\mu},$$
$$q_2=\frac{4\mu(b_1l_{11}+b_2l_{21})}{\kappa_1(\lambda+2\mu)}\quad q_3=\frac{4\mu(b_1l_{12}+b_2l_{22})}{\kappa_2(\lambda+2\mu)},$$
$$q_4=\frac{4\mu(b_1e_5+b_2e_6+\beta_1(\kappa_2+\kappa_3)-\beta_2(\kappa_1+\kappa_3))}{\nu^2(\lambda+2\mu)},$$

$e_1$, $e_2$, $e_3$, $e_4$, $e_5$ and $e_6$ equal

$$e_1=A\bar e_1,\quad e_2=A\bar e_2\quad A:=\frac{2(\lambda+2\mu)}{\lambda+\mu-2\mu(b_1e_1+b_2e_2)},$$
$$\bar e_1=\frac{b_1\alpha_2-b_2\alpha_3}{2((\alpha_1\alpha_2-\alpha_3^2)(\lambda+2\mu)-\alpha_1b_2^2-\alpha_2b_1^2+2\alpha_3b_1b_2)},$$
$$\bar e_2=\frac{b_1\alpha_1-b_1\alpha_3}{2((\alpha_1\alpha_2-\alpha_3^2)(\lambda+2\mu)-\alpha_1b_2^2-\alpha_2b_1^2+2\alpha_3b_1b_2)},$$
$$e_3=T_{11}^*+T_{12}^*,\quad e_4=T_{12}^*+T_{22}^*,$$
$$e_5=X_{11}(\kappa_2+\kappa_3)-X_{12}(\kappa_1+\kappa_3),$$
$$e_6=X_{21}(\kappa_2+\kappa_3)-X_{22}(\kappa_1+\kappa_3),$$

where $T_{11}^*$, $T_{12}^*$ and $T_{22}^*$ are elements of the matrix $T^*=C^{-1}T$, $X_{11}$ , $X_{12}$, $X_{21}$ and $X_{22}$ are elements of the matrix $X=(\zeta^2I-C)^{-1}T$,

$$T=\begin{pmatrix} a_1 & a_3\\ a_3 & a_2\end{pmatrix}^{-1}\cdot\begin{pmatrix}\dfrac{b_1\beta_1}{\lambda+2\mu}-m_1 & \dfrac{b_1\beta_2}{\lambda+2\mu}-m_3\\ \dfrac{b_2\beta_1}{\lambda+2\mu}-m_3 & \dfrac{b_2\beta_2}{\lambda+2\mu}-m_2\end{pmatrix}.$$

Substituting formula (3.1) into the first equation of (2.6), we obtain

$$\begin{aligned} t_{11}-t_{22}+2it_{12}&=-2z\overline{f''(z)}-2\overline{h'(z)}+2q_1z\overline{g''(z)}-2q_2\partial_{\bar z\bar z}^2\chi_1\,(z,\tilde z)\\ &\quad-2q_3\partial_{\bar z\bar z}^2\chi_2\,(z,\bar z)+2q_4\partial_{\bar z\bar z}^2\eta\,(z,\tilde z)\,,\\ t_{11}+t_{22}&=2\delta_1(f'(z)+\overline{f'(z)})+2\delta_2(g'(z)+\overline{g'(z)})-2\delta_3\chi_1(z,\tilde z)\\ &\quad-2\delta_4\chi_2(z,\tilde z)+2\delta_5\eta(z,\tilde z),\end{aligned}\tag{3.2}$$

where

$$\delta_1=\frac{(\lambda+\mu)(\varkappa-1)}{2\mu}-b_1e_1-b_2e_2,\quad \delta_2=\frac{(\lambda+\mu)q_1}{2\mu}-b_1e_3-b_2e_4-\beta_1-\beta_2,$$
$$\delta_3=\frac{(\lambda+\mu)\varkappa_1}{4\mu}-b_1l_{11}-b_2l_{21},\quad \delta_4=\frac{(\lambda+\mu)\varkappa_2}{4\mu}-b_1l_{12}-b_2l_{22},$$
$$\delta_5=\frac{(\lambda+\mu)\nu^2}{4\mu}-b_1e_5-b_2e_6-\beta_1(k_2+k_3)+\beta_2(k_1+k_3).$$

Assume that mutually perpendicular unit vectors $\mathbf{l}$ and $\mathbf{s}$ are such that

$$\mathbf{l}\times\mathbf{s}=\mathbf{e}_3,$$

where $\mathbf{e}_3$ is the unit vector directed along the $x_3$-axis. The vector $\mathbf{l}$ forms the angle $\alpha$ with the positive direction of the $x_1$-axis. Then the displacement components $u_l = \mathbf{u}\cdot\mathbf{l}$, $u_s = \mathbf{u}\cdot\mathbf{s}$, as well as the stress and moment stress components acting on an area of arbitrary orientation are expressed by the formulas

$$
\begin{aligned}
u_l + iu_s &= e^{-i\alpha}u_+,\\
t_{ll} - it_{ls} &= \frac{1}{2}\left[t_{11} + t_{22} + (t_{11} - t_{22} + 2it_{12})e^{-2i\alpha}\right],\\
\sigma_l &= \frac{1}{2}\left[\sigma_+ e^{-i\alpha} + \overline{\sigma}_+ e^{i\alpha}\right],\\
\tau_l &= \frac{1}{2}\left[\tau_+ e^{-i\alpha} + \overline{\tau}_+ e^{i\alpha}\right].
\end{aligned}
\tag{3.3}
$$

## 4. A Problem for a Concentric Circular Ring

Let a porous elastic body with double porosity occupy the domain $V$ which is bounded by the concentric circumferences $L_1$ and $L_2$ with radii $R_1$ and $R_2$, respectively, ($R_1 < R_2$) (see Figure 1). The origin is located at the center of the circle.

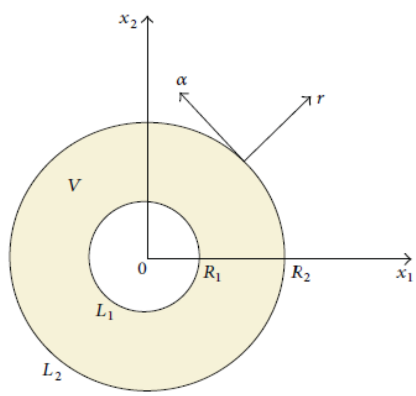


Figure 1

We consider the following problem:

$$
\begin{gathered}
t_{rr} + it_{r\alpha} = \begin{cases} A', & r = R_1,\\ A'', & r = R_2,\end{cases}\\
\varphi_1 = \begin{cases} B', & r = R_1,\\ B'', & r = R_2,\end{cases} \qquad \varphi_2 = \begin{cases} C', & r = R_1,\\ C'', & r = R_2,\end{cases}\\
p_1 = \begin{cases} D', & r = R_1,\\ D'', & r = R_2,\end{cases} \qquad p_2 = \begin{cases} E', & r = R_1,\\ E'', & r = R_2,\end{cases}
\end{gathered}
\tag{4.1}
$$

where $A'$, $A''$, $B'$, $B''$, $C'$, $C''$, $D'$, $D''$, $E'$ and $E''$ are sufficiently smooth functions.

The analytic functions $f'(z)$, $h'(z)$, $g'(z)$ and the metaharmonic functions $\chi_1(z,\bar z)$, $\chi_2(z,\bar z)$, $\eta(z,\bar z)$ are represented as the series

$$
\begin{gathered}
f'(z) = A^* \ln z + \sum_{-\infty}^{\infty} a_n z^n, \quad h'(z) = \sum_{-\infty}^{\infty} b_n z^n, \quad g'(z) = \sum_{-\infty}^{\infty} c_n z^n, \\
\chi_1(z,\bar z) = \sum_{-\infty}^{+\infty} (\alpha_n' I_n(\sqrt{\varkappa_1} r) + \alpha_n'' K_n(\sqrt{\varkappa_1} r))\, e^{in\alpha}, \\
\chi_2(z,\bar z) = \sum_{-\infty}^{+\infty} (\beta_n' I_n(\sqrt{\varkappa_2} r) + \beta_n'' K_n(\sqrt{\varkappa_2} r))\, e^{in\vartheta}, \\
\eta(z,\bar z) = \sum_{-\infty}^{+\infty} (\gamma_n' I_n(\nu r) + \gamma_n'' K_n(\nu r))\, e^{in\alpha},
\end{gathered}
\tag{4.2}
$$

where $I_n(\cdot)$ and $K_n(\cdot)$ are the modified Bessel functions of the first and second kind of $n$-th order.

Expand the functions $A'$, $A''$, $B'$, $B''$, $C'$, $C''$, $D'$, $D''$, $E'$ and $E''$, in a complex Fourier series given on $r = R_1$ and $r = R_1$,

$$
\begin{gathered}
A' = \sum_{-\infty}^{\infty} A_n' e^{in\alpha}, \quad A'' = \sum_{-\infty}^{\infty} A_n'' e^{in\alpha}, \quad B' = \sum_{-\infty}^{\infty} B_n' e^{in\alpha}, \quad B'' = \sum_{-\infty}^{\infty} B_n'' e^{in\alpha}, \\
C' = \sum_{-\infty}^{\infty} C_n' e^{in\alpha}, \quad C'' = \sum_{-\infty}^{\infty} C_n'' e^{in\alpha}, \quad D' = \sum_{-\infty}^{\infty} D_n' e^{in\alpha}, \quad D'' = \sum_{-\infty}^{\infty} D_n'' e^{in\alpha}, \\
E' = \sum_{-\infty}^{\infty} E_n' e^{in\alpha}, \quad E'' = \sum_{-\infty}^{\infty} E_n'' e^{in\alpha}.
\end{gathered}
$$

The condition of single-valuedness of the displacements in the present case is expressed as

$$
A^* = 0, \quad \varkappa a_{-1} + \bar b_{-1} + q_1 c_{-1} = 0. \tag{4.3}
$$

Substituting (4.2) in (3.1)–(3.3), taking into account the boundary conditions (4.1) and assuming that the series converge on the circumference $r = R_1$, $r = R_2$, one finds

$$
\begin{gathered}
\delta_1 \sum_{-\infty}^{\infty} r^n a_n e^{in\alpha} + \sum_{-\infty}^{\infty} (\delta_1 - n) r^n \bar a_n e^{-in\alpha} - \sum_{-\infty}^{\infty} r^n \bar b_n e^{-i(n+2)\alpha} + \delta_2 \sum_{-\infty}^{\infty} r^n c_n e^{in\alpha} \\
+ \sum_{-\infty}^{\infty} (\delta_2 + n q_1) r^n \bar c_n e^{-in\alpha} - \delta_3 \sum_{-\infty}^{\infty} (I_n(\sqrt{\varkappa_1} r)\alpha_n' + K_n(\sqrt{\varkappa_1} r)\alpha_n'')\, e^{in\alpha} \\
-\delta_4 \sum_{-\infty}^{\infty} (I_n(\sqrt{\varkappa_2} r)\beta_n' + K_n(\sqrt{\varkappa_2} r)\beta_n'')\, e^{in\alpha} + \delta_5 \sum_{-\infty}^{\infty} (I_n(\nu r)\gamma_n' + K_n(\nu r)\gamma_n'')\, e^{in\alpha} \\
-\frac{q_2 \varkappa_1}{4} \sum_{-\infty}^{\infty} (I_{n+2}(\sqrt{\varkappa_1} r)\alpha_n' + K_{n+2}(\sqrt{\varkappa_1} r)\alpha_n'')\, e^{in\alpha} \\
-\frac{q_3 \varkappa_2}{4} \sum_{-\infty}^{\infty} (I_{n+2}(\sqrt{\varkappa_2} r)\beta_n' + K_{n+2}(\sqrt{\varkappa_2} r)\beta_n'')\, e^{in\alpha} \\
+\frac{q_4 \nu^2}{4} \sum_{-\infty}^{\infty} (I_{n+2}(\nu r)\gamma_n' + K_{n+2}(\nu r)\gamma_n'')\, e^{in\alpha} = \begin{cases} A', & r = R_1, \\ A'', & r = R_2, \end{cases}
\end{gathered}
\tag{4.4}
$$

$$
l_{11} \sum_{-\infty}^{\infty} (I_n(\sqrt{\varkappa_1} r)\alpha_n' + K_n(\sqrt{\varkappa_1} r)\alpha_n'')\, e^{in\alpha} - e_1 \sum_{-\infty}^{\infty} \left( r^n a_n e^{in\alpha} + r^n \bar a_n e^{-in\alpha} \right)
$$

$$+l_{12}\sum_{-\infty}^{\infty}\left(I_n(\sqrt{\varkappa_2}r)\beta_n'+K_n(\sqrt{\varkappa_2}r)\beta_n''\right)e^{in\alpha}-e_3\sum_{-\infty}^{\infty}\left(r^nc_ne^{in\alpha}+r^n\bar{c}_ne^{-in\alpha}\right)$$

$$-e_5\sum_{-\infty}^{\infty}\left(I_n(\nu r)\gamma_n'+K_n(\nu r)\gamma_n''\right)e^{in\alpha}=\begin{cases}B', & r=R_1,\\ B'', & r=R_2,\end{cases} \tag{4.5}$$

$$l_{21}\sum_{-\infty}^{\infty}\left(I_n(\sqrt{\varkappa_1}r)\alpha_n'+K_n(\sqrt{\varkappa_1}r)\alpha_n''\right)e^{in\alpha}-e_2\sum_{-\infty}^{\infty}\left(r^na_ne^{in\alpha}+r^n\bar{a}_ne^{-in\alpha}\right)$$

$$+l_{22}\sum_{-\infty}^{\infty}\left(I_n(\sqrt{\varkappa_2}r)\beta_n'+K_n(\sqrt{\varkappa_2}r)\beta_n''\right)e^{in\alpha}-e_4\sum_{-\infty}^{\infty}\left(r^nc_ne^{in\alpha}+r^n\bar{c}_ne^{-in\alpha}\right)$$

$$-e_6\sum_{-\infty}^{\infty}\left(I_n(\nu r)\gamma_n'+K_n(\nu r)\gamma_n''\right)e^{in\alpha}=\begin{cases}C', & r=R_1,\\ C'', & r=R_2,\end{cases} \tag{4.6}$$

$$\sum_{-\infty}^{\infty}\left(r^nc_ne^{in\alpha}+r^n\bar{c}_ne^{-in\alpha}\right)+(k_2+k_3)\sum_{-\infty}^{\infty}\left(I_n(\nu r)\gamma_n'+K_n(\nu r)\gamma_n''\right)e^{in\alpha}$$

$$=\begin{cases}D', & r=R_1,\\ D'', & r=R_2,\end{cases} \tag{4.7}$$

$$\sum_{-\infty}^{\infty}\left(r^nc_ne^{in\alpha}+r^n\bar{c}_ne^{-in\alpha}\right)-(k_1+k_3)\sum_{-\infty}^{\infty}\left(I_n(\nu r)\gamma_n'+K_n(\nu r)\gamma_n''\right)e^{in\alpha}$$

$$=\begin{cases}E', & r=R_1,\\ E'', & r=R_2.\end{cases} \tag{4.8}$$

Comparing in (4.4)–(4.8) the coefficients of $e^{in\alpha}$, we have

$$\begin{gathered}
\delta_1R_1^na_n+\frac{\delta_1-n}{R_1^n}\bar{a}_{-n}-\frac{1}{R_1^{n+2}}\bar{b}_{-n-2}-\left(\delta_3I_n(\sqrt{\varkappa_1}R_1)+\frac{q_2\varkappa_1}{4}I_{n+2}(\sqrt{\varkappa_1}R_1)\right)\alpha_n'\\
+\delta_2R_1^nc_n+\frac{\delta_2+nq_1}{R_1^n}\bar{c}_{-n}-\left(\delta_3K_n(\sqrt{\varkappa_1}R_1)+\frac{q_2\varkappa_1}{4}K_{n+2}(\sqrt{\varkappa_1}R_1)\right)\alpha_n''\\
-\left(\delta_4I_n(\sqrt{\varkappa_2}R_1)+\frac{q_3\varkappa_2}{4}I_{n+2}(\sqrt{\varkappa_2}R_1)\right)\beta_n'+\left(\delta_5I_n(\nu R_1)+\frac{q_4\nu^2}{4}I_{n+2}(\nu R_1)\right)\gamma_n'\\
-\left(\delta_4K_n(\sqrt{\varkappa_2}R_1)+\frac{q_3\varkappa_2}{4}K_{n+2}(\sqrt{\varkappa_2}R_1)\right)\beta_n''+\left(\delta_5K_n(\nu R_1)+\frac{q_4\nu^2}{4}K_{n+2}(\nu R_1)\right)\gamma_n''=A_n',\\
\delta_1R_2^na_n+\frac{\delta_1-n}{R_2^n}\bar{a}_{-n}-\frac{1}{R_2^{n+2}}\bar{b}_{-n-2}-\left(\delta_3I_n(\sqrt{\varkappa_1}R_2)+\frac{q_2\varkappa_1}{4}I_{n+2}(\sqrt{\varkappa_1}R_2)\right)\alpha_n'\\
+\delta_2R_2^nc_n+\frac{\delta_2+nq_1}{R_2^n}\bar{c}_{-n}-\left(\delta_3K_n(\sqrt{\varkappa_1}R_2)+\frac{q_2\varkappa_1}{4}K_{n+2}(\sqrt{\varkappa_1}R_2)\right)\alpha_n''\\
-\left(\delta_4I_n(\sqrt{\varkappa_2}R_2)+\frac{q_3\varkappa_2}{4}I_{n+2}(\sqrt{\varkappa_2}R_2)\right)\beta_n'+\left(\delta_5I_n(\nu R_2)+\frac{q_4\nu^2}{4}I_{n+2}(\nu R_2)\right)\gamma_n'\\
-\left(\delta_4K_n(\sqrt{\varkappa_2}R_2)+\frac{q_3\varkappa_2}{4}K_{n+2}(\sqrt{\varkappa_2}R_2)\right)\beta_n''+\left(\delta_5K_n(\nu R_2)+\frac{q_4\nu^2}{4}K_{n+2}(\nu R_2)\right)\gamma_n''=A_n''.
\end{gathered} \tag{4.9}$$

$$\begin{aligned}
&l_{11}I_n(\sqrt{\varkappa_1}R_1)\alpha_n'+l_{11}K_n(\sqrt{\varkappa_1}R_1)\alpha_n''+l_{12}I_n(\sqrt{\varkappa_2}R_1)\beta_n'+l_{12}K_n(\sqrt{\varkappa_2}R_1)\beta_n''\\
&-e_1R_1^na_n-e_1\frac{1}{R_1^n}\bar a_{-n}-e_3R_1^nc_n-e_3\frac{1}{R_1^n}\bar c_{-n}-e_5I_n(\nu R_1)\gamma_n'-e_5K_n(\nu R_1)\gamma_n''=B_n',\\
&l_{11}I_n(\sqrt{\varkappa_1}R_2)\alpha_n'+l_{11}K_n(\sqrt{\varkappa_1}R_2)\alpha_n''+l_{12}I_n(\sqrt{\varkappa_2}R_2)\beta_n'+l_{12}K_n(\sqrt{\varkappa_2}R_2)\beta_n''\\
&-e_1R_2^na_n-e_1\frac{1}{R_2^n}\bar a_{-n}-e_3R_2^nc_n-e_3\frac{1}{R_2^n}\bar c_{-n}-e_5I_n(\nu R_2)\gamma_n'-e_5K_n(\nu R_2)\gamma_n''=B_n'',\\
&l_{21}I_n(\sqrt{\varkappa_1}R_1)\alpha_n'+l_{21}K_n(\sqrt{\varkappa_1}R_1)\alpha_n''+l_{22}I_n(\sqrt{\varkappa_2}R_1)\beta_n'+l_{22}K_n(\sqrt{\varkappa_2}R_1)\beta_n''\\
&-e_2R_1^na_n-e_2\frac{1}{R_1^n}\bar a_{-n}-e_4R_1^nc_n-e_4\frac{1}{R_1^n}\bar c_{-n}-e_6I_n(\nu R_1)\gamma_n'-e_6K_n(\nu R_1)\gamma_n''=C_n',\\
&l_{21}I_n(\sqrt{\varkappa_1}R_2)\alpha_n'+l_{21}K_n(\sqrt{\varkappa_1}R_2)\alpha_n''+l_{22}I_n(\sqrt{\varkappa_2}R_2)\beta_n'+l_{22}K_n(\sqrt{\varkappa_2}R_2)\beta_n''\\
&-e_2R_2^na_n-e_2\frac{1}{R_2^n}\bar a_{-n}-e_4R_2^nc_n-e_4\frac{1}{R_2^n}\bar c_{-n}-e_6I_n(\nu R_2)\gamma_n'-e_6K_n(\nu R_2)\gamma_n''=C_n'',
\end{aligned}\tag{4.10}$$

$$\begin{aligned}
&R_1^nc_n+\frac{1}{R_1^n}\bar c_{-n}+(\kappa_2+\kappa_3)I_n(\nu R_1)\gamma_n'+(\kappa_2+\kappa_3)K_n(\nu R_1)\gamma_n''=D_n',\\
&R_2^nc_n+\frac{1}{R_2^n}\bar c_{-n}+(\kappa_2+\kappa_3)I_n(\nu R_2)\gamma_n'+(\kappa_2+\kappa_3)K_n(\nu R_2)\gamma_n''=D_n'',\\
&R_1^nc_n+\frac{1}{R_1^n}\bar c_{-n}-(\kappa_1+\kappa_3)I_n(\nu R_1)\gamma_n'-(\kappa_1+\kappa_3)K_n(\nu R_1)\gamma_n''=E_n',\\
&R_2^nc_n+\frac{1}{R_2^n}\bar c_{-n}-(\kappa_1+\kappa_3)I_n(\nu R_2)\gamma_n'-(\kappa_1+\kappa_3)K_n(\nu R_2)\gamma_n''=E_n''.
\end{aligned}\tag{4.11}$$

From (4.3) and (4.9)–(4.11), we can find all coefficients $a_n$, $b_n$, $c_n$ $\alpha_n'$, $\alpha_n''$, $\beta_n'$, $\beta_n''$ [17].

It is easy to prove the absolute and uniform convergence of the series obtained in the circular ring (including the contours) when the set of functions on the boundaries has sufficient smoothness, in particular, when the function defined on $L_1$ and $L_2$ has second-order derivatives satisfying the Dirichlet condition [17].

The procedure of solving a boundary value problem remains the same when the displacement vector, the equilibrated stress vectors, the change in volume fractions and the fluid pressures on the boundary of the domain are defined arbitrarily, but the condition of equality to zero of the principal vector and the principal moment of external forces is satisfied.

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