Transactions of A. Razmadze
Mathematical Institute
Vol. **179** (2025), issue 1, 73–83

# ON THE CONTINUITY OF SOLUTION OF ONE CLASS CONTROLLED NEUTRAL FUNCTIONAL-DIFFERENTIAL EQUATION WITH RESPECT TO INITIAL DATA

GIVI BERIKELASHVILI[1,2], IA RAMISHVILI[2], TEA SHAVADZE[3,5*] AND TAMAZ TADUMADZE[4]

**Abstract.** For the neutral functional-differential equation, whose right-hand side is linear with respect to the prehistory of the phase velocity and depends on a control function, the theorem on the continuous dependence of a solution with respect to perturbations in the initial data is proved. Such type theorems play an important role in studying the neutral type optimization problems, in proving formulas for the analytic representation of solutions, in constructing an approximate solution, and in analyzing the sensitivity of mathematical models. Under the initial data we imply the set of delay parameter contained in a nonlinear addend of the right-hand side of the equation, the initial vector, and the initial and control functions.

## 1. Introduction

The neutral functional-differential equation is a mathematical model of such system whose behavior at a given moment depends on system's past velocity. Many real processes are described by neutral functional-differential equations [1, 3, 6, 7, 12, 15] and many works are devoted to the investigation of such type equations, including [1–4, 6, 7, 9, 10, 12, 15, 16]. In the present paper, for the controlled neutral functional-differential equation

$$\dot{x}(t) = A(t, x(t), u(t))\dot{x}(t-\sigma) + f(t, x(t), x(t-\tau), u(t)), \quad t \in [t_0, t_1],$$

with the initial condition

$$x(t) = \varphi(t), \quad t < t_0, \quad x(t_0) = x_0 \tag{1.1}$$

the theorem on the continuous dependence of a solution on the initial data is proved. Theorems of this type play an important role in studying neutral type optimization problems, in proving formulas for the analytic representation of solutions, in constructing an approximate solution and in analyzing the sensitivity of mathematical models [1, 9, 10, 12, 15]. Under the initial data we mean the set of the delay parameter $\tau$, the initial vector $x_0$, the initial function $\varphi(t)$ and the control function $u(t)$. Condition (1.1) is called the discontinuous initial condition, since $\varphi(t_0) \neq x_0$. The discontinuity at the initial moment may be due to an instantaneous change in a dynamic process, such as shifts in investment or environmental factors. Finally, we note that the case $A(t, x, u) = A(t)$ is considered in [2, 4, 15, 16], and the case $A(t, x, u) = 0$ is considered in [5, 8, 13, 14].

The paper is organized as follows: in Section 2, the main theorem is formulated, in Section 3, the auxiliary assertions are given, in Section 4, the main theorem is proved.

## 2. Formulation of the Main Result

Let $\mathbb{R}^n$ be the $n$-dimensional vector space of points $x = (x^1, \dots, x^n)^T$, where $T$ is the sign of transposition; let $I = [t_0, t_1]$ be a fixed interval and let $\sigma > 0, \tau_2 > \tau_1 > 0$ be the given numbers, with

$$\max\{t_0 + \sigma, t_0 + \tau_2\} < t_1. \tag{2.1}$$

---

2020 *Mathematics Subject Classification.* 34K40, 34K24.

*Key words and phrases.* Neutral controlled functional-differential equation; Continuity of solution with respect to initial data; Discontinuous initial condition; Perturbations.

*Corresponding author.

The $n \times n$-dimensional matrix-function $A(t,x,u)$ and the $n$-dimensional vector-function $f(t,x,y,u)$ are continuous and bounded on the sets $I \times \mathbb{R}^n \times \mathbb{R}^r$ and $I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^r$, respectively. Moreover, there exist the numbers $L_A > 0$ and $L_f > 0$ such that the following conditions

$$|A(t,x_1,u_1) - A(t,x_2,u_2)| \le L_A\Big(|x_1 - x_2| + |u_1 - u_2|\Big)$$
$$\forall t \in I, (x_i,u_i) \in \mathbb{R}^n \times \mathbb{R}^r, \quad i = 1,2$$

and

$$|f(t,x_1,y_1,u_1) - f(t,x_2,y_2,u_2)| \le L_f\Big(|x_1 - x_2| + |y_1 - y_2| + |u_1 - u_2|\Big)$$
$$\forall t \in I, (x_i,y_i,u_i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^r, \quad i = 1,2$$

hold.

Let us introduce the notations: denote by $\Phi$ and $\Omega$ the sets of continuous differentiable initial functions $\varphi(t) \in \mathbb{R}^n, t \in I_1 = [\hat\tau, t_0]$, where $\hat\tau = t_0 - \max\{\sigma, \tau_2\}$ and piecewise continuous control functions $u(t) \in \mathbb{R}^r, t \in I$, with the set $clu(I)$ is compact; the initial data is called the set of the delay parameter $\tau$, the initial vector $x_0$, the initial function $\varphi(t)$ and the control function $u(t)$. This set we denote by $w$ and the set of such elements $w = (\tau, x_0, \varphi(t), u(t))$ we denote by $W = (\tau_1,\tau_2) \times R^n \times \Phi \times \Omega$. Further,

$$|w| = |\tau| + |x_0| + \|\varphi\|_1 + \|u\|,$$

where

$$\|\varphi\|_1 = \sup\{|\varphi(t)| + |\dot\varphi(t)| : t \in I_1\}, \quad \|u\| = \sup\{|u(t)| : t \in I\}.$$

To each element $w = (\tau, x_0, \varphi(t), u(t)) \in W$ we assign the controlled neutral functional-differential equation

$$\dot x(t) = A(t,x(t),u(t))\dot x(t-\sigma) + f(t,x(t),x(t-\tau),u(t)), \quad t \in I \tag{2.2}$$

with the discontinuous initial condition

$$x(t) = \varphi(t), \quad t \in [\hat\tau, t_0), \quad x(t_0) = x_0. \tag{2.3}$$

**Definition 2.1.** Let $w \in W$. A function $x(t) = x(t;w), t \in [\hat\tau, t_1]$, is called a solution of problem (2.2), (2.3) or a solution corresponding to the element $w$, if it satisfies condition (2.3) and $x(t)$ is absolutely continuous on the interval $I$ and satisfies equation (2.2) almost everywhere on $I$.

**Theorem 2.1.** *For each element $w = (\tau, x_0, \varphi(t), u(t)) \in W$, there exists the unique solution $x(t) = x(t;w)$ of problem (2.2), (2.3). Let $w_0 = (\tau_0, x_{00}, \varphi_0(t), u_0(t)) \in W$ be a fixed element and let $x_0(t) = x(t;w_0)$ be the corresponding solution. For an arbitrary $\varepsilon > 0$, there exists a number $\delta = \delta(\varepsilon) > 0$ such that for $\forall w = (\tau, x_0, \varphi(t), u(t)) \in W$, the inequality*

$$|x_0(t) - x(t)| < \varepsilon, \quad \forall t \in I$$

*holds, when*

$$|w - w_0| = |\tau - \tau_0| + |x_0 - x_{00}| + \|\varphi - \varphi_0\|_1 + \|u - u_0\| < \delta.$$

## 3. Auxiliary Assertions

**Theorem 3.1.** *For any $w \in W$, there exists the unique solution $x(t) = x(t;w)$, $t \in [\hat\tau, t_1]$.*

*Proof.* The existence of the unique global solution will be proved by the step method from the left to the right with respect to $\sigma$.

**Step 3.1.** Let $t \in [t_0, t_0 + \sigma] \subset I$ (see (2.1)), then we get the following delay functional-differential equation

$$\dot x(t) = A(t,x(t),u(t))\dot\varphi(t-\sigma) + f(t,x(t),x(t-\tau),u(t)) \tag{3.1}$$

with the initial condition

$$x(t) = \varphi(t), \quad t \in [\hat\tau, t_0), \quad x(t_0) = x_0. \tag{3.2}$$

It is clear that the functions $A(t,x,u(t))$ and $f(t,x,y,u(t))$ satisfy the Lipschitz condition on the spaces with respect to $x \in R^n$ and $(x,y) \in R^n \times R^n$, respectively. Therefore, the existence of the unique solution $x_1(t), t \in [\hat{\tau}, t_0+\sigma]$ of problem (3.1), (3.2) (see Definition 2.1) can be proved by using the method of successive approximations and the Arzela–Ascoli lemma.

**Step 3.2.** Let $t_0 + 2\sigma < t_1$, then on the interval $[t_0+\sigma, t_0+2\sigma]$, we have the problem

$$\begin{cases} \dot{x}(t) = A(t, x(t), u(t))\dot{x}_1(t-\sigma) + f(t, x(t), x(t-\tau), u(t)), \\ t \in [t_0+\sigma, t_0+2\sigma], \\ x(t) = x_1(t), \quad t \in [\hat{\tau}, t_0+\sigma]. \end{cases} \tag{3.3}$$

Analogously can be proved the existence of the solution $x_2(t)$, $t \in [\hat{\tau}, t_0+2\sigma]$ of problem (3.3). Thus, the function

$$x(t) = \begin{cases} \varphi(t), & t \in [\hat{\tau}, t_0), \\ x_1(t), & t \in [t_0, t_0+\sigma], \\ x_2(t), & t \in (t_0+\sigma, t_0+2\sigma] \end{cases}$$

will be the solution of problem (3.1), (3.2) on the interval $[\hat{\tau}, t_0+2\sigma]$. Continuing this procedure, we establish the existence of the unique solution $x(t)$ on the interval $[\hat{\tau}, t_1]$. □

**Theorem 3.2** ([11]). *The solution $x(t), t \in [\hat{\tau}, t_1]$ of problem (2.2), (2.3) can be represented on the interval $I$ in the following integral form:*

$$x(t) = x_0 + \int_{t_0}^{t_0+\sigma} Y(\xi; t, x(\cdot), u(\cdot))A(\xi, x(\xi), u(\xi))\dot{\varphi}(\xi-\sigma)d\xi$$
$$+ \int_{t_0}^{t} Y(\xi; t, x(\cdot), u(\cdot))f(\xi, x(\xi), x(\xi-\tau), u(\xi))d\xi, t \in I, \tag{3.4}$$

*with the condition*

$$x(\xi) = \varphi(\xi), \xi \in [\hat{\tau}, t_0), \tag{3.5}$$

*where $Y(\xi, t, x(\cdot), u(\cdot))$ is the matrix-function satisfying the difference equation*

$$Y(\xi; t, x(\cdot), u(\cdot)) = E + Y(\xi+\sigma; t, x(\cdot), u(\cdot))A(\xi+\sigma, x(\xi+\sigma), u(\xi+\sigma)) \tag{3.6}$$

*on $(t_0, t)$ for any fixed $t \in (t_0, t_1]$ and the condition*

$$Y(\xi; t, x(\cdot), u(\cdot)) = \begin{cases} E, & \xi = t, \\ \Theta, & \xi > t. \end{cases}$$

*Here, $E$ is the identity matrix and $\Theta$ is the zero matrix.*

**Theorem 3.3.** *The solution of the difference equation (3.6) can be represented by the following formula*

$$Y(\xi; t, x(\cdot), u(\cdot)) = \chi(\xi; t)E$$
$$+ \sum_{m=1}^{k} \chi(\xi+m\sigma; t) \prod_{q=m}^{1} A(\xi+q\sigma), x(\xi+q\sigma), u(\xi+q\sigma)), \quad (\xi, t) \in I^2, \tag{3.7}$$

*where*

$$\chi(\xi; t) = \begin{cases} 1, & t_0 \le \xi \le t, \\ 0, & \xi > t, \end{cases} \tag{3.8}$$

*and $k$ is a minimal natural number satisfying the condition $t_1 - k\sigma < t_0$.*

Theorem 3.3 is proved by the step method from the right to the left.

**Remark 3.1.** Using formula (3.7) and applying (3.8), we obtain

$$|Y(\xi;t,x(\cdot),u(\cdot))| \le M, \tag{3.9}$$

where

$$M=\sqrt{n}+\|A\|\sum_{m=1}^{k} m!, \quad \|A\|=\sup\Big\{|A(t,x,u)| : (t,x,u)\in I\times\mathbb{R}_x^n\times\mathbb{R}_u^r\Big\}.$$

Expression (3.4) under condition (3.5) is referred to as the functional-integral equation corresponding to problem (2.2), (2.3). It is not difficult to see that representation (3.7) is not dependent on the initial function $\varphi(t), t\in I_1$.

**Theorem 3.4** ([11])**.** *Problems* (2.2), (2.3) *and* (3.4), (3.5) *are equivalent.*

**Theorem 3.5.** *Let* $x_0(t)=x(t;w_0)$, $t\in[\hat\tau,t_1]$ *and* $x(t)=x(t;w)$, $t\in[\hat\tau,t_1]$, *where* $w_0=(\tau_0,x_{00},\varphi_0(t),u_0(t))\in W$ *and* $w=(\tau,x_0,\varphi(t),u(t))\in W$. *For* $\forall(\xi,t)\in I^2$ *the following inequality*

$$\begin{aligned}|Y(\xi;t,&x(\cdot),u(\cdot))-Y(\xi;t,x_0(\cdot),u_0(\cdot))|\\ &\le L_A\sum_{m=1}^{k}\chi(\xi+m\sigma;t)\|A\|^{m-1}\Big(\sum_{q=m}^{1}\Big[|x(\xi+q\sigma)-x_0(\xi+q\sigma)|\\ &\qquad +|u(\xi+q\sigma)-u_0(\xi+q\sigma)|\Big]\Big)\end{aligned} \tag{3.10}$$

*holds.*

*Proof.* Using formula (3.7), we get

$$\begin{aligned}|Y(\xi;t,&x(\cdot),u(\cdot))-Y(\xi;t,x_0(\cdot),u_0(\cdot))|\\ &\le\sum_{m=1}^{k}\chi(\xi+m\sigma;t)\Big|\prod_{q=m}^{1}A(\xi+q\tau,x(\xi+q\sigma),u(\xi+q\sigma))\\ &\qquad-\prod_{q=m}^{1}A(\xi+q\tau,x_0(\xi+q\sigma),u_0(\xi+q\sigma))\Big|.\end{aligned}$$

Now, we estimate every addend in the right-hand side of the last inequality and shown some regularity in the evaluations. For the first addend, we obtain

$$\begin{aligned}\chi(\xi+\sigma;t)&\Big|A(\xi+\sigma,x(\xi+\sigma),u(\xi+\sigma))-A(\xi+\sigma,x_0(\xi+\sigma),u_0(\xi+\sigma))\Big|\\ &\le\chi(\xi+\sigma;t)L_A\Big(|x(\xi+\sigma)-x_0(\xi+\sigma)|+|u(\xi+\sigma)-u_0(\xi+\sigma)|\Big)\\ &=L_A\chi(\xi+\sigma;t)\|A\|^{1-1}\sum_{m=1}^{1}\Big(|x(\xi+m\sigma)-x_0(\xi+q\sigma)|\\ &\qquad+|u(\xi+m\sigma)-u_0(\xi+m\sigma)|\Big).\end{aligned}$$

For the second addend, we have

$$\begin{aligned}\chi(\xi+2\sigma;t)&\Big|A(\xi+2\sigma,x(\xi+2\sigma),u(\xi+2\sigma))A(\xi+\sigma,x(\xi+\sigma),u(\xi+\sigma))\\ &-A(\xi+2\sigma,x_0(\xi+2\sigma),u_0(\xi+2\sigma))A(\xi+\sigma,x_0(\xi+\sigma),u_0(\xi+\sigma))\Big|\\ &\le\chi(\xi+2\sigma;t)\Big[\Big|A(\xi+2\sigma,x(\xi+2\sigma),u(\xi+2\sigma))-A(\xi+2\sigma,x_0(\xi+2\sigma),\\ &\qquad u_0(\xi+2\sigma))\Big|\Big|A(\xi+\sigma,x(\xi+\sigma),u(\xi+\sigma))\Big|\\ &+\Big|A(\xi+2\sigma,x_0(\xi+2\sigma),u_0(\xi+2\sigma))\Big|\Big|A(\xi+\sigma,x(\xi+\sigma),u(\xi+\sigma))\\ &-A(\xi+\sigma,x_0(\xi+\sigma),u_0(\xi+\sigma))\Big|\le L_A\chi(\xi+2\sigma;t)\|A\|\Big[|x(\xi+2\sigma)-x_0(\xi+2\sigma)|\end{aligned}$$

$$+|u(\xi+2\sigma)-u_0(\xi+2\sigma)|+|x(\xi+\sigma)-x_0(\xi+\sigma)|$$

$$+|u(\xi+\sigma)-u_0(\xi+\sigma)|\Big]=L_A\chi(\xi+2\sigma;t)\|A\|^{2-1}\Big(\sum_{q=2}^{1}\Big[|x(\xi+q\sigma)-x_0(\xi+q\sigma)|$$

$$+|u(\xi+q\sigma)-u_0(\xi+q\sigma)|\Big]\Big).$$

Continuing this process for the $k$ addend, we obtain the following estimation:

$$\chi(\xi+k\sigma;t)\Big|\prod_{q=k}^{1}A(\xi+q\sigma,x(\xi+q\sigma),u_1(\xi+q\tau))$$

$$-\prod_{q=k}^{1}A(\xi+q\sigma,x_0(\xi+q\sigma),u_0(\xi+q\sigma))\Big|$$

$$\leq L_A\chi(\xi+k\sigma;t)\|A\|^{k-1}\Big(\sum_{q=k}^{1}\Big[|x(\xi+q\sigma)-x_0(\xi+q\sigma)|$$

$$+|u(\xi+q\sigma)-u_0(\xi+q\sigma)|\Big]\Big).$$

After summarizing the evaluations, we get formula (3.10). □

**Remark 3.2.** It is not difficult to see that after elementary transformations on the right-hand side of inequality (3.10), we obtain the following formula:

$$|Y(\xi;t,x(\cdot),u(\cdot))-Y(\xi;t,x_0(\cdot),u_0(\cdot))|$$

$$\leq L_A\sum_{m=1}^{k}\sum_{q=m}^{k}\Big(\chi(\xi+q\sigma;t)\|A\|^{q-1}\Big)\Big(|x(\xi+m\sigma)-x_0(\xi+m\sigma)|$$

$$+|u(\xi+m\sigma)-u_0(\xi+m\sigma)|\Big). \quad (3.11)$$

## 4. Proof of Theorem 2.1

For any $w=(\tau,x_0,\varphi(t),u(t))\in W$, there exists the unique solution $x(t)=x(t;w)$ of problem (2.2), (2.3) on the basis of Theorem 3.1. Let $w_0=(\tau_0,x_{00},\varphi_0(t),u_0(t))\in W$ be a fixed element and let $x_0(t)=x(t;w_0)$ be the corresponding solution. By virtue of Theorems 3.2 and 3.3 we have the following functional-integral equalities:

$$x(t)=x_0+\int_{t_0}^{t_0+\sigma}Y(\xi;t,x(\cdot),u(\cdot))A(\xi,x(\xi),u(\xi))\dot{\varphi}(\xi-\sigma)d\xi$$

$$+\int_{t_0}^{t}Y(\xi;t,x(\cdot),u(\cdot))f(\xi,x(\xi),x(\xi-\tau),u(\xi))d\xi,\ \ t\in I,$$

$$x_0(t)=x_{00}+\int_{t_0}^{t_0+\sigma}Y(\xi;t,x_0(\cdot),u_0(\cdot))A(\xi,x_0(\xi),u_0(\xi))\dot{\varphi}_0(\xi-\sigma)d\xi$$

$$+\int_{t_0}^{t}Y(\xi;t,x_0(\cdot),u_0(\cdot))f(\xi,x_0(\xi),x_0(\xi-\tau_0),u_0(\xi))d\xi,\ \ t\in I,$$

with

$$x(\xi)=\varphi(\xi),x_0(\xi)=\varphi_0(\xi),\quad \xi\in[\hat{\tau},t_0).$$

Consequently, for any $t \in I$, we obtain

$$\begin{aligned}|x(t)-x_0(t)| &\le |x_0-x_{00}|+\alpha[t]+\beta[t]\\ &\le |w-w_0|+\alpha[t]+\beta[t],\end{aligned} \tag{4.1}$$

where

$$\alpha[t]=\int_{t_0}^{t_0+\sigma}\Big|Y(\xi;t,x(\cdot),u(\cdot))A(\xi,x(\xi),u(\xi))\dot\varphi(\xi-\sigma)$$
$$-Y(\xi;t,x_0(\cdot),u_0(\cdot))A(\xi,x_0(\xi),u_0(\xi))\dot\varphi_0(\xi-\sigma)\Big|d\xi$$

and

$$\beta[t]=\int_{t_0}^{t}\Big|Y(\xi;t,x(\cdot),u(\cdot))f(\xi,x(\xi),x(\xi-\tau),u(\xi))$$
$$-Y(\xi;t,x_0(\cdot),u_0(\cdot))f(\xi,x_0(\xi),x_0(\xi-\tau_0),u_0(\xi))\Big|d\xi.$$

We now estimate $\alpha[t]$. We have

$$\begin{aligned}\alpha[t] &\le \int_{t_0}^{t_0+\sigma}|Y(\xi;t,x(\cdot),u(\cdot))A(\xi),x(\xi),u(\xi))||\dot\varphi(\xi-\sigma)-\dot\varphi_0(\xi-\sigma)|d\xi\\ &\quad+\int_{t_0}^{t_0+\sigma}|Y(\xi;t,x(\cdot),u(\cdot))A(\xi),x(\xi),u(\xi))\\ &\quad-Y(\xi;t,x_0(\cdot),u_0(\cdot))A(\xi,x_0(\xi),u_0(\xi))|\dot\varphi_0(\xi-\sigma)|d\xi\\ &\le \sigma M\|A\|\|\varphi-\varphi_0\|_1+\|\varphi_0\|_1\alpha_1[t]\le \sigma M\|A\||w-w_0|+\|\varphi_0\|_1\alpha_1[t],\end{aligned} \tag{4.2}$$

where

$$\alpha_1[t]=\int_{t_0}^{t_0+\sigma}|Y(\xi;t,x(\cdot),u(\cdot))A(\xi,x(\xi),u(\xi))$$
$$-Y(\xi;t,x_0(\cdot),u_0(\cdot))A(\xi,x_0(\xi),u_0(\xi))|d\xi$$

(see (3.9)). Further,

$$\begin{aligned}\alpha_1[t] &\le \int_{t_0}^{t_0+\sigma}\Big|Y(\xi;t,x(\cdot),u(\cdot))\Big|\Big|A(\xi,x(\xi),u(\xi))-A(\xi,x_0(\xi),u_0(\xi))\Big|d\xi\\ &\quad+\int_{t_0}^{t_0+\sigma}\Big|Y(\xi;t,x(\cdot),u(\cdot))-Y(\xi;t,x_0(\cdot),u_0(\cdot))\Big|\Big|A(\xi,x_0(\xi),u_0(\xi))\Big|d\xi\\ &\le L_AM\int_{t_0}^{t}\Big(|x(\xi)-x_0(\xi)|+|u(\xi)-u_0(\xi|\Big)d\xi+\|A\|\alpha_2[t]\\ &\le L_AM\int_{t_0}^{t}|x(\xi)-x_0(\xi)|d\xi+L_AM(t_1-t_0)|w-w_0|+\|A\|\alpha_2[t],\end{aligned} \tag{4.3}$$

where

$$\alpha_2[t] = \int_{t_0}^{t} \Big| Y(\xi, x(\xi), u(\xi)) - Y(\xi, x_0(\xi), u_0(\xi)) \Big| d\xi.$$

Using formula (3.11), we get

$$\alpha_2[t] \le L_A \sum_{m=1}^{k} \sum_{q=m}^{k} \int_{t_0}^{t} \Big( \chi(\xi + q\sigma; t) \|A\|^{q-1} \Big) \Big( |x(\xi + m\sigma) - x_0(\xi + m\sigma)|$$

$$+ |u(\xi + m\sigma) - u_0(\xi + m\sigma)| \Big) d\xi \le L_A \sum_{m=1}^{k} \sum_{q=m}^{k} \int_{t_0+m\sigma}^{t+m\sigma} \Big( \chi(\xi + (q-m)\sigma; t) \|A\|^{q-1} \Big)$$

$$\Big( |x(\xi) - x_0(\xi| + |u(\xi) - u_0(\xi)| \Big) d\xi \le L_A \alpha_3[t]$$

$$+ L_A \Big( \sum_{m=1}^{k} \sum_{q=m}^{k} \|A\|^{q-1} \Big) (t_1 - t_0) |w - w_0|, \tag{4.4}$$

where

$$\alpha_3[t] = \sum_{m=1}^{k} \sum_{q=m}^{k} \int_{t_0+m\sigma}^{t+m\sigma} \Big( \chi(\xi + (q-m)\sigma; t) \|A\|^{q-1} \Big) |x(\xi) - x_0(\xi)| d\xi.$$

Obviously,

$$\alpha_3[t] \le \sum_{m=1}^{k} \sum_{q=m}^{k} \int_{t_0}^{t} \Big( \chi(\xi + (q-m)\sigma; t) \|A\|^{q-1} \Big) |x(\xi) - x_0(\xi)| d\xi$$

$$+ \sum_{m=1}^{k} \sum_{q=m}^{k} \int_{t}^{t+m\sigma} \Big( \chi(\xi + (q-m)\sigma; t) \|A\|^{q-1} \Big) |x(\xi) - x_0(\xi)| d\xi.$$

It is clear that if $\xi \in (t, t + m\sigma)$, then $\xi + (q-m)\sigma > t$, i.e., $\chi(\xi + (q-m)\sigma; t) = 0$ (see (3.8)). Thus,

$$\alpha_3[t] \le \sum_{m=1}^{k} \sum_{q=m}^{k} \int_{t_0}^{t} \Big( \chi(\xi + (q-m)\sigma; t) \|A\|^{q-1} \Big) |x(\xi) - x_0(\xi)| d\xi$$

$$\le \sum_{m=1}^{k} \sum_{q=m}^{k} \|A\|^{q-1} \int_{t_0}^{t} |x(\xi) - x_0(\xi)| d\xi. \tag{4.5}$$

According to (4.3)–(4.5), inequality (4.2) implies

$$\alpha[t] \le \alpha_4 |w - w_0| + \alpha_5 \int_{t_0}^{t} |x(\xi) - x_0(\xi)| d\xi, \tag{4.6}$$

where

$$\alpha_4 = \sigma M \|A\| + \|\varphi_0\|_1 L_A (t_1 - t_0)(M + \sum_{m=1}^{k} \sum_{q=1}^{k} \|A\|^q)$$

and

$$\alpha_5 = \|\varphi_0\|_1 L_A (M + \sum_{m=1}^{k} \sum_{q=1}^{k} \|A\|^q).$$

Now, let us estimate $\beta[t]$. We have

$$\beta[t] \leq \int_{t_0}^{t} \Big|Y(\xi;t,x(\cdot),u(\cdot)) - Y(\xi;t,x_0(\cdot),u_0(\cdot))\Big|\Big|f(\xi,x(\xi),x(\xi-\tau))\Big|d\xi$$

$$+\int_{t_0}^{t} \Big|Y(\xi;t,x_0(\cdot),u_0(\cdot))\Big|\Big|f(\xi,x(\xi),x(\xi-\tau),u(\xi))$$

$$-f(\xi,x_0(\xi),x_0(\xi-\tau_0),u_0(\xi))\Big|d\xi \leq \|f\|\sum_{m=1}^{k}\sum_{q=m}^{k}\|A\|^{q-1}\int_{t_0}^{t}\chi(\xi+q\sigma;t)$$

$$\times\Big(|x(\xi+q\sigma)-x_0(\xi+q\sigma)|+|u(\xi+q\sigma)-u_0(\xi+q\sigma)|\Big)d\xi$$

$$+ML_f\int_{t_0}^{t}\Big(|x(\xi)-x_0(\xi)|+|x(\xi-\tau)-x_0(\xi-\tau_0)|+|u(\xi+q\sigma)-u_0(\xi+q\sigma)|\Big)d\xi$$

$$\leq\Big(\|f\|\sum_{m=1}^{k}\sum_{q=m}^{k}\|A\|^{q-1}+ML_f\Big)(t_1-t_0)|w-w_0|+ML_f\int_{t_0}^{t}|x(\xi)-x_0(\xi)|d\xi$$

$$+\sum_{m=1}^{k}\sum_{q=m}^{k}\|A\|^{q-1}\beta_1[t]+ML_f\beta_2[t], \tag{4.7}$$

where

$$\|f\| = \Big\{|f(t,x,y,u)| : t\in I,(x,y,u)\in\mathbb{R}^n_x\times\mathbb{R}^n_x\times\mathbb{R}^r_u\Big\}$$

$$\beta_1[t] = \int_{t_0}^{t}\chi(\xi+q\sigma;t)|x(\xi+q\sigma)-x_0(\xi+q\sigma)|d\xi$$

and

$$\beta_2[t] = \int_{t_0}^{t}|x(\xi-\tau)-x_0(\xi-\tau_0)|d\xi.$$

Further,

$$\beta_1[t] = \int_{t_0+q\sigma}^{t+q\sigma}\chi(\xi;t)|x(\xi)-x_0(\xi)|d\xi \leq \int_{t_0}^{t}|x(\xi)-x_0(\xi)|d\xi \tag{4.8}$$

and

$$\beta_2[t]\leq\beta_3[t]+\beta_4[t],$$

where

$$\beta_3[t]=\int_{t_0}^{t}|x(\xi-\tau)-x_0(\xi-\tau)|d\xi,\quad \beta_4[t]=\int_{t_0}^{t}|x_0(\xi-\tau)-x_0(\xi-\tau_0)|d\xi.$$

Now, let us estimate $\beta_3[t]$. We have

$$\beta_3[t]=\int_{t_0-\tau}^{t-\tau}|x(\xi)-x_0(\xi)|d\xi.$$

It is clear that if $t-\tau<t_0$, then

$$\beta_3[t]\leq\|\varphi-\varphi_0\|_1(t_1-t_0)\leq(t_1-t_0)|w-w_0|,$$

if $t-\tau>t_0$, then

$$\beta_3[t]=\int_{t_0-\tau}^{t_0}|\varphi(\xi)-\varphi_0(\xi)|d\xi+\int_{t_0}^{t-\tau}|x(\xi)-x_0(\xi)|d\xi$$

$$\le\tau\|\varphi-\varphi_0\|_1+\int_{t_0}^{t}|x(\xi)-x_0(\xi)|d\xi\le\tau_2|w-w_0|+\int_{t_0}^{t}|x(\xi)-x_0(\xi)|d\xi.$$

Thus

$$\beta_3[t]\le(t_1-t_0+\tau_2)|w-w_0|)+\int_{t_0}^{t}|x(\xi)-x_0(\xi)|d\xi.$$

We introduce the notations

$$s_1=\min\{t_0+\tau,t_0+\tau_0\},\quad s_2=\max\{t_0+\tau,t_0+\tau_0\}.$$

It is clear that

$$s_2-s_1<|\tau-\tau_0|<|w-w_0|.$$

If $t\in[t_0,s_1]$, then $t-\tau\le t_0$ and $t-\tau_0\le t_0$; if $t\in[s_2,t_1]$, then $t-\tau\ge t_0$ and $t-\tau_0\ge t_0$. Moreover, there exists a number $K>0$ such that

$$|\dot x_0(t)|\le K \text{ almost everywhere on } I,$$

[9, Lemma 3.4]. Therefore,

$$\beta_4[t]\le\int_{t_0}^{s_1}\Big|\int_{\xi-\tau}^{\xi-\tau_0}|\dot\varphi_0(\varsigma)|d\varsigma\Big|d\xi+\int_{s_1}^{s_2}|x_0(\xi-\tau)-x_0(\xi-\tau_0)|d\xi$$

$$+\int_{s_2}^{t}\Big|\int_{\xi-\tau}^{\xi-\tau_0}|\dot x_0(\varsigma)|d\varsigma\Big|d\xi\le\|\varphi_0\|_1|\tau-\tau_0|(t_1-t_0)+2\|x_0\|(s_2-s_1)$$

$$+K|\tau-\tau_0|(t_1-t_0)\le\Big(\|\varphi_0\|_1|(t_1-t_0)+2\|x_0\|+K(t_1-t_0)\Big)|w-w_0|,$$

where

$$\|x_0\|=\sup\{|x_0(t)|:t\in[\hat\tau,t_1]\}.$$

On the basis of estimates $\beta_3[t]$ and $\beta_4[t]$, we get

$$\beta_2[t]\le\Big(t_1-t_0+\tau_2+\|\varphi_0\|_1(t_1-t_0)+2\|x_0\|+K(t_1-t_0)\Big)|w-w_0|$$
$$+\int_{t_0}^{t}|x(\xi)-x_0(\xi)|d\xi. \tag{4.9}$$

From (4.7), by virtue of (4.8) and (4.9), we obtain

$$\beta[t]\le\beta_5|w-w_0|+\beta_6\int_{t_0}^{t}|x(\xi)-x_0(\xi)|d\xi, \tag{4.10}$$

where

$$\beta_5=\Big(\|f\|\sum_{m=1}^{k}\sum_{q=m}^{k}\|A\|^{q-1}+ML_f\Big)(t_1-t_0)+ML_f\Big(t_1-t_0+\tau_2+\|\varphi_0\|_1(t_1-t_0)$$
$$+2\|x_0\|+K(t_1-t_0)\Big)$$

and

$$\beta_6 = 2ML_f + \sum_{m=1}^{k}\sum_{q=m}^{k} \|A\|^{q-1}.$$

From (4.1), according to (4.6) and (4.10), we get the following estimate:

$$|x(t) - x_0(t)| \le (1 + \alpha_4 + \beta_5)|w - w_0| + (\alpha_5 + \beta_6)\int_{t_0}^{t} |x(\xi) - x_0(\xi)|d\xi, \quad t \in I.$$

By the Gronwall–Bellman inequality, from the last expression, we obtain

$$|x(t) - x_0(t)| \le (1 + \alpha_4 + \beta_5)|w - w_0|e^{(\alpha_5+\beta_6)(t_1-t_0)}.$$

It is clear that if

$$|w - w_0| < \delta(\varepsilon) = \frac{\varepsilon}{1 + \alpha_4 + \beta_5}\, e^{-(\alpha_5+\beta_6)(t_1-t_0)},$$

then

$$|x(t) - x_0(t)| < \varepsilon, \quad \forall t \in I.$$

## Acknowledgement

This work was supported partly by the Shota Rustaveli National Science Foundation of Georgia (SRNSFG), Grant No. YS-21-554.

## References

1. C. T. H. Baker, G. A. Bocharov, F. A. Rihan, Neutral delay differential equations in the modelling of cell growth. *J. Egyptian Math. Soc.* **16** (2008), no. 2, 133–160.
2. N. Gorgodze, I. Ramishvili, T. Tadumadze, Continuous dependence of a solution of a neutral functional differential equation on the right-hand side and initial data taking into account perturbations of variable delays. *Georgian Math. J.* **23** (2016), no. 4, 519–535.
3. J. Hale, *Theory of Functional Differential Equations.* Second edition. Applied Mathematical Sciences, vol. 3. Springer-Verlag, New York-Heidelberg, 1977.
4. G. Kharatishvili, T. Tadumadze, N. Gorgodze, Continuous dependence and differentiability of solution with respect to initial data and right-hand side for differential equations with deviating argument. *Mem. Differential Equations Math. Phys.* **19** (2000), 3–105.
5. G. L. Kharatishvili, T. A. Tadumadze, Formulas for the variation of a solution and optimal control problems for differential equations with retarded arguments. (Russian) *translated from Sovrem. Mat. Prilozh. No. 25, Optimal. Upr.* (2005), 3–166; *J. Math. Sci. (N.Y.)* **140** (2007), no. 1, 1–175.
6. G. Kharatishvili, T. Tadumadze, N. Gorgodze, Dynamic mathematical model of output the production and neutral optimal problem. *Rep. Enlarged Sess. Semin. I. Vekua Appl. Math.* **17** (2002), no. 1-2, 36–38.
7. V. Kolmanovskii, A. Myshkis, *Introduction to the Theory and Applications of Functional-differential Equations.* Mathematics and its Applications, 463. Kluwer Academic Publishers, Dordrecht, 1999.
8. A. Nachaoui, T. Shavadze, T. Tadumadze, The local representation formula of solution for the perturbed controlled differential equation with delay and discontinuous initial condition. *Mathematics* **8** (2020), no. 10, 1845.
9. A. Nachaoui, T. Shavadze, T. Tadumadze, On the representation of solution for the perturbed quasi-linear controlled neutral functional-differential equation with the discontinuous initial condition. *Georgian Math. J.* **31** (2024), no. 5, 847–860.
10. F. A. Rihan, Adjoint sensitivity analysis of neutral delay differential models. *JNAIAM. J. Numer. Anal. Ind. Appl. Math.* **5** (2010), no. 1-2, 95–101.
11. T. Shavadze, I. Ramishvili, T. Tadumadze, A controlled integral equation and properties of its kernel. *Bull. TICMI* **26** (2022), no. 2, 81–88.
12. T. Shavadze, T. Tadumadze, Existence of an optimal element for a class of neutral optimal problems. *Mem. Differ. Equ. Math. Phys.* **86** (2022), 127–138.
13. T. Tadumadze, Variation formulas of solutions for functional differential equations with several constant delays and their applications in optimal control problems. *Mem. Differ. Equ. Math. Phys.* **70** (2017), 7–97.
14. T. Tadumadze, Ph. Dvalishvili, T. Shavadze, On the representation of solution of the perturbed controlled differential equation with delay and continuous initial condition. *Appl. Comput. Math.* **18** (2019), no. 3, 305–315.
15. T. Tadumadze, N. Gorgodze, Variation formulas of a solution and initial data optimization problems for quasi-linear neutral functional differential equations with discontinuous initial condition. *Mem. Differ. Equ. Math. Phys.* **63** (2014), 1–77.

16. T. A. Tadumadze, N. Z. Gorgodze, I. V. Ramishvili, On the well-posedness of the Cauchy problem for quasilinear differential equations of neutral type. (Russian) *translated from Sovrem. Mat. Fundam. Napravl.* **19** (2006), 179–197; *J. Math. Sci. (N.Y.)* **151** (2008), no. 6, 3611–3630.

(Received 07.07.2023)

[1]A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 2 Merab Aleksidze II Lane, Tbilisi 0193, Georgia

[2]Department of Mathematics Georgian Technical University, 75 Kostava Str., Tbilisi 0175, Georgia

[3]Alte University, 2 University Str., Tbilisi 0177, Georgia

[4]I. Javakhishvili Tbilisi State University, Department of Mathematics and I. Vekua Institute of Applied Mathematics, 11 University Str., Tbilisi 0186, Georgia

[5]I. Vekua Institute of Applied Mathematics of Iv. Javakhishvili Tbilisi State University, 11 University Str., Tbilisi 0186, Georgia

*Email address*: givi.berikelashvili@ tsu.ge
*Email address*: ia.ramis@yahoo.com
*Email address*: tea.shavadze@gmail.com
*Email address*: tamaz.tadumadze@tsu.ge