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WEIGHTED NORM INEQUALITIES IN THE VARIABLE LEBESGUE SPACES FOR THE BERGMAN PROJECTOR ON THE UNIT BALL OF $\mathbb{C}^n$

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Abstract. In this work we extend the theory of Békollè–Bonami B_p weights. Here, we replace the constant p by a non-negative measurable function $p(\cdot)$, which is a log-Hölder continuous with lower bound 1. We show that the Bergman projector on the unit ball of $\mathbb{C}^n$ is continuous on the weighted variable Lebesgue spaces $L^{p(\cdot)}(w)$ if and only if w belongs to the generalised Békollè–Bonami class $B_{p(\cdot)}$. To achieve this, we define a maximal function and show that it is bounded on $L^{p(\cdot)}(w)$ if $w \in B_{p(\cdot)}$. We next state and prove a weighted extrapolation theorem that allows us to draw a conclusion.

1. Introduction

The purpose of this work is to generalise the Békollè–Bonami theorem [2] for the Bergman projector on the unit ball $\mathbb{B}$ of $\mathbb{C}^n$ to the case of weighted variable Lebesgue spaces. The case of the unweighted variable Lebesgue spaces was treated by Chacon and Rafeiro [5, 15]. These authors showed that the Bergman projector is bounded on variable Lebesgue spaces for exponent functions $p(\cdot)$, which are log-Hölder continuous, with lower bound 1 (cf. Definition 1.1, below). The ingredients of their proof are: the classical Békollè–Bonami theorem, the boundedness of the Hardy–Littlewood maximal function on variable Lebesgue spaces and an extrapolation theorem. For the basic properties of variable Lebesgue spaces, e.g., the boundedness of the Hardy–Littlewood maximal function and an extrapolation theorem, we refer to [9, 11].

The σ-algebra on $\mathbb{B}$ is the Borel σ-algebra. Let ν be a positive measure on $\mathbb{B}$. The variable Lebesgue space on $\mathbb{B}$, denoted by $L^{p(\cdot)}(\nu)$, is a generalisation of the classical Lebesgue spaces obtained by replacing the constant exponent p by a measurable exponent function $p(\cdot) : \mathbb{B} \to [0, \infty)$. We denote by $\mathcal{P}(\mathbb{B})$ this family of all exponent functions $p(\cdot)$ on $\mathbb{B}$. For a measurable subset E of $\mathbb{B}$, we introduce the following notation:

$$p_-(E) = \operatorname{ess\,inf}_{z\in E} p(z) \quad \text{and} \quad p_+(E) = \operatorname{ess\,sup}_{z\in E} p(z)$$

and use the notation $p_- = p_-(\mathbb{B})$ and $p_+ = p_+(\mathbb{B})$. We denote by $\mathcal{P}_+(\mathbb{B})$ the subfamily of $\mathcal{P}(\mathbb{B})$ consisting of such $p(\cdot)$ that $p_+ < \infty$. More precisely, for $p(\cdot) \in \mathcal{P}(\mathbb{B})$, we say that $f \in L^{p(\cdot)}(\nu)$ if for some $\lambda > 0$, $\rho_{p(\cdot)}(\frac{f}{\lambda}) < \infty$, where

$$\rho_{p(\cdot)}(f) := \int_{\mathbb{B}} |f(z)|^{p(z)} d\nu(z).$$

For $p(\cdot) \in \mathcal{P}_+(\mathbb{B})$, this definition can be simplified as follows: $f \in L^{p(\cdot)}(\nu)$ if $\rho_{p(\cdot)}(f) < \infty$. When ν is a σ-finite measure and $p(\cdot) \in \mathcal{P}(\mathbb{B})$ is such that $p(\cdot) \geq 1$, the functional

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \quad \rho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1\right\}$$

is a norm on the space $L^{p(\cdot)}(\nu)$, equipped with this norm, $L^{p(\cdot)}(\nu)$ is a Banach space.

We denote by μ the Lebesgue measure on $\mathbb{B}$. A non-negative locally integrable function on $\mathbb{B}$ is called a weight. If $d\nu = w d\mu$ for a weight w, we call $L^{p(\cdot)}(\nu)$ a weighted variable Lebesgue space. In

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the sequel, α is a positive number and we set $d\mu_\alpha(z) = (1-|z|^2)^{\alpha-1}d\mu(z)$. We shall focus on the weighted variable Lebesgue space $L^{p(\cdot)}(wd\mu_\alpha)$, which we simply denote by $L^{p(\cdot)}(w)$.

In this paper, we take

$$d(z,\zeta) = \begin{cases} ||z|-|\zeta|| + \left|1 - \frac{\langle z,\zeta\rangle}{|z||\zeta|}\right| & \text{if } z,\zeta \in \overline{\mathbb{B}} \setminus \{0\}, \\ |z|+|\zeta| & \text{if } z=0 \text{ or } \zeta = 0. \end{cases}$$

Here, for $z=(z_1,\ldots,z_n)$ and $\zeta = (\zeta_1,\ldots,\zeta_n) \in \mathbb{B}$, we have set

$$\langle z,\zeta\rangle = z_1\overline{\zeta_1} + \cdots + z_n\overline{\zeta_n} \qquad \text{and} \qquad |z| = \langle z,z\rangle^{\frac{1}{2}}.$$

This application d is a pseudo-distance on $\mathbb{B}$. Explicitly, for all $z,\zeta,\xi \in \mathbb{B}$, we have $d(z,\zeta) \le 2\left(d(z,\xi)+d(\xi,\zeta)\right)$ and $0 \le d(z,\zeta) < 3$.

In addition, for $z \in \mathbb{B}$ and $r > 0$, we denote by

$$B(z,r) = \{\zeta \in \mathbb{B} : \, d(z,\zeta) < r\}$$

the open pseudo-ball centred at z and of radius $r > 0$.

Definition 1.1. A function $p(\cdot) \in \mathcal{P}(\mathbb{B})$ is log-Hölder continuous on $\mathbb{B}$ if there exists a positive constant $c > 0$ such that for all $z,\zeta \in \mathbb{B}$,

$$|p(z)-p(\zeta)| \le \frac{c}{\ln(e + \frac{1}{d(z,\zeta)})} \quad \text{if} \quad z \neq \zeta.$$

We denote by $\mathcal{P}^{\log}(\mathbb{B})$ the space of all log-Hölder continuous functions on $\mathbb{B}$. It is easily checked that $\mathcal{P}^{\log}(\mathbb{B}) \subset \mathcal{P}_+(\mathbb{B})$. As usual, we set $\mathcal{P}^{\log}_{\pm}(\mathbb{B}) = \left\{p(\cdot) \in \mathcal{P}^{\log}(\mathbb{B}) : \, p_- > 1\right\}$.

An example of a member of $\mathcal{P}^{\log}_{\pm}(\mathbb{B})$ is $p(z) = 2 + \sin|z|$. We denote by $\mathcal{P}_-(\mathbb{B})$ the subfamily of $\mathcal{P}(\mathbb{B})$ consisting of such $p(\cdot)$ that $p_- > 1$. So, $\mathcal{P}^{\log}_{\pm}(\mathbb{B}) = \mathcal{P}^{\log}(\mathbb{B}) \cap \mathcal{P}_-(\mathbb{B})$.

Definition 1.2. We denote by $\mathcal{B}$ the collection of pseudo-balls B of $\mathbb{B}$ such that $\overline{B} \cap \partial\mathbb{B} \neq \emptyset$. Next, we define the maximal function m_α by

$$m_\alpha f(z) = \sup_{B\in\mathcal{B}} \frac{\chi_B(z)}{\mu_\alpha(B)} \int_B |f(\zeta)| d\mu_\alpha(\zeta).$$

Observe that $\mathcal{B}$ from Lemma 2.1 is the set of pseudo-balls that are tangent to the boundary of $\mathbb{B}$.

In the classical Lebesgue spaces, we have the following

Definition 1.3. Let $p > 1$ be a constant exponent. The Békollè–Bonami B_p−weight class consists of weights w such that

$$\sup_{B\in\mathcal{B}} \left(\frac{1}{\mu_\alpha(B)} \int_B w d\mu_\alpha\right)\left(\frac{1}{\mu_\alpha(B)} \int_B w^{-\frac{1}{p-1}} d\mu_\alpha\right)^{p-1} < \infty.$$

This definition of B_p is equivalent to the following definition:

$$\sup_{B\in\mathcal{B}} \frac{1}{\mu_\alpha(B)} \|w^{\frac{1}{p}}\chi_B\|_p \|w^{-\frac{1}{p}}\chi_B\|_{p'} < \infty,$$

where p' is the conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$.

In the same spirit, we introduce a variable generalisation of the B_p−weight class. Analogously to the classical case, for $p(\cdot) \in \mathcal{P}_-(\mathbb{B})$, we say that $p'(\cdot)$ is the conjugate exponent function of $p(\cdot)$ if for all $z \in \mathbb{B}$, we have

$$\frac{1}{p(z)} + \frac{1}{p'(z)} = 1.$$

Moreover, we set $\mathcal{P}_{\pm}(\mathbb{B}) = \mathcal{P}_-(\mathbb{B}) \cap \mathcal{P}_+(\mathbb{B})$, the subfamily of $\mathcal{P}(\mathbb{B})$ consisting of those exponent functions $p(\cdot)$ such that $1 < p_- \le p_+ < \infty$. We now define the variable Békollè–Bonami classes of weights.

Definition 1.4. Let $p(\cdot) \in \mathcal{P}_{\pm}(\mathbb{B})$. A weight w belongs to the variable Békollè–Bonami class on $\mathbb{B}$, denoted by $B_{p(\cdot)}$ if

$$[w]_{B_{p(\cdot)}} := \sup_{B\in\mathcal{B}} \frac{1}{\mu_\alpha(B)} \|w^{\frac{1}{p(\cdot)}}\chi_B\|_{p(\cdot)}\|w^{-\frac{1}{p(\cdot)}}\chi_B\|_{p'(\cdot)} < \infty.$$

We define the operator P_α on $L^1(\mathbb{B}, d\mu_\alpha)$ by

$$P_\alpha f(z) = \int_{\mathbb{B}} \frac{f(\zeta)}{(1-\langle z,\zeta\rangle)^{n+\alpha}} d\mu_\alpha(\zeta).$$

The restriction to $L^2(\mathbb{B}, d\mu_\alpha)$ of the operator P_α is called the Bergman projector of $\mathbb{B}$. We also define the positive Bergman operator P_α^+ by

$$P_\alpha^+ f(z) = \int_{\mathbb{B}} \frac{f(\zeta)}{|1-\langle z,\zeta\rangle|^{n+\alpha}} d\mu_\alpha(\zeta).$$

We now recall the classical Békollè–Bonami theorem:

Theorem 1.5 ([2]). *Let w be a non-negative measurable function and let* $1 < p < \infty$ (p is a constant exponent). *The following two assertions are equivalent:*

1. *The Bergman operator P_α is bounded on $L^p(wd\mu_\alpha)$;*
2. *$w \in B_p$.*

Moreover, P_α^+ is bounded on $L^p(wd\mu_\alpha)$ if $w \in B_p$.

The purpose of this work is to prove the following generalisation of the previous theorem.

Theorem 1.6. *Let w be a non-negative measurable function and $p(\cdot) \in \mathcal{P}_{\pm}^{\log}(\mathbb{B})$. The following two assertions are equivalent:*

1. *The Bergman operator P_α is bounded on $L^{p(\cdot)}(wd\mu_\alpha)$;*
2. *$w \in B_{p(\cdot)}$.*

Moreover, P_α^+ is bounded on $L^{p(\cdot)}(wd\mu_\alpha)$ if $w \in B_{p(\cdot)}$.

The problem under study is trivial if $w(\mathbb{B}) = 0$, i.e., $w \equiv 0$ a.e. on $\mathbb{B}$. We assume that $w(\mathbb{B}) > 0$.

In [12], Diening and Hästo introduced the variable Muckenhoupt weight class $A_{p(\cdot)}$ on $\mathbb{R}^n$ and showed that for $p(\cdot) \in \mathcal{P}_{\pm}^{\log}(\mathbb{R}^n)$, the Hardy–Littlewood maximal function is bounded on $L^{p(\cdot)}(w)$ only if $w \in A_{p(\cdot)}$. In order to manage the necessary condition, they introduced a new class $A_{p(\cdot)}^+$ which coincides with $A_{p(\cdot)}$ when $p(\cdot) \in \mathcal{P}_{\pm}^{\log}(\mathbb{R}^n)$, but whose condition is easier to check. Later, Cruz–Uribe and these two authors [8] gave a new proof of this result using the Calderón–Zygmund decomposition and they also proved the reverse implication. Very recently, Cruz–Uribe and Cummings [7] extended the result of [8] to the spaces of homogeneous type.

In this paper, we use the technique of [8, 12] to manage the proof of the necessary condition in Theorem 1.6. Precisely, we shall introduce a new class denoted by $B_{p(\cdot)}^+$ which coincides with $B_{p(\cdot)}$ when $p(\cdot) \in \mathcal{P}_{\pm}^{\log}(\mathbb{B})$. To deal with the sufficient condition, we rely on the result of [7] about the boundedness on $L^{p(\cdot)}(w)$ of the Hardy–Littlewood maximal function on the space of homogeneous type $\mathbb{B}$, for $p(\cdot) \in \mathcal{P}_{\pm}^{\log}(\mathbb{B})$ and w in the corresponding class of weights $A_{p(\cdot)}$. The proof then follows two steps. First, we use this result to show that, for $p(\cdot) \in \mathcal{P}_{\pm}^{\log}(\mathbb{B})$, the maximal function m_α is bounded on $L^{p(\cdot)}(w)$ if $w \in B_{p(\cdot)}$. Secondly, we lean on the first step to define a new extrapolation theorem which allows us to complete the proof of the sufficient condition in Theorem 1.6.

The rest of our paper is organised as follows. In Section 2, we recall some preliminaries. Next, in Section 3, we review the properties of weighted variable Lebesgue spaces, variable Békollè–Bonami and Muckenhoupt classes of weights. In the end of this section, we state the theorem of Cruz–Uribe and Cummings about the boundedness on $L^{p(\cdot)}(w)$ of the Hardy–Littlewood maximal function on $\mathbb{B}$. In Section 4, we prove the necessity of the conditions $w^{\frac{1}{p(\cdot)}} \in L^{p(\cdot)}(d\mu_\alpha)$ and $w^{-\frac{1}{p(\cdot)}} \in L^{p'(\cdot)}(d\mu_\alpha)$ in Theorem 1.6. In Section 5, we define and study the class $B_{p(\cdot)}^+$ and show the identity $B_{p(\cdot)}^+ = B_{p(\cdot)}$.

In Section 6, we prove the necessary condition in Theorem 1.6. In Section 7, we show that the maximal function m_α is bounded on $L^{p(\cdot)}(w)$ if $w \in B_{p(\cdot)}$. Finally, in Section 8, we prove a weighted extrapolation theorem from which we deduce a proof of the sufficient condition in Theorem 1.6.

Let a and b be two positive numbers. Throughout the paper, we write $a \lesssim b$ if there exists $C > 0$ such that $a \leq Cb$. We write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$.

2. Preliminaries

In this section, we present some background material regarding the unit ball of $\mathbb{C}^n$ as a space of homogeneous type and the variable exponent Lebesgue spaces.

2.1. **The unit ball is a space of homogeneous type.** In this subsection, we recall some lemmas from [2], where $(\mathbb{B}, d, \mu_\alpha)$ was first considered.

Lemma 2.1. *Let $z \in \mathbb{B}$ and $R > 0$. The pseudo-ball $B(z, r)$ meets the boundary of $\mathbb{B}$ (in other words, $B \in \mathcal{B}$) if and only if $R > 1 - |z|$.*

Lemma 2.2. *Let $z \in \mathbb{B}$ and $0 < R \leq 3$, we have*

$$\mu_\alpha(B(z, R)) \simeq R^{n+1} \left(\max(R, 1 - |z|)\right)^{\alpha - 1}.$$

Remark 2.3. From Lemma 2.1 and Lemma 2.2, if $B(z, R) \in \mathcal{B}$, we have

$$\mu_\alpha(B(z, R)) \simeq R^{n+\alpha}.$$

From Lemma 2.2, we deduce that $(\mathbb{B}, d, \mu_\alpha)$ is a space of homogeneous type. Next, we have the following

Lemma 2.4 ([7])**.** *There exist two positive constants C and γ such that for all $\zeta \in B(z, R)$, we have*

$$\mu_\alpha(B(\zeta, r)) \geq C\left(\frac{r}{R}\right)^\gamma \mu_\alpha(B(z, R))$$

for every $0 < r \leq R < \infty$.

2.2. **Variable exponent Lebesgue spaces.** We denote by $\mathcal{M}$ the space of complex-valued measurable functions defined on $\mathbb{B}$. Let ν be a positive measure on $\mathbb{B}$. The family $\mathcal{P}(\mathbb{B})$ of variable exponents is defined in Introduction. In the rest of the paper, we take $p(\cdot) \in \mathcal{P}(\mathbb{B})$. The next definitions, properties and propositions are stated in [9, 11, 15]. We first recall some properties of the modular functional $\rho_{p(\cdot)} : \mathcal{M} \to [0, \infty]$, defined in the introduction as

$$\rho_{p(\cdot)}(f) = \int_{\mathbb{B}} |f(z)|^{p(z)} d\nu(z).$$

Proposition 2.5. *Let $p(\cdot) \in \mathcal{P}(\mathbb{B})$ be such that $p(\cdot) \geq 1$.*

1. *For all $f \in \mathcal{M}$, $\rho_{p(\cdot)}(f) \geq 0$ and $\rho_{p(\cdot)}(f) = \rho_{p(\cdot)}(|f|)$.*
2. *For all $f \in \mathcal{M}$ if $\rho_{p(\cdot)}(f) < \infty$, then $|f(z)| < \infty$ a.e. on $\mathbb{B}$.*
3. *$\rho_{p(\cdot)}$ is convex. In particular, for $0 < \alpha \leq 1$ and $f \in \mathcal{M}$, $\rho_{p(\cdot)}(\alpha f) \leq \alpha \rho_{p(\cdot)}(f)$ and for $\alpha \geq 1$, $\alpha \rho_{p(\cdot)}(f) \leq \rho_{p(.)}(\alpha f)$.*
4. *$\rho_{p(\cdot)}(f) = 0$ if and only if $f(z) = 0$ a.e. on $\mathbb{B}$.*
5. *If for almost all $z \in \mathbb{B}$, $|f(z)| \leq |g(z)|$, then $\rho_{p(\cdot)}(f) \leq \rho_{p(\cdot)}(g)$.*
6. *If there exists $\beta > 0$ such that $\rho_{p(\cdot)}(\frac{f}{\beta}) < \infty$, then the function $\lambda \longmapsto \rho_{p(\cdot)}(\frac{f}{\lambda})$ is continuous and non-increasing on $[\beta, \infty[$. In addition,*

$$\lim_{\lambda \longrightarrow \infty} \rho_{p(.)}\left(\frac{f}{\lambda}\right) = 0.$$

For $p(\cdot) \in \mathcal{P}(\mathbb{B})$, the variable Lebesgue space $L^{p(\cdot)}(d\nu)$ is defined in the introduction.

Proposition 2.6 ([9, Theorem 2.7.2])**.** *Let $p(\cdot) \in \mathcal{P}_+(\mathbb{B})$ be such that $p(\cdot) \geq 1$. Then the subspace of continuous functions with a compact support in $\mathbb{B}$ is dense in the space $L^{p(\cdot)}(d\nu)$.*

We next recall the Hölder inequality in the variable exponent context.

Proposition 2.7 ([9, Theorem 2.26, Corollary 2.28]).

1. *Let $p(\cdot) \in \mathcal{P}(\mathbb{B})$ be such that $p(\cdot) \geq 1$. Then for all $f, g \in \mathcal{M}$, we have*
$$\int_{\mathbb{B}} |fg| d\nu \leq 2\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}.$$
2. *Let $r(\cdot)$, $q(\cdot) \in \mathcal{P}(\mathbb{B})$ such that $r(\cdot), q(\cdot) \geq 1$ and $\frac{1}{q(x)} + \frac{1}{r(x)} \leq 1$ for all $x \in \mathbb{B}$. Define $p(\cdot) \in \mathcal{P}(\mathbb{B})$ such that $p(\cdot) \geq 1$, by*
$$\frac{1}{p(x)} = \frac{1}{q(x)} + \frac{1}{r(x)}.$$
Then there exists a constant K such that for all $f \in L^{q(\cdot)}$ and $g \in L^{r(\cdot)}$, $fg \in L^{p(\cdot)}$ and
$$\|fg\|_{p(\cdot)} \leq K\|f\|_{q(\cdot)}\|g\|_{r(\cdot)}.$$

We record the following useful

Remark 2.8. The property $p(\cdot) \in \mathcal{P}^{\log}_{\pm}(\mathbb{B})$ is also true for $p'(\cdot)$.

The following lemma will be useful.

Lemma 2.9. *Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{B})$. Let $B = B(x, R)$ be a pseudo-ball of $\mathbb{B}$ such that $R < \frac{1}{4}$. Then*
$$p_+(B) - p_-(B) \leq \frac{c}{\ln(\frac{1}{4R})}.$$

Lemma 2.10. *Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{B})$. There exist two positive constants $C_1 = C_1(\alpha, n, p(\cdot))$ and $C_2 = C_2(\alpha, n, p(\cdot))$ such that for every pseudo-ball B of $\mathbb{B}$, we have*
$$\mu_\alpha(B)^{p_-(B)-p_+(B)} \leq C_1 \text{ and } \mu_\alpha(B)^{p_+(B)-p_-(B)} \leq C_2.$$

Proof. Since $p_+(B)-p_-(B) \geq 0$, $\mu_\alpha(\mathbb{B}) < \infty$, $B \subset \mathbb{B}$ and $p(\cdot)$ bounded, we have the second inequality. We are going to prove the first inequality. Suppose that $B = B(z, R)$.

1. If $R \geq \frac{1}{16}$, from Lemma 2.2, there exists $C > 0$ such that $\mu_\alpha(B) \geq CR^{n+\alpha}$ and as $p_-(B) - p_+(B) \leq 0$, we obtain
$$\mu_\alpha(B)^{p_-(B)-p_+(B)} \leq (CR^{n+\alpha})^{(p_-(B)-p_+(B))} \leq (C(16)^{n+1})^{(p_+(B)-p_-(B))}.$$
2. If $R < \frac{1}{16}$, from Lemma 2.4 and Lemma 2.2, there exist positive constants C, C' and γ such that
$$\mu_\alpha(B(z,R)) \geq C\,(4R)^\gamma\,\mu_\alpha\Big(B\Big(z, \frac{1}{16}\Big)\Big) \geq C'R^\gamma. \tag{2.1}$$
Thus, since $p_-(B) - p_+(B) \leq 0$, by Lemma 2.9, it follows from (2.1) that
$$\begin{aligned}\mu_\alpha(B)^{p_-(B)-p_+(B)} &\leq (C'R^\gamma)^{p_-(B)-p_+(B)}\\ &\lesssim R^{-\frac{\gamma c}{\ln\left(\frac{1}{4R}\right)}}\\ &\leq \exp(2\gamma c).\end{aligned}$$
□

From the previous lemma, we easily deduce the following corollary.

Corollary 2.11. *Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{B})$. There exists a constant $C = C(\alpha, n, p(\cdot)) > 1$ such that for every pseudo-ball B of $\mathbb{B}$ and every $z \in B$, we have*
$$\frac{1}{C} \leq \mu_\alpha(B)^{p_-(B)-p(z)} \leq C.$$

3. Weighted Variable Lebesgue Spaces, Variable Békollè–Bonami and Muckenhoupt Classes of Weights

3.1. **Weighted variable Lebesgue spaces.** Let w be a weight and let $p(\cdot) \in \mathcal{P}(\mathbb{B})$ be such that $p(\cdot) \geq 1$. The corresponding weighted variable exponent Lebesgue space $L^{p(\cdot)}(wd\mu_\alpha)$ consists of those $f \in \mathcal{M}$ which satisfy the estimate

$$\rho_{p(\cdot),w}(f) = \int_{\mathbb{B}} |f(z)|^{p(z)} w(z) d\mu_\alpha(z) < \infty.$$

We also denote it by $L^{p(\cdot)}(\mathbb{B}, w)$ or, simply, $L^{p(\cdot)}(w)$, and denote its norm by $\|\cdot\|_{p(\cdot),w}$. It is easy to check that

$$\|f\|_{p(\cdot),w} = \|f w^{\frac{1}{p(\cdot)}}\|_{p(\cdot)}.$$

In the sequel, we shall adopt the following notation:

$$w' := w^{1-p'(\cdot)}.$$

We recall the notion of a subordinate norm on $L^{p(\cdot)}(w)$ defined by

$$\|f\|'_{p(\cdot),w} := \sup_{\|g\|_{p'(\cdot),w'}=1} \left| \int_{\mathbb{B}} f(\zeta)\overline{g(\zeta)} d\mu_\alpha(\zeta) \right|.$$

We next recall the following

Proposition 3.1 ([9, 11, Corollary 2.7.5])**.** *Let $p(\cdot) \in \mathcal{P}(\mathbb{B})$ such that $p(\cdot) \geq 1$ and let w be a weight. Then*

$$\|f\|_{p(\cdot),w} \leq \|f\|'_{p(\cdot),w} \leq 2\|f\|_{p(\cdot),w}.$$

The following lemma will be very useful.

Lemma 3.2 ([9, 11])**.** *Let w be a non-negative measurable function and let $p(\cdot) \in \mathcal{P}_+(\mathbb{B})$ be such that $p_- > 0$. Then for every $f \in \mathcal{M}$ whose support is E, the following double inequality*

$$\min\left(\rho_{p(\cdot),w}(f)^{\frac{1}{p_-(E)}}, \rho_{p(\cdot),w}(f)^{\frac{1}{p_+(E)}}\right) \leq \|f\|_{p(\cdot),w} \leq \max\left(\rho_{p(\cdot),w}(f)^{\frac{1}{p_-(E)}}, \rho_{p(\cdot),w}(f)^{\frac{1}{p_+(E)}}\right)$$

holds.

It is equivalent to

$$\min\left(\|f\|_{p(\cdot),w}^{p_-(E)}, \|f\|_{p(\cdot),w}^{p_+(E)}\right) \leq \rho_{p(\cdot),w}(f) \leq \max\left(\|f\|_{p(\cdot),w}^{p_-(E)}, \|f\|_{p(\cdot),w}^{p_+(E)}\right).$$

3.2. **Variable Békollè–Bonami classes of weights.** Concerning the variable Békollè–Bonami weight class $B_{p(\cdot)}$, we record the following elementary

Remark 3.3. Let $p(\cdot) \in \mathcal{P}_\pm(\mathbb{B})$. If $w \in B_{p(\cdot)}$, the following two assertions are valid:

1) $\|w^{\frac{1}{p(\cdot)}}\|_{p(\cdot)} < \infty$ and $\|w^{-\frac{1}{p(\cdot)}}\|_{p'(\cdot)} < \infty$.
2) The functions w and w' are integrable on $\mathbb{B}$.

Proof. Otherwise, if $\|w^{\frac{1}{p(\cdot)}}\|_{p(\cdot)} = \infty$, then necessarily $\|w^{-\frac{1}{p(\cdot)}}\|_{p'(\cdot)} = 0$ and this would imply that $w \equiv \infty$ a.e. Alternatively, if $\|w^{-\frac{1}{p(\cdot)}}\|_{p(\cdot)} = \infty$, then necessarily $\|w^{\frac{1}{p(\cdot)}}\|_{p'(\cdot)} = 0$ and this would imply that $w \equiv 0$ a.e. Furthermore, by Lemma 3.2, for $p(\cdot) \in \mathcal{P}_\pm(\mathbb{B})$, assertions 1) and 2) are equivalent. □

We also have the following

Proposition 3.4. *Let $p(\cdot) \in \mathcal{P}_\pm(\mathbb{B})$. For a weight w, the following two assertions are equivalent:*

(1) $w \in B_{p(\cdot)}$.
(2) $w' \in B_{p'(\cdot)}$.

Moreover, $[w]_{B_{p(\cdot)}} = [w']_{B_{p'(\cdot)}}$.

The following simple lemma will be useful.

Lemma 3.5. *Let $p(\cdot) \in \mathcal{P}_\pm(\mathbb{B})$. For a weight w, the following two assertions are equivalent:*

(1) $w \in B_{p(\cdot)}$;
(2) $\sup\limits_{B \in \mathcal{B}} \frac{1}{\mu_\alpha(B)} \|\chi_B\|_{p(\cdot),w} \|\chi_B\|_{p'(\cdot),w'} < \infty$.

3.3. Variable Muckenhoupt classes of weights.

Definition 3.6. The Hardy–Littlewood maximal function M_α on the space of homogeneous type $(\mathbb{B}, d, \mu_\alpha)$ is defined by

$$M_\alpha f(z) = \sup_B \frac{\chi_B(z)}{\mu_\alpha(B)} \int\limits_B |f(\zeta)| d\mu_\alpha(\zeta),$$

where the supremum is taken over all pseudo-balls of $\mathbb{B}$.

When p is a constant greater than 1, the Muckenhoupt class A_p consists of weights w which satisfy the estimate

$$\sup_B \left(\frac{1}{\mu_\alpha(B)} \int\limits_B w d\mu_\alpha \right) \left(\frac{1}{\mu_\alpha(B)} \int\limits_B w^{-\frac{1}{p-1}} d\mu_\alpha \right)^{p-1} < \infty,$$

where the sup is taken over all pseudo-balls B of $\mathbb{B}$. This definition is equivalent to the following definition:

$$\sup_B \frac{1}{\mu_\alpha(B)} \|w^{\frac{1}{p}} \chi_B\|_p \|w^{-\frac{1}{p}} \chi_B\|_{p'} < \infty,$$

where the sup is taken again over all pseudo-balls of $\mathbb{B}$.

We next have the following variable generalisation of the variable Muckenhoupt weight classes. This generalisation was given first by Diening and Hästo [12].

Definition 3.7. Let $p(\cdot) \in \mathcal{P}_\pm(\mathbb{B})$. A weight w belongs to the variable Muckenhoupt class $A_{p(\cdot)}$ on $\mathbb{B}$ if

$$[w]_{A_{p(\cdot)}} := \sup_B \frac{1}{\mu_\alpha(B)} \|w^{\frac{1}{p(\cdot)}} \chi_B\|_{p(\cdot)} \|w^{-\frac{1}{p(\cdot)}} \chi_B\|_{p'(\cdot)} < \infty,$$

where the sup is taken over all pseudo-balls of $\mathbb{B}$.

Let $p(\cdot) \in \mathcal{P}_\pm(\mathbb{B})$. The following proposition is similar to Proposition 3.4.

Proposition 3.8. *The following two assertions are equivalent:*

(1) $w \in A_{p(\cdot)}$;
(2) $w' \in A_{p'(\cdot)}$.

We record the following properties of $A_{p(\cdot)}$ and $B_{p(\cdot)}$.

Proposition 3.9.

1. *The inclusion* $A_{p(\cdot)} \subset B_{p(\cdot)}$ *holds with* $[w]_{B_{p(\cdot)}} \le [w]_{A_{p(\cdot)}}$.
2. $[w]_{B_{p(\cdot)}} \ge \frac{1}{2}$ *and* $[w]_{A_{p(\cdot)}} \ge \frac{1}{2}$.

Proof. 1. This follows directly from the definitions of $A_{p(\cdot)}$ and $B_{p(\cdot)}$.

2. We first give the proof for $B_{p(\cdot)}$. Let $B \in \mathcal{B}$. From the Hölder inequality and the definition of $B_{p(\cdot)}$, we have

$$1 = \frac{1}{\mu_\alpha(B)} \int\limits_B w^{\frac{1}{p(\cdot)}} w^{-\frac{1}{p(\cdot)}} d\mu_\alpha \le \frac{2}{\mu_\alpha(B)} \|w^{\frac{1}{p(\cdot)}} \chi_B\|_{p(\cdot)} \|w^{-\frac{1}{p(\cdot)}} \chi_B\|_{p'(\cdot)} \le 2[w]_{B_{p(\cdot)}}.$$

The proof for $A_{p(\cdot)}$ then follows from assertion 1. □

In [7], Cruz–Uribe and Cummings proved the following fundamental result for the maximal Hardy–Littlewood function. This variable theorem generalises a well-known theorem of Muckenhoupt [14] in the Euclidean space $\mathbb{R}^n$. For the spaces of homogeneous type, the analogous theorem for constant exponents was proved later by A. P. Calderón [4].

Theorem 3.10. *Let* $p(\cdot) \in \mathcal{P}_\pm^{\log}(\mathbb{B})$. *The following two assertions are equivalent:*

1. *There exists a positive constant C such that for all $f \in L^{p(\cdot)}(w)$, we have*
$$\|M_\alpha f\|_{p(\cdot),w} \leq C\|f\|_{p(\cdot),w}.$$
2. $w \in A_{p(\cdot)}$.

In fact, these authors [7] proved their result in the general setting of spaces of homogeneous type. There, in addition to the condition $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{B})$, they have a condition at infinity which has the following expression on the unit ball $\mathbb{B}$: there are two constants c and p_∞ such that
$$|p(z) - p_\infty| \leq \frac{c}{\ln(e + |z|)}$$
for every $z \in \mathbb{B}$. It is easy to show that this extra condition is satisfied on $\mathbb{B}$.

Since $m_\alpha f \leq M_\alpha f$, we deduce the following

Corollary 3.11. *Let $p(\cdot) \in \mathcal{P}^{\log}_{\pm}(\mathbb{B})$ and $w \in A_{p(\cdot)}$. For the same constant C as in Theorem* 3.10*, we have*
$$\|m_\alpha f\|_{p(\cdot),w} \leq C\|f\|_{p(\cdot),w}$$
for all $f \in L^{p(\cdot)}(w)$.

4. Necessity of the Conditions $w^{\frac{1}{p(\cdot)}} \in L^{p(\cdot)}(d\mu_\alpha)$ and $w^{-\frac{1}{p(\cdot)}} \in L^{p'(\cdot)}(d\mu_\alpha)$ in Theorem 1.6

Proposition 4.1. *Let w be a weight and let $p(\cdot) \in \mathcal{P}_-(\mathbb{B})$. If the operator P_α is bounded on $L^{p(\cdot)}(w)$, then $w^{-\frac{1}{p(\cdot)}} \in L^{p'(\cdot)}(d\mu_\alpha)$.*

Proof. If the operator P_α is bounded on $L^{p(\cdot)}(w)$, then for every $f \in L^{p(\cdot)}(w)$, we have
$$|P_\alpha(f)(z)| < \infty \quad \text{a.e. on } \mathbb{B}.$$
We show that for every non-negative $f \in L^{p(\cdot)}(w)$, we have
$$|P_\alpha(f)(0)| < \infty. \tag{4.1}$$

Indeed, it was proved in [2] that the Bergman projector P_α is a singular integral on the space of homogeneous type $(\mathbb{B}, d, \mu_\alpha)$; in particular, there exist two positive constants c and C such that for all $z, \zeta \in \mathbb{B}$ with $d(z, 0) < c$, the following estimate holds:
$$\left|\frac{1}{(1 - \langle z, \zeta\rangle)^{n+\alpha}} - 1\right| \leq Cd(z, 0) = C|z|. \tag{4.2}$$

Let ϵ be a positive number such that $\epsilon \leq \min\left(\frac{1}{2C}, c\right)$. There exists $z \in B(0, \epsilon)$ such that $|P_\alpha(f)(z)| < \infty$. So,
$$\begin{aligned}|P_\alpha(f)(0)| &\leq |P_\alpha(f)(z)| + |P_\alpha(f)(0) - P_\alpha(f)(z)| \\ &\leq |P_\alpha(f)(z)| + \int_{\mathbb{B}} f(\zeta)\left|1 - \frac{1}{(1 - \langle z, \zeta\rangle)^{n+\alpha}}\right| d\mu_\alpha(\zeta) \\ &\leq |P_\alpha(f)(z)| + C\epsilon\int_{\mathbb{B}} f(\zeta)d\mu_\alpha(\zeta) = |P_\alpha(f)(z)| + C\epsilon|P_\alpha(f)(0)|.\end{aligned}$$
For the latter inequality, we have used (4.2). This implies that $|P_\alpha(f)(0)| \leq 2|P_\alpha(f)(z)| < \infty$. This proves (4.1).

Now, by a contradiction argument, suppose that $w^{-\frac{1}{p(\cdot)}}$ does not belong to $L^{p'(\cdot)}(d\mu_\alpha)$. Then applying the closed graph theorem and using Proposition 3.1 above and Proposition 2.67 of [9][1], there exists a non-negative $g \in L^{p(\cdot)}(d\mu_\alpha)$ such that
$$\int_{\mathbb{B}} g(\zeta)w(\zeta)^{-\frac{1}{p(\zeta)}} d\mu_\alpha(\zeta) = \infty.$$
Let $f = gw^{-\frac{1}{p(\cdot)}}$. We have $f \in L^{p(\cdot)}(w)$, but f does not belong to $L^1(\mathbb{B})$; in other words, $|P_\alpha(f)(0)| = \infty$.

[1]Also refer to Exercise 4.7 of [3], for the analogous result for the classical Lebesgue spaces.

This contradicts (4.1). □

Proposition 4.2. *Let w be a weight and $p(\cdot) \in \mathcal{P}_-(\mathbb{B})$. If the operator P_α is bounded on $L^{p(\cdot)}(wd\mu)$, then $w^{\frac{1}{p(\cdot)}} \in L^{p(\cdot)}(d\mu_\alpha)$.*

Proof. Let $0 < r < 1$ and define the function $f(z) = (1-|z|^2)^{1-\alpha}\chi_{B(0,r)}(z)$ on $\mathbb{B}$. We have

$$\begin{aligned} P_\alpha f(z) &= \int_{\mathbb{B}} \frac{f(\zeta)}{(1-\langle z,\zeta\rangle)^{n+\alpha}} d\mu_\alpha(\zeta) \\ &= \int_{B(0,r)} \frac{1}{(1-\langle z,\zeta\rangle)^{n+\alpha}} d\mu(\zeta) \\ &= \overline{\int_{B(0,r)} \frac{1}{(1-\langle \zeta,z\rangle)^{n+\alpha}} d\mu(\zeta)}. \end{aligned}$$

Since the function $\zeta \longmapsto \frac{1}{(1-\langle\zeta,z\rangle)^{n+\alpha}}$ is analytic on $\mathbb{B}$ and $B(0,r)$ is the Euclidean ball centred at 0 and of radius r, it follows from the mean value property that $P_\alpha f(z) \equiv C_{r,n}$ and so,

$$|C_{r,n}| \| w^{\frac{1}{p(\cdot)}} \|_{p(\cdot)} = \|P_\alpha(f)\|_{p(\cdot),w}. \tag{4.3}$$

In addition,

$$\rho_{p(\cdot),w}(f) = \int_{B(0,r)} w(z)(1-|z|^2)^{(\alpha-1)(1-p(z))} d\mu(z).$$

On the one hand, if $\alpha \le 1$, we have $(1-|z|^2)^{(\alpha-1)(1-p(z))} \le 1$ because $(\alpha-1)(1-p(z)) > 0$ and $1-|z|^2 \le 1$. Consequently,

$$\rho_{p(\cdot),w}(f) \le \int_{B(0,r)} w(z)d\mu(z) < \infty$$

because w is locally integrable.
On the other hand, if $\alpha > 1$, we have $(1-|z|^2)^{(\alpha-1)(1-p(z))} \le (1-|z|^2)^{(\alpha-1)(1-p_+)}$. So,

$$\rho_{p(\cdot),w}(f) \le \sup_{z\in B(0,r)} (1-|z|^2)^{(\alpha-1)(1-p_+)} \int_{B(0,r)} w(\eta)d\mu(\eta) < \infty$$

because w is locally integrable and $\sup_{z\in B(0,r)}(1-|z|^2)^{(\alpha-1)(1-p_+)} = (1-r^2)^{(\alpha-1)(1-p_+)}$.
Thus, since $\rho_{p(\cdot),w}(f) < \infty$ in both cases, by Lemma 3.2, we obtain $\|f\|_{p(\cdot),w} < \infty$ and as P_α is bounded on $L^{p(\cdot)}(w)$, we deduce from (4.3) that there exists a positive constant $c_{r,\alpha,n}$ such that

$$\|w^{\frac{1}{p(\cdot)}}\|_{p(\cdot)} \le c_{r,\alpha,n}\|f\|_{p(\cdot),w} < \infty.$$

Hence we have the result. □

5. The Weight Classes $B^+_{p(\cdot)}$ and $B^{++}_{p(\cdot)}$

Definition 5.1. Let $p(\cdot) \in \mathcal{P}_\pm(\mathbb{B})$ and let w be a weight. We say that w is in the $B^+_{p(\cdot)}$ class if

$$[w]_{B^+_{p(\cdot)}} := \sup_{B\in\mathcal{B}} \frac{1}{\mu_\alpha(B)^{p_B}} \|w\chi_B\|_1 \|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}} < \infty,$$

where

$$p_B = \left(\frac{1}{\mu_\alpha(B)} \int_B \frac{1}{p(x)} d\mu_\alpha(x)\right)^{-1}.$$

This class coincides with the B_p class when $p(\cdot) = p$ (p constant). We also adopt the following notation:

$$\langle p\rangle_B = \frac{1}{\mu_\alpha(B)}\int_B p(x)d\mu_\alpha(x).$$

Remark 5.2. Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{B})$ and let B be a pseudo-ball of $\mathbb{B}$. As $p_-(B) \le p_B, \langle p\rangle_B \le p_+(B)$, it follows from Lemma 2.10 and Corollary 2.11 that

$$\mu_\alpha(B)^{p_-(B)} \simeq \mu_\alpha(B)^{p_B} \simeq \mu_\alpha(B)^{p_B} \simeq \mu_\alpha(B)^{\langle p\rangle_B} \simeq \mu_\alpha(B)^{p_+(B)}.$$

Lemma 5.3 ([11, Theorem 4.5.7])**.** *Let $p(\cdot) \in \mathcal{P}^{log}(\mathbb{B})$ be such that $p_- > 0$. Let B be a pseudo-ball of $\mathbb{B}$. Then*

$$\|\chi_B\|_{p(\cdot)} \simeq \mu_\alpha(B)^{\frac{1}{p_B}}.$$

Proof. From Lemma 3.2, we have

$$\min\left(\mu_\alpha(B)^{\frac{1}{p_-(B)}}, \mu_\alpha(B)^{\frac{1}{p_+(B)}}\right) \le \|\chi_B\|_{p(\cdot)} \le \max\left(\mu_\alpha(B)^{\frac{1}{p_-(B)}}, \mu_\alpha(B)^{\frac{1}{p_+(B)}}\right).$$

Next, from Remark 5.2, we have $\mu_\alpha(B)^{\frac{1}{p_-(B)}} \simeq \mu_\alpha(B)^{\frac{1}{p_+(B)}} \simeq \mu_\alpha(B)^{\frac{1}{p_B}}$ because $p(\cdot)$ is bounded away from zero. The conclusion follows. □

Lemma 5.4. *Let $p(\cdot) \in \mathcal{P}_\pm^{\log}(\mathbb{B})$ and let q be a constant exponent greater than $p_+ + 1$. There exists a positive constant C depending only on the log-Hölder constant of $p(\cdot)$ such that*

$$[w]_{B_q} \le C[w]_{B^+_{p(\cdot)}}.$$

Proof. As $p(\cdot) < p_+ + 1 < q$, we have $\frac{q'}{q} < \frac{p'(\cdot)}{p(\cdot)}$. Hence from the Hölder inequality (assertion 2 of Proposition 2.7), we obtain

$$\|w^{-1}\chi_B\|_{\frac{q'}{q}} \le K\|\chi_B\|_{\beta(\cdot)}\|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}},$$

where

$$\frac{1}{\beta(\cdot)} = \frac{q}{q'} - \frac{p(\cdot)}{p'(\cdot)} = q - p(\cdot) > 1.$$

It is easy to check that $\beta(\cdot)$ is a member of $\mathcal{P}^{\log}(\mathbb{B})$ such that $\beta_- > 0$. Consequently, from Lemma 5.3 and Remark 5.2, we have

$$\|\chi_B\|_{\beta(\cdot)} \simeq \mu_\alpha(B)^{\frac{1}{\beta_B}} \simeq \mu_\alpha(B)^{q-\langle p\rangle_B} \simeq \mu_\alpha(B)^{q-p_B}.$$

Thus there exists a positive constant C such that

$$\begin{aligned}\tfrac{1}{\mu_\alpha(B)^q}\|w\chi_B\|_1\|w^{-1}\chi_B\|_{\frac{q'}{q}} &\le C\tfrac{1}{\mu_\alpha(B)^q}\|w\chi_B\|_1\|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}}\mu_\alpha(B)^{q-p_B}\\ &= C\tfrac{1}{\mu_\alpha(B)^{p_B}}\|w\chi_B\|_1\|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}} \le C[w]_{B^+_{p(\cdot)}}\end{aligned}$$

for all pseudo-balls $B \in \mathcal{B}$. The conclusion follows. □

We recall the following

Definition 5.5. The weight class B_∞ is defined by $B_\infty = \bigcup_{q\in(1,\infty)} B_q$.

Remark 5.6. It follows from Lemma 5.4 that if $p(\cdot) \in \mathcal{P}_\pm^{\log}(\mathbb{B})$, we have

$$B^+_{p(\cdot)} \subset B_\infty.$$

In the rest of this article, to simplify the notation, we denote $w(B) = \|w\chi_B\|_1$.

We next define another class of weights Λ, which contains the class B_∞. For a reference, cf., e.g., [1].

Definition 5.7. We call Λ the class consisting of those integrable weights w satisfying the following property. There exist two positive constants C and δ such that the following inequality

$$\frac{\mu_\alpha(E)}{\mu_\alpha(B)} \le C\Big(\frac{w(E)}{w(B)}\Big)^\delta \tag{5.1}$$

holds whenever $B \in \mathcal{B}$ and E is a measurable subset of B.

Remark 5.8.

1. For $w \in \Lambda$, the weighted measure $wd\mu_\alpha$ is doubling in the following sense. There exists a positive constant C such that for every pseudo-ball B of $\mathbb{B}$ whose pseudo-ball $\widetilde{B}$ of the same centre and of double radius is a member of $\mathcal{B}$, we have
$$w(\widetilde{B}) \le Cw(B).$$
This result easily follows from the definition of Λ.
2. We recall that $w(\mathbb{B}) > 0$. For $w \in \Lambda$, this implies that $w(B) > 0$ for every pseudo-ball B of $\mathbb{B}$. Indeed, take $\mathbb{B}$ for B and B for E in (5.1).

Lemma 5.9. *Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{B})$ be such that $p_- > 0$. Let $w \in \Lambda$. Then*

$$\|\chi_B\|_{p(\cdot),w} \simeq w(B)^{\frac{1}{p_+(B)}} \simeq w(B)^{\frac{1}{p_-(B)}} \simeq w(B)^{\frac{1}{p(x)}} \simeq w(B)^{\frac{1}{p_B}}$$

for all pseudo-balls B of $\mathbb{B}$ such that $w(B) > 0$ and for all $x \in B$.

Proof. Take $\mathbb{B}$ for B and B for E in Definition 5.7. We have

$$\Big(C^{-1}\frac{\mu_\alpha(B)}{\mu_\alpha(\mathbb{B})}\Big)^{\frac{1}{\delta}} w(\mathbb{B}) \le w(B) \le w(\mathbb{B}).$$

So,

$$w(\mathbb{B})^{p_-(B)-p_+(B)} \le w(B)^{p_-(B)-p_+(B)} \lesssim \mu_\alpha(B)^{\frac{1}{\delta}(p_-(B)-p_+(B))} w(\mathbb{B})^{p_-(B)-p_+(B)}.$$

It is easy to check that

$$\min(1, w(\mathbb{B})^{p_--p_+}) \le w(\mathbb{B})^{p_-(B)-p_+(B)} \le \max(1, w(\mathbb{B})^{p_--p_+}).$$

Next, combining with Lemma 2.10 gives

$$\min(1, w(\mathbb{B})^{p_--p_+}) \le w(B)^{p_-(B)-p_+(B)} \lesssim C_\delta \max(1, w(\mathbb{B})^{p_--p_+}).$$

Thus we have proved the estimates $w(B)^{\frac{1}{p_+(B)}} \simeq w(B)^{\frac{1}{p_-(B)}} \simeq w(B)^{\frac{1}{p(x)}} \simeq w(B)^{\frac{1}{p_B}}$ for all $x \in B$.

On the other hand, from Lemma 3.2, we have

$$\min\left(w(B)^{\frac{1}{p_-(B)}}, w(B)^{\frac{1}{p_+(B)}}\right) \le \|\chi_B\|_{p(\cdot),w} \le \max\left(w(B)^{\frac{1}{p_-(B)}}, w(B)^{\frac{1}{p_+(B)}}\right).$$

Hence

$$\|\chi_B\|_{p(\cdot),w} \simeq w(B)^{\frac{1}{p_-(B)}} \simeq w(B)^{\frac{1}{p_+(B)}}.$$ □

We recall again the notation $w'(y) = w(y)^{1-p'(y)}$.

Lemma 5.10. *Let $p(\cdot) \in \mathcal{P}_\pm^{\log}(\mathbb{B})$ and $w \in B_{p(\cdot)}^+$. Then*

$$\|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}} \simeq \left(\rho_{\frac{p'(\cdot)}{p(\cdot)}}(w^{-1}\chi_B)\right)^{p_B-1} = w'(B)^{p_B-1}. \tag{5.2}$$

Proof. Let $w \in B_{p(\cdot)}^+$ and $B \in \mathcal{B}$. By the definition, we have

$$\frac{1}{\mu_\alpha(B)^{p_B}} w(B)\|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}} \le [w]_{B_{p(\cdot)}^+}. \tag{5.3}$$

On the other hand, by the Hölder inequality (Proposition 2.7, assertion 1) and Lemma 5.9, we have

$$\mu_\alpha(B) = \int_B w(y)^{\frac{1}{p(y)}} w(y)^{-\frac{1}{p(y)}} d\mu_\alpha(y) \le 2\|w^{\frac{1}{p(\cdot)}}\chi_B\|_{p(\cdot)}\|w^{-\frac{1}{p(\cdot)}}\chi_B\|_{p'(\cdot)} \simeq w(B)^{\frac{1}{p_B}}\|w^{-\frac{1}{p(\cdot)}}\chi_B\|_{p'(\cdot)}.$$

Hence

$$\left\| \frac{w(B)^{\frac{1}{p_B}}}{\mu_\alpha(B)} w^{-\frac{1}{p(\cdot)}} \chi_B \right\|_{p'(\cdot)} \gtrsim 1. \tag{5.4}$$

Consequently, from (5.4), Lemma 3.2, Lemma 5.9 and Corollary 2.11, we have:

$$\begin{aligned} 1 \lesssim \rho_{p'(\cdot)}\left(\frac{w(B)^{\frac{1}{p_B}}}{\mu_\alpha(B)} w^{-\frac{1}{p(\cdot)}} \chi_B \right) &= \int_B \left(\frac{w(B)^{\frac{1}{p_B}}}{\mu_\alpha(B)} \right)^{p'(y)} w(y)^{-\frac{p'(y)}{p(y)}} d\mu_\alpha(y) \\ &\simeq \int_B \left(\frac{w(B)}{\mu_\alpha(B)^{p_B}} \right)^{\frac{p'(y)}{p(y)}} w(y)^{-\frac{p'(y)}{p(y)}} d\mu_\alpha(y) \\ &= \rho_{\frac{p'(\cdot)}{p(\cdot)}} \left(\frac{w(B)}{\mu_\alpha(B)^{p_B}} w^{-1} \chi_B \right). \end{aligned}$$

So, by Lemma 3.2, we have

$$\left\| \frac{w(B)}{\mu_\alpha(B)^{p_B}} w^{-1} \chi_B \right\|_{\frac{p'(\cdot)}{p(\cdot)}} \gtrsim 1. \tag{5.5}$$

Thus from (5.3) and (5.5), we have

$$\|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}} \simeq \frac{\mu_\alpha(B)^{p_B}}{w(B)}. \tag{5.6}$$

Furthermore, from Remark 5.2 and as $p(\cdot) \in \mathcal{P}_\pm$, we have the equivalences

$$\begin{aligned} \mu_\alpha(B)^{p_+(B)} \simeq \mu_\alpha(B)^{p_-(B)} &\iff \mu_\alpha(B)^{\frac{1}{p_-(B)-1}} \simeq \mu_\alpha(B)^{\frac{1}{p_+(B)-1}} \\ &\iff \mu_\alpha(B)^{\frac{p_B}{p_-(B)-1}} \simeq \mu_\alpha(B)^{\frac{p_B}{p_+(B)-1}}, \end{aligned}$$

from which we deduce that

$$\mu_\alpha(B)^{\frac{p_B}{p_+(B)-1}} \simeq \mu_\alpha(B)^{\frac{p_B}{p_-(B)-1}} \simeq \mu_\alpha(B)^{\frac{p_B}{p_B-1}},$$

since $p_-(B) \le p_B \le p_+(B)$. Similarly, by Lemma 5.9, we deduce from the estimate

$$w(B)^{p_B} \simeq w(B)^{p_+(B)} \simeq w(B)^{p_-(B)}$$

that

$$w(B)^{\frac{1}{p_-(B)-1}} \simeq w(B)^{\frac{1}{p_+(B)-1}} \simeq w(B)^{\frac{1}{p_B-1}}.$$

So, from (5.6), we have

$$\|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}}^{\frac{1}{p_+(B)-1}} \simeq \|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}}^{\frac{1}{p_-(B)-1}} \simeq \|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}}^{\frac{1}{p_B-1}}.$$

Since $\rho_{\frac{p'(\cdot)}{p(\cdot)}}\left(w^{-1}\chi_B\right) = w'(B)$, combining with Lemma 3.2 where $\frac{p'(\cdot)}{p(\cdot)}$ replaces $p(\cdot)$, we obtain the required result. □

Proposition 5.11. *Let $p(\cdot) \in \mathcal{P}_\pm^{\log}(\mathbb{B})$ and $w \in B_{p(\cdot)}^+$. Then $w' \in B_{p'(\cdot)}^+$ and*

$$\left(\frac{1}{\mu_\alpha(B)^{p'_B}} \|w'\chi_B\|_1 \|w'^{-1}\chi_B\|_{\frac{p(\cdot)}{p'(\cdot)}} \right)^{p_B-1} \simeq \frac{1}{\mu_\alpha(B)^{p_B}} \|w\chi_B\|_1 \|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}} \simeq \frac{w(B)}{\mu_\alpha(B)} \left(\frac{w'(B)}{\mu_\alpha(B)} \right)^{p_B-1}$$

for all pseudo-balls $B \in \mathcal{B}$.

Proof. We recall that $w \in \Lambda$ by Remark 5.6. Hence, from Lemma 5.9 used with $\frac{p(\cdot)}{p'(\cdot)}$ replacing $p(\cdot)$, equation (5.2) and the property $w \in B^+_{p(\cdot)}$, we obtain

$$\begin{aligned}
\frac{1}{\mu_\alpha(B)^{p'_B}} w'(B) \|w'^{-1}\chi_B\|_{\frac{p(\cdot)}{p'(\cdot)}} &= \frac{1}{\mu_\alpha(B)^{p'_B}} w'(B) \|\chi_B\|_{\frac{p(\cdot)}{p'(\cdot)},w} \\
&\simeq \frac{1}{\mu_\alpha(B)^{p'_B}} w'(B) w(B)^{\frac{1}{p_B-1}} \\
&= \left(\frac{w(B)}{\mu_\alpha(B)^{p_B}} w'(B)^{p_B-1}\right)^{\frac{1}{p_B-1}} \qquad (5.7) \\
&\simeq \left(\frac{w(B)}{\mu_\alpha(B)^{p_B}} \|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}}\right)^{\frac{1}{p_B-1}} \qquad (5.8) \\
&\leq [w]^{\frac{1}{p_B-1}}_{B^+_{p(\cdot)}}.
\end{aligned}$$

Hence $w' \in B^+_{p'(\cdot)}$ and from (5.7) and (5.8), we deduce that

$$\frac{w(B)}{\mu_\alpha(B)^{p_B}} \|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}} \simeq \frac{w(B)}{\mu_\alpha(B)} \left(\frac{w'(B)}{\mu_\alpha(B)}\right)^{p_B-1}. \qquad \square$$

Definition 5.12. Let $p(\cdot) \in \mathcal{P}_\pm(\mathbb{B})$ and let w be a weight. We say that w is in the $B^{++}_{p(\cdot)}$ class if

$$[w]_{B^{++}_{p(\cdot)}} := \sup_{B\in\mathcal{B}} \frac{w(B)}{\mu_\alpha(B)} \left(\frac{w'(B)}{\mu_\alpha(B)}\right)^{p_B-1} < \infty.$$

It is easy to check the following Proposition.

Proposition 5.13. *Let $p(\cdot) \in \mathcal{P}_\pm(\mathbb{B})$. The following two assertions are equivalent:*

1. $w \in B^{++}_{p(\cdot)}$;
2. $w' \in B^{++}_{p'(\cdot)}$.

Lemma 5.14. *Let $p(\cdot) \in \mathcal{P}^{\log}_\pm(\mathbb{B})$ and $w \in B^{++}_{p(\cdot)}$. Then for all pseudo-balls B of $\mathbb{B}$,*

$$\|\chi_B\|_{p(\cdot),w} \simeq w(B)^{\frac{1}{p_+(B)}} \simeq w(B)^{\frac{1}{p_-(B)}} \simeq w(B)^{\frac{1}{p_B}}.$$

Proof. Since $w \in B^{++}_{p(\cdot)}$, we have $w'(\mathbb{B}) < \infty$ and it follows from Lemma 3.2 that $\|\chi_\mathbb{B}\|_{p'(\cdot),w'} < \infty$. Hence, by the Hölder inequality, we obtain

$$\begin{aligned}
\mu_\alpha(B) &\leq 2\|\chi_B\|_{p(\cdot),w}\|\chi_B\|_{p'(\cdot),w'} \\
&\leq 2\|\chi_B\|_{p(\cdot),w}\|\chi_\mathbb{B}\|_{p'(\cdot),w'}.
\end{aligned}$$

Therefore, from Lemma 2.10, we get

$$\begin{aligned}
\|\chi_B\|^{p_-(B)-p_+(B)}_{p(\cdot),w} &\lesssim \mu_\alpha(B)^{p_-(B)-p_+(B)} \|\chi_\mathbb{B}\|^{p_+(B)-p_-(B)}_{p'(\cdot),w'} \\
&\lesssim \max\left(1, \|\chi_\mathbb{B}\|^{p_+-p_-}_{p'(\cdot),w'}\right). \qquad (5.9)
\end{aligned}$$

On the other hand, using again $w \in B^{++}_{p(\cdot)}$, we have $w(\mathbb{B}) < \infty$ and hence $\|\chi_\mathbb{B}\|_{p(\cdot),w} < \infty$. Then

$$\|\chi_B\|^{p_+(B)-p_-(B)}_{p(\cdot),w} \leq \|\chi_\mathbb{B}\|^{p_+(B)-p_-(B)}_{p(\cdot),w} \leq \max\left(1, \|\chi_\mathbb{B}\|^{p_+-p_-}_{p(\cdot),w}\right). \qquad (5.10)$$

Thus from (5.9) and (5.10), we have

$$\|\chi_B\|^{p_+(B)}_{p(\cdot),w} \simeq \|\chi_B\|^{p_-(B)}_{p(\cdot),w},$$

and from Lemma 3.2, we conclude the proof. $\square$

Lemma 5.15. *Let $p(\cdot) \in \mathcal{P}^{\log}_\pm(\mathbb{B})$. Then $B^{++}_{p(\cdot)} \subset \Lambda$.*

Proof. Let $w \in B^{++}_{p(\cdot)}$. Let $B \in \mathcal{B}$ and E be a measurable subset of B. By the Hölder inequality and from Lemma 3.2, we have

$$\begin{aligned}\mu_\alpha(E) &\leq 2\|\chi_E\|_{p(\cdot),w}\|\chi_E\|_{p'(\cdot),w'}\\ &\leq 2\|\chi_E\|_{p(\cdot),w}\|\chi_B\|_{p'(\cdot),w'}\\ &\leq 2\max\left(w(E)^{\frac{1}{p_+(B)}}, w(E)^{\frac{1}{p_-(B)}}\right)\|\chi_B\|_{p'(\cdot),w'}.\end{aligned}$$

However, since $w' \in B^{++}_{p'(\cdot)}$ by Proposition 5.13, from Lemma 5.14, we have

$$\|\chi_B\|_{p'(\cdot),w'} \simeq w'(B)^{\frac{1}{p'_+(B)}} \simeq w'(B)^{\frac{1}{p'_-(B)}} \simeq w'(B)^{\frac{1}{p'_B}}.$$

Hence, using $w \in B^{++}_{p(\cdot)}$, we deduce that

$$\begin{aligned}\mu_\alpha(E) &\lesssim \max\left(w(E)^{\frac{1}{p_+(B)}} w'(B)^{\frac{1}{p'_+(B)}}, w(E)^{\frac{1}{p_-(B)}} w'(B)^{\frac{1}{p'_-(B)}}\right)\\ &\leq \max\left([w]^{\frac{1}{p_+(B)}}_{B^{++}_{p(\cdot)}}, [w]^{\frac{1}{p_-(B)}}_{B^{++}_{p(\cdot)}}\right)\max\left(\left(\frac{w(E)}{w(B)}\right)^{\frac{1}{p_+(B)}}, \left(\frac{w(E)}{w(B)}\right)^{\frac{1}{p_-(B)}}\right)\mu_\alpha(B)\\ &\leq \max\left([w]^{\frac{1}{p_+}}_{B^{++}_{p(\cdot)}}, [w]^{\frac{1}{p_-}}_{B^{++}_{p(\cdot)}}\right)\left(\frac{w(E)}{w(B)}\right)^{\frac{1}{p_+(B)}}\mu_\alpha(B)\\ &\leq \max\left([w]^{\frac{1}{p_+}}_{B^{++}_{p(\cdot)}}, [w]^{\frac{1}{p_-}}_{B^{++}_{p(\cdot)}}\right)\left(\frac{w(E)}{w(B)}\right)^{\frac{1}{p_+}}\mu_\alpha(B).\end{aligned}$$

Therefore

$$\frac{\mu_\alpha(E)}{\mu_\alpha(B)} \lesssim \left(\frac{w(E)}{w(B)}\right)^{\frac{1}{p_+}}.$$ □

Proposition 5.16. *Let $p(\cdot) \in \mathcal{P}^{\log}_{\pm}(\mathbb{B})$. Then $B^{+}_{p(\cdot)} = B^{++}_{p(\cdot)}$.*

Proof. For the inclusion $B^{+}_{p(\cdot)} \subset B^{++}_{p(\cdot)}$, apply Proposition 5.11. For the reverse inclusion, apply Lemma 5.15 and Lemma 5.9. □

Remark 5.17. From Proposition 5.11, Remark 5.6 and Lemma 5.9, we have the inclusion $B^{+}_{p(\cdot)} \subset B_{p(\cdot)}$ for $p(\cdot) \in \mathcal{P}^{\log}_{\pm}(\mathbb{B})$.

Now, we prove the reverse inclusion. In this direction, we first state the following result.

Lemma 5.18. *Let $p(\cdot) \in \mathcal{P}^{\log}_{\pm}(\mathbb{B})$ and $w \in B_{p(\cdot)}$. Then there exists a constant $C > 1$,*

$$\frac{1}{C} \leq \|\chi_B\|^{p_-(B)-p_+(B)}_{p(\cdot),w} \leq C$$

for all $B \in \mathcal{B}$.

Proof. By the Hölder inequality, we have

$$\mu_\alpha(B) \leq 2\|\chi_B\|_{p(\cdot),w}\|\chi_B\|_{p'(\cdot),w'}$$

and as $w \in B_{p(\cdot)}$, from Lemma 2.10 and according to the estimate $\|\chi_{\mathbb{B}}\|_{p'(\cdot),w'} < \infty$ given by Remark 3.3, we have

$$\begin{aligned}\|\chi_B\|^{p_-(B)-p_+(B)}_{p(\cdot),w} &\lesssim \mu_\alpha(B)^{p_-(B)-p_+(B)}\|\chi_B\|^{p_+(B)-p_-(B)}_{p'(\cdot),w'}\\ &\simeq \|\chi_{\mathbb{B}}\|^{p_+(B)-p_-(B)}_{p'(\cdot),w'}\\ &\lesssim \max\left(1, \|\chi_{\mathbb{B}}\|^{p_+-p_-}_{p'(\cdot),w'}\right).\end{aligned}$$

On the other hand, according to the estimate $\|\chi_{\mathbb{B}}\|_{p(\cdot),w} < \infty$ given by Remark 3.3, we have

$$\|\chi_B\|^{p_+(B)-p_-(B)}_{p(\cdot),w} \leq \max(1, \|\chi_{\mathbb{B}}\|^{p_+-p_-}_{p(\cdot),w}).$$ □

From Lemma 3.2 and Lemma 5.18, we deduce the following

Corollary 5.19. *Let* $p(\cdot) \in \mathcal{P}_{\pm}^{\log}(\mathbb{B})$ *and* $w \in B_{p(\cdot)}$. *Then*

$$\|\chi_B\|_{p(\cdot),w} \simeq w(B)^{\frac{1}{p_+(B)}} \simeq w(B)^{\frac{1}{p_-(B)}} \simeq w(B)^{\frac{1}{p_B}},$$

for all $B \in \mathcal{B}$.

We next state the following

Theorem 5.20. *Let* $p(\cdot) \in \mathcal{P}_{\pm}^{\log}(\mathbb{B})$. *Then* $B_{p(\cdot)} = B_{p(\cdot)}^{+} = B_{p(\cdot)}^{++}$.

Proof. From Remark 5.17, we have $B_{p(\cdot)}^{+} \subset B_{p(\cdot)}$. Let $w \in B_{p(\cdot)}$. By Proposition 3.4, $w' \in B_{p'(\cdot)}$. It follows from Corollary 5.19 that

$$\frac{w(B)}{\mu_\alpha(B)}\left(\frac{w'(B)}{\mu_\alpha(B)}\right)^{p_B-1} \simeq \left(\frac{1}{\mu_\alpha(B)}\|\chi_B\|_{p(\cdot),w}\|\chi_B\|_{p'(\cdot),w'}\right)^{p_B} \leq \max\left(1, [w]_{B_{p(\cdot)}}^{p_+}\right). \qquad \square$$

To end this section, we record with the same proof the following analogous theorem for the variable Muckenhoupt weight classes.

Theorem 5.21. *Let* w *be a weight and let* $p(\cdot) \in \mathcal{P}_{\pm}^{\log}(\mathbb{B})$. *The following three assertions are equivalent:*

1. $w \in A_{p(\cdot)}$;
2. $\sup_B \frac{1}{\mu_\alpha(B)^{p_B}}\|w\chi_B\|_1\|w^{-1}\chi_B\|_{\frac{p'(\cdot)}{p(\cdot)}} < \infty$, *where the* sup *is taken over all pseudo-balls of* $\mathbb{B}$;
3. $\sup_B \frac{w(B)}{\mu_\alpha(B)}\left(\frac{w'(B)}{\mu_\alpha(B)}\right)^{p_B-1} < \infty$, *where the* sup *is taken over all pseudo-balls of* $\mathbb{B}$.

6. Proof of the Necessary Condition in Theorem 1.6

The aim of this section is to prove the following result.

Proposition 6.1. *Let* w *be a weight and let* $p(\cdot) \in \mathcal{P}_{\pm}^{\log}(\mathbb{B})$. *If the Bergman projector is bounded on* $L^{p(\cdot)}(wd\mu_\alpha)$, *then* $w \in B_{p(\cdot)}$.

Proof. According to Theorem 5.20, it suffices to prove that $w \in B_{p(\cdot)}^{++}$, i.e., the following estimate

$$\sup_{B\in\mathcal{B}} \frac{w(B)}{\mu_\alpha(B)}\left(\frac{w'(B)}{\mu_\alpha(B)}\right)^{p_B-1} < \infty \tag{6.1}$$

holds. From Proposition 4.1, we have $w^{-\frac{1}{p(\cdot)}} \in L^{p'(\cdot)}(d\mu_\alpha)$ and from Proposition 4.2, we have $w^{\frac{1}{p(\cdot)}} \in L^{p(\cdot)}(d\mu_\alpha)$. In particular, $w(\mathbb{B}) < \infty$ and $w'(\mathbb{B}) < \infty$. Thus we just have to show the estimate (6.1) for the pseudo-balls of radius smaller than a positive constant R_0, because if the radius of B is larger, then B can be identified with $\mathbb{B}$. We use the following

Lemma 6.2 ([2])**.** *There exist three positive numbers* R_0, c *and* C_α *such that the following holds. For every pseudo-ball* $B^1 \in \mathcal{B}$ *of radius* $R < R_0$, *there exists a pseudo-ball* $B^2 \in \mathcal{B}$ *of the same radius such that* $d(B^1, B^2) = cR$, *that satisfies the following property: for every non-negative measurable function* f *supported in* B^i *and for two distinct superscripts* $i, j \in \{1, 2\}$, *we have*

$$|P_\alpha f| \geq C_\alpha \chi_{B^j}\mu_\alpha(B^i)^{-1}\int_{B^i} f d\mu_\alpha. \tag{6.2}$$

Thus, by taking $f = \chi_{B^i}$ in (6.2), we obtain

$$|P_\alpha \chi_{B^i}(z)| \geq \chi_{B^j}(z)C_\alpha\mu_\alpha(B^i)^{-1}\int_{B^i}\chi_{B^i}d\mu_\alpha \simeq \chi_{B^j}(z).$$

Using the growth of the norm $\|\cdot\|_{p(\cdot),w}$, we obtain

$$\|P_\alpha\chi_{B^i}\|_{p(\cdot),w} = \|w^{\frac{1}{p(\cdot)}}P_\alpha\chi_{B^i}\|_{p(\cdot)} \gtrsim \|\chi_{B^j}w^{\frac{1}{p(\cdot)}}\|_{p(\cdot)}.$$

So, using the fact that P_α is bounded on $L^{p(\cdot)}(wd\mu_\alpha)$, we obtain

$$\|\chi_{B^j} w^{\frac{1}{p(\cdot)}}\|_{p(\cdot)} \lesssim \|P_\alpha\| \|\chi_{B^i}\|_{p(\cdot),w}.$$

We then deduce that

$$\|\chi_{B^1} w^{\frac{1}{p(\cdot)}}\|_{p(\cdot)} \simeq \|\chi_{B^2} w^{\frac{1}{p(\cdot)}}\|_{p(\cdot)}. \tag{6.3}$$

In the rest of the proof, we shall take $f = w'\chi_{B^1}$. We have $f \in L^{p(\cdot)}(wd\mu_\alpha)$ since

$$\rho_{p(\cdot),w}(f) = \int_{\mathbb{B}} w(z)^{-p'(z)}\chi_{B^1}(z)w(z)d\mu_\alpha(z) = \rho_{p'(\cdot)}(w^{-\frac{1}{p(\cdot)}}\chi_{B^1}) < \infty$$

by Proposition 4.1. However, $\rho_{p(\cdot),w}(f) = \int_{B^1} w' d\mu_\alpha = w'(B^1)$. Also, from (6.2) and the previous equality, we have

$$\chi_{B^2}(z)w'(B^1) \leq C_\alpha^{-1}\mu_\alpha(B^1)|P_\alpha f(z)|.$$

Moving to the norm $\|\cdot\|_{p(\cdot),w}$, we obtain

$$\|w^{\frac{1}{p(\cdot)}}\chi_{B^2}\|_{p(\cdot)}w'(B^1) \leq C_\alpha^{-1}\mu_\alpha(B^1)\|P_\alpha f\|_{p(\cdot),w}.$$

Then using the boundedness of P_α on $L^{p(\cdot)}(wd\mu_\alpha)$, the previous inequality implies

$$\|w^{\frac{1}{p(\cdot)}}\chi_{B^2}\|_{p(\cdot)}w'(B^1) \leq C_\alpha^{-1}\mu_\alpha(B^1)\|P_\alpha\|\|f\|_{p(\cdot),w}$$

and combining with (6.3), we obtain the following

Lemma 6.3 (Main Lemma)**.** *Suppose that P_α is bounded on $L^{p(\cdot)}(wd\mu_\alpha)$. Then*

$$\|w^{\frac{1}{p(\cdot)}}\chi_B\|_{p(\cdot)}w'(B) \leq CC_\alpha^{-1}\mu_\alpha(B)\|P_\alpha\|\|w'\chi_B\|_{p(\cdot),w} \tag{6.4}$$

for every pseudo-ball $B \in \mathcal{B}$ of radius smaller than R_0. The absolute constants R_0, C and C_α were respectively defined in Lemma 6.2*,* (6.3) *and* (6.2)*.*

At this level, we need to calculate $\|w^{\frac{1}{p(\cdot)}}\chi_B\|_{p(\cdot)}$ and $\|w'\chi_B\|_{p(\cdot),w}$. This calculation is not as obvious as in the case where $p(\cdot)$ is constant.

Lemma 6.4. *Let $p(\cdot) \in \mathcal{P}(\mathbb{B})$. If P_α is bounded on $L^{p(\cdot)}(w)$, then P_α is bounded on $L^{p'(\cdot)}(w')$.*

Proof. We first recall that the weighted Bergman projector P_α is the orthogonal projector from the (Hilbert)–Lebesgue space $L^2(d\mu_\alpha)$ to its closed subspace $L^2(d\mu_\alpha) \cap Hol(\mathbb{B})$ (the standard weighted Bergman space). We call $\mathcal{C}_c(\mathbb{B})$ the space of continuous functions with a compact support in $\mathbb{B}$. By Proposition 2.6, $\mathcal{C}_c(\mathbb{B})$ is a dense subspace of $L^{p(\cdot)}(w)$ and $L^{p'(\cdot)}(w')$. From Proposition 3.1 and the boundedness of P_α on $L^{p(\cdot)}(w)$, for all $f \in \mathcal{C}_c(\mathbb{B})$, we have

$$\begin{aligned}
\|P_\alpha f\|_{p'(\cdot),w'} &= \sup_{g\in\mathcal{C}_c(\mathbb{B}):\|g\|_{p(\cdot),w}=1} \left|\int_{\mathbb{B}} P_\alpha f(\zeta)\overline{g(\zeta)}d\mu_\alpha(\zeta)\right| \\
&= \sup_{g\in\mathcal{C}_c(\mathbb{B}):\|g\|_{p(\cdot),w}=1} \left|\int_{\mathbb{B}} f(\zeta)\overline{P_\alpha g(\zeta)}d\mu_\alpha(\zeta)\right| \\
&\leq 2 \sup_{g\in\mathcal{C}_c(\mathbb{B}):\|g\|_{p(\cdot),w}=1} \|f\|_{p'(\cdot),w'}\|P_\alpha g\|_{p(\cdot),w} \\
&\leq 2\,\|P_\alpha\|\,\|f\|_{p'(\cdot),w'}.
\end{aligned}$$

We have used the elementary fact that $\mathcal{C}_c(\mathbb{B})$ is contained in $L^2(d\mu_\alpha)$. For the last but one inequality, we have used the Hölder inequality. □

Lemma 6.5. *Let $p(\cdot) \in \mathcal{P}(\mathbb{B})$ and let w be a weight. If P_α is bounded on $L^{p(\cdot)}(w)$, then for all $t > 0$,*

$$\|t\chi_{\{|P_\alpha f|>t\}}\|_{p(\cdot),w} \leq \|P_\alpha\|\,\|f\|_{p(\cdot),w}.$$

Proof. It suffices to remark that for all $t > 0$, $t\chi_{\{|P_\alpha f|>t\}} \leq |P_\alpha f|$. □

Lemma 6.6. *Let $p(\cdot) \in \mathcal{P}_{\pm}^{\log}(\mathbb{B})$ and w be a weight. If P_α is bounded on $L^{p(\cdot)}(w)$, then*

$$\|\chi_B\|_{p(\cdot),w} \simeq w(B)^{\frac{1}{p_+(B)}} \simeq w(B)^{\frac{1}{p_-(B)}}$$

for all pseudo-balls B of $\mathbb{B}$.

Proof. If $\|\chi_B\|_{p(\cdot),w} \geq 1$, then $\|\chi_{\mathbb{B}}\|_{p(\cdot),w}^{p_- - p_+} \leq \|\chi_B\|_{p(\cdot),w}^{p_- - p_+} \leq \|\chi_B\|_{p(\cdot),w}^{p_-(B)-p_+(B)} \leq 1$. So,

$$\|\chi_B\|_{p(\cdot),w}^{p_-(B)-p_+(B)} \simeq 1. \tag{6.5}$$

Otherwise, if $\|\chi_B\|_{p(\cdot),w} < 1$, then by the Hölder inequality, we have

$$\begin{aligned}\mu_\alpha(B) &\leq 2\|\chi_B\|_{p(\cdot),w}\|\chi_B\|_{p'(\cdot),w'}\\ &\leq 2\|\chi_B\|_{p(\cdot),w}\|\chi_{\mathbb{B}}\|_{p'(\cdot),w'}.\end{aligned} \tag{6.6}$$

Hence from (6.6) and Lemma 2.10, we have

$$\begin{aligned}\|\chi_B\|_{p(\cdot),w}^{p_-(B)-p_+(B)} &\leq 2^{p_+-p_-}\mu_\alpha(B)^{p_-(B)-p_+(B)}\|\chi_{\mathbb{B}}\|_{p'(\cdot),w'}^{p_+(B)-p_-(B)}\\ &\lesssim \max\left(1, \|\chi_{\mathbb{B}}\|_{p'(\cdot),w'}^{p_+-p_-}\right).\end{aligned} \tag{6.7}$$

We point out that $\|\chi_{\mathbb{B}}\|_{p'(\cdot),w'} < \infty$ according to Proposition 4.1, since P_α is bounded on $L^{p(\cdot)}(w)$. On the other hand, using again the boundedness of P_α on $L^{p(\cdot)}(w)$, we have the estimate $\|\chi_{\mathbb{B}}\|_{p(\cdot),w} < \infty$ according to Proposition 4.2. Then

$$\|\chi_B\|_{p(\cdot),w}^{p_+(B)-p_-(B)} \lesssim \max\left(1, \|\chi_{\mathbb{B}}\|_{p(\cdot),w}^{p_+-p_-}\right) < \infty. \tag{6.8}$$

Thus, from (6.5), (6.7) and (6.8), we deduce that

$$\|\chi_B\|_{p(\cdot),w}^{p_+(B)} \simeq \|\chi_B\|_{p(\cdot),w}^{p_-(B)}$$

for all pseudo-balls of $\mathbb{B}$. Applying Lemma 3.2 gives

$$\|\chi_B\|_{p(\cdot),w} \simeq w(B)^{\frac{1}{p_+(B)}} \simeq w(B)^{\frac{1}{p_-(B)}}.$$ □

End of the proof of Proposition 6.1. We go back to the Main Lemma (Lemma 6.3). On the one hand, since P_α is bounded on $L^{p(\cdot)}(w)$, it follows from Lemma 6.4 that P_α is also bounded on $L^{p'(\cdot)}(w')$. So, from Lemma 6.6 with $p'(\cdot)$ in the place of $p(\cdot)$ and w' in the place of w, we have

$$\|\chi_B\|_{p'(\cdot),w'} \simeq w'(B)^{1-\frac{1}{p_-(B)}} \simeq w'(B)^{1-\frac{1}{p_+(B)}}.$$

This implies the estimate $w'(B)^{\frac{1}{p_-(B)}} \simeq w'(B)^{\frac{1}{p_+(B)}}$. It then follows from Lemma 6.3 that

$$\|w'\chi_B\|_{p(\cdot),w} \simeq w'(B)^{\frac{1}{p_B}}.$$

On the other hand, $\|w^{\frac{1}{p(\cdot)}}\chi_B\|_{p(\cdot)} = \|\chi_B\|_{p(\cdot),w} \simeq w(B)^{\frac{1}{p_B}}$ by Lemma 6.6. Inequality (6.4) of the Main Lemma takes the following form:

$$w(B)^{\frac{1}{p_B}}w'(B) \lesssim \mu_\alpha(B)w'(B)^{\frac{1}{p_B}}.$$

Equivalently,

$$\sup_{B\in\mathcal{B}} \frac{w(B)}{\mu_\alpha(B)}\left(\frac{w'(B)}{\mu_\alpha(B)}\right)^{p_B-1} < \infty.$$

We have shown the estimate (6.1). This finishes the proof of Proposition 6.1. □

7. Boundedness on $L^{p(\cdot)}(w)$ of the Maximal Function m_α

In this section, we prove the boundedness of the maximal function m_α on $L^{p(\cdot)}(w)$ when $w \in B_{p(\cdot)}$. As in [2], we use the regularisation operator that we recall here with some of its properties.

Definition 7.1. For all $k \in (0,1)$, we define the regularisation operator R_k^α of order k by

$$R_k^\alpha f(z) = \frac{1}{\mu_\alpha(B^k(z))} \int_{B^k(z)} f(w)d\mu_\alpha(w),$$

where $B^k(z) = \{\zeta \in \mathbb{B} : d(z,\zeta) < k(1-|z|)\}$.

Proposition 7.2. *For all $k \in (0,1)$, there exists a constant $C_k > 1$ such that for every non-negative locally integrable function f, the following two estimates*

1) $m_\alpha f \leq C_k m_\alpha R_k^\alpha f$;
2) $C_k^{-1} m_\alpha g \leq R_k^\alpha m_\alpha g \leq C_k m_\alpha g$

hold.

Lemma 7.3. *Let $k \in (0, \frac{1}{2})$. If $z' \in B^k(z)$, then $z \in B^{k'}(z')$, where $k' = \frac{k}{1-k}$, and $\chi_{B^k(z)}(z') \leq \chi_{B^{k'}(z')}(z)$. Moreover, there exists a constant $C_k > 1$ such that*

$$C_k^{-1}\mu_\alpha(B^k(z)) \leq \mu_\alpha(B^{k'}(z')) \leq C_k \mu_\alpha(B^k(z)).$$

Lemma 7.4. *Let $k \in \left(0, \frac{1}{5}\right)$ and $p(\cdot) \in \mathcal{P}_{\pm}^{\log}(\mathbb{B})$. For $w \in B_{p(\cdot)}$, there exists a constant $C_k > 1$ such that for all $z, z' \in \mathbb{B}$ such that $z' \in B^k(z)$, we have*

$$C_k^{-1} w(B^k(z)) \leq w(B^{k'}(z')) \leq C_k w(B^k(z)).$$

Proof. We have $B^k(z) \subset B^{2k'}(z')$ and $B^{k'}(z') \subset B^{6k}(z)$. From Theorem 5.20 and Remark 5.6, we have $w \in \Lambda$. Apply Remark 5.8 to conclude. □

Lemma 7.5. *Let $k \in (0, \frac{1}{2})$. There exists a positive constant C_k such that for all non-negative locally integrable f, g, we have*

$$\int_{\mathbb{B}} f(\zeta) R_k g(\zeta) d\mu_\alpha(\zeta) \leq C_k \int_{\mathbb{B}} g(z) R_k f(z) d\mu_\alpha(z).$$

We also recall the following elementary

Lemma 7.6. *Let $z_0 \in \mathbb{B}$ and $r > 1 - |z_0|$. For $z \in B(z_0, r)$ and $\zeta \in B^k(z)$, we have $\zeta \in B(z_0, ar)$ with $a = 2(2k+1)r$.*

In the rest of this section, to simplify the notation, we write $\sigma = R_k^\alpha w$. The following result is a generalisation to the variable exponent of the analogous result in [2, Lemma 10].

Proposition 7.7. *Let $p(\cdot) \in \mathcal{P}_{\pm}^{\log}(\mathbb{B})$, $k \in (0, \frac{1}{2})$ and $w \in B_{p(\cdot)}$. Then $R_k^\alpha w \in A_{p(\cdot)}$ with $[R_k^\alpha w]_{A_{p(\cdot)}} \lesssim [w]_{B_{p(\cdot)}}$.*

Proof. From Theorem 5.21, it suffices to show that

$$\frac{\sigma(B)}{\mu_\alpha(B)} \left(\frac{\sigma'(B)}{\mu_\alpha(B)} \right)^{p(z_0)-1} \lesssim [w]_{B_{p(\cdot)}}$$

for every pseudo-ball B of $\mathbb{B}$.

We write $a = 2k+1$. Let $B = B(z_0, r)$ be a pseudo-ball in $\mathbb{B}$. We set $B' = B(z_0, ar)$. We distinguish two cases: 1. $B \in \mathcal{B}$; 2. B is not a member of $\mathcal{B}$.

1. Suppose first that $B \in \mathcal{B}$. We claim that there exists a positive absolute constant C_k such that

$$\frac{\sigma(B)}{\mu_\alpha(B)} \leq C_k \frac{w(B')}{\mu_\alpha(B')}. \tag{7.1}$$

Indeed, from the Fubini-Tonelli theorem and Lemma 7.3, we have

$$\begin{aligned}
\sigma(B) &= \int_B \sigma(z)d\mu_\alpha(z) \\
&= \int_B \left(\frac{1}{\mu_\alpha(B^k(z))}\int_{B^k(z)} w(\zeta)d\mu_\alpha(\zeta)\right)d\mu_\alpha(z) \\
&= \int_{\mathbb{B}}\left(\int_{\mathbb{B}} \frac{\chi_{B^k(z)}(\zeta)\chi_B(z)}{\mu_\alpha(B^k(z))}d\mu_\alpha(z)\right)w(\zeta)d\mu_\alpha(\zeta) \\
&\lesssim \int_{\mathbb{B}}\left(\int_{\mathbb{B}} \frac{\chi_{B^{k'}(\zeta)}(z)\chi_{B'}(\zeta)}{\mu_\alpha(B^{k'}(\zeta))}d\mu_\alpha(z)\right)w(\zeta)d\mu_\alpha(\zeta) \\
&= w(B').
\end{aligned}$$

For the latter inequality, we have used Lemma 7.6. Moreover, since $B \subset B'$ and $\mu_\alpha(B) \simeq \mu_\alpha(B')$, we obtain

$$\frac{\sigma(B)}{\mu_\alpha(B)} \lesssim \frac{\sigma(B')}{\mu_\alpha(B')}.$$

Furthermore, from the Hölder inequality and Lemma 5.9, we have

$$\begin{aligned}
\sigma^{-1}(z) &= \frac{\mu_\alpha(B^k(z))}{w(B^k(z))} \\
&\leq \frac{2}{w(B^k(z))}\|w^{\frac{1}{p(\cdot)}}\chi_{B^k(z)}\|_{p(\cdot)}\|w^{-\frac{1}{p(\cdot)}}\chi_{B^k(z)}\|_{p'(\cdot)} \\
&\simeq \frac{1}{w(B^k(z))}w(B^k(z))^{\frac{1}{p(z)}}w'(B^k(z))^{\frac{1}{p'(z)}} \\
&= \left(\frac{w'(B^k(z))}{w(B^k(z))}\right)^{\frac{1}{p'(z)}}.
\end{aligned}$$

Hence

$$\sigma'(z) = \left(\sigma^{-1}\right)^{p'(z)-1}(z) \lesssim \left(\frac{w'(B^k(z))}{w(B^k(z))}\right)^{\frac{1}{p(z)}}.$$

From the Hölder inequality and Lemma 5.9, we have

$$\begin{aligned}
\sigma'(B) &= \int_B \sigma'(z)d\mu_\alpha(z) \\
&\lesssim \int_B \left(\frac{w'(B^k(z))}{w(B^k(z))}w(z)\right)^{\frac{1}{p(z)}} w(z)^{-\frac{1}{p(z)}}d\mu_\alpha(z) \\
&\leq 2\|w^{-\frac{1}{p(\cdot)}}\chi_B\|_{p'(\cdot)}\|\left(\frac{w'(B^k(.))}{w(B^k(.))}w(.)\right)^{\frac{1}{p(.)}}\chi_B\|_{p(\cdot)} \\
&\lesssim w'(B)^{\frac{1}{p'(z_0)}}\|\left(\frac{w'(B^k(.))}{w(B^k(.))}w(.)\right)^{\frac{1}{p(.)}}\chi_B\|_{p(\cdot)} \qquad (7.2)
\end{aligned}$$

Since $w' \in \Lambda$, from Lemma 5.9, we have $\beta := \|w'^{\frac{1}{p(\cdot)}}\chi_{B'}\|_{p(\cdot)} \simeq w'(B')^{\frac{1}{p(z_0)}}$. Thus as $B \subset B'$, from Lemma 7.3 and Lemma 7.4, we have

$$\begin{aligned}
&\rho_{p(.)}\left(\frac{1}{\beta}\left(\frac{w'(B^k(.))}{w(B^k(.))}w\right)^{\frac{1}{p(.)}}\chi_B\right) \\
&= \int_{\mathbb{B}} \frac{1}{\beta^{p(z)}}\frac{w'(B^k(z))}{w(B^k(z))}w(z)\chi_B(z)d\mu_\alpha(z)
\end{aligned}$$

$$\begin{aligned}
&\simeq \int\limits_{\mathbb{B}} \frac{1}{w'(B')} \frac{w'(B^k(z))}{w(B^k(z))} w(z)\chi_B(z)d\mu_\alpha(z)\\
&= w'(B')^{-1} \times \int\limits_{\mathbb{B}} \left(\frac{1}{w(B^k(z))} \int\limits_{\mathbb{B}} w'(\zeta)\chi_{B_k(z)}(\zeta)\chi_B(z)w(z)d\mu_\alpha(\zeta) \right) d\mu_\alpha(z)\\
&\leq C_k w'(B')^{-1} \times \int\limits_{\mathbb{B}} \left(\frac{1}{w(B_{k'}(\zeta))} \int\limits_{\mathbb{B}} \chi_{B_{k'}(\zeta)}(z)\chi_{B'}(\zeta)w(z)d\mu_\alpha(z) \right) w'(\zeta)d\mu_\alpha(\zeta)\\
&= C_k.
\end{aligned}$$

For the latter inequality, we have used Lemma 7.6. Hence we obtain

$$\left\| \left(\frac{w'(B^k(.))}{w(B^k(.))} w \right)^{\frac{1}{p(\cdot)}} \chi_B \right\|_{p(\cdot)} \lesssim C_k w'(B')^{\frac{1}{p(z_0)}}.$$

Consequently, we deduce from (7.2) that

$$\sigma'(B) \lesssim C_k w'(B)^{\frac{1}{p'(z_0)}} w'(B')^{\frac{1}{p(z_0)}} \leq C_k w'(B')$$

because $B \subset B'$. Moreover, as $\mu_\alpha(B) \simeq \mu_\alpha(B')$, we have

$$\frac{\sigma'(B)}{\mu_\alpha(B)} \leq C_k \frac{w'(B')}{\mu_\alpha(B')}$$

and hence

$$\left(\frac{\sigma'(B)}{\mu_\alpha(B)} \right)^{p(z_0)-1} \leq C'_k \left(\frac{w'(B')}{\mu_\alpha(B')} \right)^{p(z_0)-1}. \tag{7.3}$$

Combining (7.1) and (7.3) gives

$$\frac{\sigma(B)}{\mu_\alpha(B)} \left(\frac{\sigma'(B)}{\mu_\alpha(B)} \right)^{p(z_0)-1} \leq \gamma_k \frac{w(B')}{\mu_\alpha(B')} \left(\frac{w'(B')}{\mu_\alpha(B')} \right)^{p(z_0)-1} \leq \gamma_k [w]_{B_{p(\cdot)}} \tag{7.4}$$

by Theorem 5.20.

2. Suppose next that the pseudo-ball B is not a member of $\mathcal{B}$, i.e., $r \leq 1-|z_0|$. In the case, where $k(1-|z_0|) \leq r \leq 1-|z_0|$, we have $B \subset B(z_0, 1-|z_0|)$ and $\mu_\alpha(B) \simeq (1-|z_0|)^{n+\alpha} \simeq \mu_\alpha(B(z_0, 1-|z_0|))$. The pseudo-ball $B(z_0, 1-|z_0|)$ is a member of $\mathcal{B}$; so, we can apply to it the computations of the first case. We obtain

$$\begin{aligned}
\frac{\sigma(B)}{\mu_\alpha(B)} \left(\frac{\sigma'(B)}{\mu_\alpha(B)} \right)^{p(z_0)-1} &\lesssim \frac{\sigma(B(z_0, 1-|z_0|))}{\mu_\alpha(B(z_0, 1-|z_0|))} \left(\frac{\sigma'(B(z_0, 1-|z_0|))}{\mu_\alpha(B(z_0, 1-|z_0|))} \right)^{p(z_0)-1}\\
&\lesssim [w]_{B_{p(\cdot)}}.
\end{aligned}$$

Next, if $0 < r < k(1-|z_0|)$, then for $z \in B$, we have $(1-k)(1-|z_0|) \leq 1-|z| \leq (1+k)(1-|z_0|)$. This shows that $\mu_\alpha(B^k(z_0)) \simeq \mu_\alpha(B^k(z))$. We also claim that $w(B^k(z_0)) \simeq w(B^k(z))$. Indeed, it is easy to show the inclusions $B^k(z_0) \subset B(z, 4k(1-|z|))$ and $B^k(z) \subset B(z_0, 2k(2+k)(1-|z_0|))$. Then the claim follows with application of Remark 5.8. Combining with the estimate $\mu_\alpha(B^k(z_0)) \simeq \mu_\alpha(B^k(z))$ gives

$$\sigma(z) \simeq \sigma(z_0) \tag{7.5}$$

for every $z \in B$. Now, by Remark 2.8, $p'(\cdot)$ is a member of $\mathcal{P}_{\pm}^{\log}(\mathbb{B})$. Then by Lemma 5.9 and Corollary 2.11, we have

$$\sigma(z_0)^{1-p'(z_0)} \simeq \sigma(z)^{1-p'(z_0)} \simeq \sigma(z)^{1-p'(z)}.$$

Combining with (7.5) gives

$$\frac{\sigma(B)}{\mu_\alpha(B)} \left(\frac{\sigma'(B)}{\mu_\alpha(B)} \right)^{p(z_0)-1} \simeq 1. \tag{7.6}$$

The conclusion of the lemma follows after a combination of (7.4) and (7.6) with Theorem 5.20. □

Lemma 7.8. *Let $p(\cdot)\in\mathcal{P}_{\pm}^{\log}(\mathbb{B}), k\in(0,\frac{1}{2})$ and $w\in B_{p(\cdot)}$. Then*

$$(R_k^\alpha g(z))^{p(z)}\lesssim R_k^\alpha(g^{p(\cdot)})(z)+1$$

for all non-negative functions g such that $\|g\|_{p(\cdot),w}=1$ and all $z\in\mathbb{B}$.

Proof. As $\|g\|_{p(\cdot),w}=1$, from the Hölder inequality (assertion 1 of Proposition 2.7), we have

$$\frac{1}{2\|\chi_{B^k(z)}\|_{p'(\cdot),w'}}\int\limits_{B^k(z)}g(\zeta)d\mu_\alpha(\zeta)\le\|g\chi_{B^k(z)}\|_{p(\cdot),w}=1.$$

Therefore from the usual Hölder inequality and Lemma 2.10, we obtain

$$\begin{aligned}
(R_k^\alpha g(z))^{p(z)}&=\left(\frac{1}{2\|\chi_{B^k(z)}\|_{p'(\cdot),w'}}\int\limits_{B^k(z)}g(\zeta)d\mu_\alpha(\zeta)\right)^{p(z)}\mu_\alpha(B^k(z))^{-p(z)}2^{p(z)}\|\chi_{B^k(z)}\|_{p'(\cdot),w'}^{p(z)}\\
&\le 2^{p(z)}\left(\frac{1}{2\|\chi_{B^k(z)}\|_{p'(\cdot),w'}}\int\limits_{B^k(z)}g(\zeta)d\mu_\alpha(\zeta)\right)^{p_-(B)}\mu_\alpha(B^k(z))^{-p(z)}\|\chi_{B^k(z)}\|_{p'(\cdot),w'}^{p(z)}\\
&\lesssim\left(\frac{1}{\mu_\alpha(B^k(z))}\int\limits_{B^k(z)}g(\zeta)d\mu_\alpha(\zeta)\right)^{p_-(B)}\mu_\alpha(B^k(z))^{p_-(B)-p(z)}\|\chi_{B^k(z)}\|_{p'(\cdot),w'}^{p(z)-p_-(B)}\\
&\le\mu_\alpha(B^k(z))^{p_-(B)-p(z)}\|\chi_{B^k(z)}\|_{p'(\cdot),w'}^{p(z)-p_-(B)}\frac{1}{\mu_\alpha(B^k(z))}\int\limits_{B^k(z)}g(\zeta)^{p_-(B)}d\mu_\alpha(\zeta)\\
&\lesssim\frac{1}{\mu_\alpha(B^k(z))}\int\limits_{B^k(z)}g\chi_{g\ge1}(\zeta)^{p_-(B)}d\mu_\alpha(\zeta)+1\\
&\lesssim R_k^\alpha g^{p(\cdot)}(z)+1.
\end{aligned}$$

For the last but one inequality, we also used the following inequality:

$$\|\chi_{B^k(z)}\|_{p'(\cdot),w'}^{p(z)-p_-(B)}\le\max(1,\|\chi_{\mathbb{B}}\|_{p'(\cdot),w'}^{p_+-p_-}).$$

□

Lemma 7.9. *Let $p(\cdot)\in\mathcal{P}_{\pm}^{\log}(\mathbb{B})$, $k\in(0,\frac{1}{2})$ and $w\in B_{p(\cdot)}$. Then*

$$\|R_k^\alpha g\cdot w^{\frac{1}{p(\cdot)}}\|_{p(\cdot)}\lesssim\|g\cdot(R_k^\alpha w)^{\frac{1}{p(\cdot)}}\|_{p(\cdot)}$$

for all non-negative functions g belonging to $L^{p(\cdot)}(R_k^\alpha w d\mu_\alpha)$.

Proof. From Proposition 7.7, we have $\sigma=R_k^\alpha w\in A_{p(\cdot)}\subset B_{p(\cdot)}$ because $w\in B_{p(\cdot)}$. Without loss of generality, we assume that $\|g\|_{p(\cdot),\sigma}=1$. Thus from Lemma 7.8 and Lemma 7.5, we have

$$\begin{aligned}
\rho_{p(\cdot)}(R_k^\alpha g\cdot w^{\frac{1}{p(\cdot)}})&=\int\limits_{\mathbb{B}}(R_k^\alpha g(z))^{p(z)}w(z)d\mu_\alpha(z)\\
&\lesssim\int\limits_{\mathbb{B}}R_k^\alpha(g^{p(\cdot)})(z)w(z)d\mu_\alpha(z)+w(\mathbb{B})\\
&\lesssim\int\limits_{\mathbb{B}}g(z)^{p(z)}\sigma(z)d\mu_\alpha(z)+w(\mathbb{B})\\
&\lesssim 1+w(\mathbb{B}).
\end{aligned}$$

For the last inequality, apply Lemma 3.2. □

We still use the notation $\sigma=R_k^\alpha w$, with $k\in(0,\frac{1}{2})$.

Lemma 7.10. *Let $k \in (0, \frac{1}{2})$, $p(\cdot) \in \mathcal{P}_{\pm}^{\log}(\mathbb{B})$ and $w \in B_{p(\cdot)}$. Then there exists a positive constant $C([w']_{B_{p'(\cdot)}})$ depending on $[w']_{B_{p'(\cdot)}}$ such that for every non-negative function $g \in L^{p'(\cdot)}(\sigma')$, we have*

$$\|R_k^\alpha g\|_{p'(\cdot),w'} \leq C([w']_{B_{p'(\cdot)}})\|g\|_{p'(\cdot),\sigma'}$$

for all non-negative functions $g \in L^{p'(\cdot)}(\sigma')$.

Proof. Without loss of generality, we assume that $\|g\|_{p'(\cdot),\sigma'} = 1$. From Proposition 3.8 and Proposition 7.7, we have $\sigma' \in A_{p'(\cdot)} \subset B_{p'(\cdot)}$ because $w \in B_{p(\cdot)}$. Thus since $\|g\|_{p'(\cdot),\sigma'} = 1$, using Lemma 7.8 and Lemma 7.5, we have

$$\begin{aligned}
\rho_{p'(\cdot),w'}(R_k^\alpha g) &= \int_{\mathbb{B}} (R_k^\alpha g(z))^{p'(z)} w'(z) d\mu_\alpha(z) \\
&\lesssim \int_{\mathbb{B}} R_k^\alpha (g^{p'(\cdot)})(z) w'(z) d\mu_\alpha(z) + w'(\mathbb{B}) \\
&\lesssim \int_{\mathbb{B}} g(z)^{p'(z)} R_k^\alpha w'(z) d\mu_\alpha(z) + w'(\mathbb{B}) \\
&= \int_{\mathbb{B}} g(z)^{p'(z)} \sigma'(z)\sigma'(z)^{-1} R_k^\alpha w'(z) d\mu_\alpha(z) + w'(\mathbb{B}) \\
&\leq [w']_{B_{p'(\cdot)}} \int_{\mathbb{B}} g(z)^{p'(z)} \sigma'(z) d\mu_\alpha(z) + w'(\mathbb{B}) \\
&\lesssim [w']_{B_{p'(\cdot)}} + w'(\mathbb{B}).
\end{aligned}$$

Indeed, the last inequality follows from Lemma 3.2; for the last but one inequality, using Theorem 5.20 and Lemma 5.9 for w' in the place of w, we get

$$\sigma'(z)^{-1} R_k^\alpha w'(z) = (R_k^\alpha w(z))^{\frac{p'(z)}{p(z)}} R_k^\alpha w'(z) \lesssim [w']_{B_{p'(\cdot)}}$$

because $B^k(z)$ is 'almost' a member of $\mathcal{B}$, as it is a subset of the member $B(z, 1-|z|)$ of $\mathcal{B}$, $\mu_\alpha(B^k(z)) \simeq \mu_\alpha(B(z, 1-|z|))$ and $w' \in B_{p'(\cdot)}$. So, by Lemma 3.2, we get

$$\|R_k^\alpha g\|_{p'(\cdot),w'} \lesssim C([w']_{B_{p'(\cdot)}}).$$

□

Lemma 7.11. *Let $p(\cdot) \in \mathcal{P}_{\pm}^{\log}(\mathbb{B})$, f non-negative in $L^{p(\cdot)}(w)$, $k \in \left(0, \frac{1}{2}\right)$ and $w \in B_{p(\cdot)}$. Then*

$$\|R_k^\alpha f (R_k^\alpha w)^{\frac{1}{p(\cdot)}}\|_{p(\cdot)} \leq C([w']_{B_{p'(\cdot)}})\|f\|_{p(\cdot),w}.$$

Proof. We still write $\sigma = R_k^\alpha w$. By the duality (Proposition 3.1), there exists a non-negative function g satisfying $\|g\|_{p'(\cdot),\sigma'} = 1$ and such that

$$\|R_k^\alpha f\|_{p(\cdot),\sigma} \leq 2 \int_{\mathbb{B}} g(z) R_k^\alpha f(z) d\mu_\alpha(z).$$

Next, from Lemma 7.5, the Hölder inequality and Lemma 7.10, we obtain

$$\begin{aligned}
\|R_k^\alpha f\|_{p(\cdot),\sigma} &\lesssim \int_{\mathbb{B}} f(z) R_k^\alpha g(z) d\mu_\alpha(z) \\
&\leq 2\|f\|_{p(\cdot),w} \|R_k^\alpha g\|_{p'(\cdot),w'} \\
&\leq 2C([w']_{B_{p'(\cdot)}})\|f\|_{p(\cdot),w}\|g\|_{p'(\cdot),\sigma'} \\
&= 2C([w']_{B_{p'(\cdot)}})\|f\|_{p(\cdot),w}.
\end{aligned}$$

Hence we have the result. □

Theorem 7.12. *Let $p(\cdot) \in \mathcal{P}_{\pm}^{\log}(\mathbb{B})$. If $w \in B_{p(\cdot)}$, there exists a non-negative function C defined on $(0,\infty)$ such that for all $f \in L^{p(\cdot)}(w)$, we have*

$$\|m_\alpha f\|_{p(\cdot),w} \leq C([w]_{B_{p(\cdot)}})\|f\|_{p(\cdot),w}.$$

Proof. By Proposition 3.4, we have the equality $[w]_{B_{p(\cdot)}} = [w']_{B_{p'(\cdot)}}$. So, from Lemma 7.11, we have

$$\|R_k^\alpha f \cdot (R_k^\alpha w)^{\frac{1}{p(\cdot)}}\|_{p(\cdot)} \leq C([w]_{B_{p(\cdot)}})\|f\|_{p(\cdot),w}. \tag{7.7}$$

Hence $R_k^\alpha f \in L^{p(\cdot)}(\sigma)$. Next, since $\sigma \in A_{p(\cdot)}$ by Proposition 7.7, Corollary 3.11 gives

$$\|m_\alpha(R_k^\alpha f) \cdot (R_k^\alpha w)^{\frac{1}{p(\cdot)}}\|_{p(\cdot)} \lesssim \|(R_k^\alpha f) \cdot (R_k^\alpha w)^{\frac{1}{p(\cdot)}}\|_{p(\cdot)}. \tag{7.8}$$

Hence from (7.7) and (7.8), we have $m_\alpha(R_k^\alpha f) \in L^{p(\cdot)}(R_k^\alpha w d\mu_\alpha)$. Now, Lemma 7.9 gives

$$\|R_k^\alpha(m_\alpha(R_k^\alpha f))w^{\frac{1}{p(\cdot)}}\|_{p(\cdot)} \lesssim \|m_\alpha(R_k^\alpha f) \cdot (R_k^\alpha w)^{\frac{1}{p(\cdot)}}\|_{p(\cdot)}.$$

Next, by Proposition 7.2, there exists a positive constant C such that

$$m_\alpha f \leq C R_k^\alpha(m_\alpha(R_k^\alpha f)).$$

This implies that

$$\|m_\alpha f\|_{p(\cdot),w} \lesssim \|m_\alpha(R_k^\alpha f) \cdot (R_k^\alpha w)^{\frac{1}{p(\cdot)}}\|_{p(\cdot)}.$$

Finally, applying (7.8) and (7.7) successively, we have the result. □

8. A Weighted Extrapolation Theorem and the Proof of the Sufficient Condition in Theorem 1.6

We are now ready to prove the sufficient condition in Theorem 1.6, we adapt the strategy used in [10].

8.1. Preliminary results.

We recall the B_1 class of weights. A weight w belongs to B_1 if

$$[w]_{B_1} := \operatorname*{ess\,sup}_{z\in\mathbb{B}} \frac{m_\alpha w(z)}{w(z)} < \infty.$$

In $\mathbb{R}^n$, the analogue of the following factorisation theorem was proved for the Muckenhoupt classes $A_p, 1 < p < \infty$, by Jones [13].

Theorem 8.1. *For a constant exponent p such that $1 < p < \infty$, the following two assertions are equivalent:*

1) $w \in B_p$;
2) *there exist $w_1 \in B_1$ and $w_2 \in B_1$ such that $w = w_1 w_2^{1-p}$.*

Proof. We first show the implication 2) ⇒ 1). Suppose that $w = w_1 w_2^{1-p}$ with $w_1, w_2 \in B_1$. For all $B \in \mathcal{B}$ and $z \in B$, we have

$$\frac{1}{\mu_\alpha(B)}\int_B w_i d\mu_\alpha \leq [w_i]_{B_1} w_i(z), \quad i = 1,2. \tag{8.1}$$

Thus as $(1-p')(1-p)=1$, we have $w^{1-p'}=\left(w_1 w_2^{1-p}\right)^{1-p'}=w_1^{1-p'}w_2$. So, from (8.1), we have

$$\begin{aligned}
&\left(\frac{1}{\mu_\alpha(B)}\int\limits_B w d\mu_\alpha\right)\left(\frac{1}{\mu_\alpha(B)}\int\limits_B w^{1-p'}d\mu_\alpha\right)^{p-1}\\
&=\left(\frac{1}{\mu_\alpha(B)}\int\limits_B w_1 w_2^{1-p}d\mu_\alpha\right)\left(\frac{1}{\mu_\alpha(B)}\int\limits_B w_1^{1-p'}w_2 d\mu_\alpha\right)^{p-1}\\
&\le [w_1]_{B_1}[w_2]_{B_1}^{p-1}\left(\frac{1}{\mu_\alpha(B)}\int\limits_B w_2 d\mu_\alpha\right)^{1-p}\left(\frac{1}{\mu_\alpha(B)}\int\limits_B w_1 d\mu_\alpha\right)\\
&\times\left(\frac{1}{\mu_\alpha(B)}\int\limits_B w_2 d\mu_\alpha\right)^{p-1}\left(\frac{1}{\mu_\alpha(B)}\int\limits_B w_1 d\mu_\alpha\right)^{-1}\\
&=[w_1]_{B_1}[w_2]_{B_1}^{p-1}.
\end{aligned}$$

Hence $w\in B_p$.

We next show the converse implication 2) $\Rightarrow$ 1). Suppose that $w\in B_p$. Set $q=pp'$ and define the operator S_1 on the space $\mathcal{M}$ by

$$S_1 f(z)=w(z)^{\frac{1}{q}}\left(m_\alpha\left(f^{p'}w^{-\frac{1}{p}}\right)(z)\right)^{\frac{1}{p'}}.$$

By the Minkowski inequality, S_1 is sublinear. Moreover, from the constant exponent version of Theorem 7.12 [2, Proposition 3], we have

$$\int\limits_{\mathbb{B}} S_1 f(z)^q d\mu_\alpha(z)=\int\limits_{\mathbb{B}}\left(m_\alpha\left(f^{p'}w^{-\frac{1}{p}}\right)(z)\right)^p w(z)d\mu_\alpha(z)\lesssim C\left([w]_{B_p}\right)\int\limits_{\mathbb{B}} f^q(z)d\mu_\alpha(z).$$

In other words, $\|S_1\|_q\lesssim\left(C\left([w]_{B_p}\right)\right)^{\frac{1}{q}}$.

Similarly, denote again $w'=w^{1-p'}\in B_{p'}$ and define the operator S_2 on the space $\mathcal{M}$ by

$$S_2 f(z)=w'(z)^{\frac{1}{q}}\left(m_\alpha\left(f^p w'^{-\frac{1}{p'}}\right)(z)\right)^{\frac{1}{p}}.$$

By the Minkowski inequality, S_2 is also sublinear. Moreover,

$$\int\limits_{\mathbb{B}} S_2 f(z)^q d\mu_\alpha(z)\lesssim C\left([w']_{B_{p'}}\right)\int\limits_{\mathbb{B}} f^q(z)d\mu_\alpha(z).$$

In other words, $\|S_2\|_q\lesssim C\left(\left([w']_{B_{p'}}\right)\right)^{\frac{1}{q}}=\left(C\left([w]_{B_p}\right)\right)^{\frac{1}{q}}$.

We use the following

Lemma 8.2. *Set $S=S_1+S_2$ and define the operator $\mathcal{R}$ on $\mathcal{M}$ by*

$$\mathcal{R}h(z)=\sum_{k=0}^{\infty}\frac{S^k h(z)}{2^k\|S\|_q^k}$$

with $S^0 h=|h|$. Then

a) $|h|\le\mathcal{R}h$;
b) $\|\mathcal{R}h\|_q\le 2\|h\|_q$;
c) $S(\mathcal{R}h)\le 2\|S\|_q\mathcal{R}h$.

Proof of Lemma 8.2. By the definition of $\mathcal{R}h$, we have $h\le\mathcal{R}h$. Moreover,

$$\|\mathcal{R}h\|_q\le\sum_{k=0}^{\infty}\frac{\|S^k h\|_q}{2^k\|S\|_q^k}\le\|h\|_q\sum_{k=0}^{\infty}\frac{1}{2^k}=2\|h\|_q.$$

Next, the sublinearity of S gives $S(\mathcal{R}h)\le 2\|S\|_q\mathcal{R}h$. □

Applying assertion c) of Lemma 8.2, we obtain

$$w(z)^{\frac{1}{q}}\left(m_\alpha\left((\mathcal{R}h)^{p'}w^{-\frac{1}{p}}\right)(z)\right)^{\frac{1}{p'}} = S_1(\mathcal{R}h)(z) \leq S(\mathcal{R}h)(z) \leq 2\|S\|_q\mathcal{R}h(z). \tag{8.2}$$

Now, set $w_2 = (\mathcal{R}h)^{p'}w^{-\frac{1}{p}}$. By (8.2), we have $w_2 \in B_1$.

Similarly, we have

$$w'(z)^{\frac{1}{q}}\left(m_\alpha\left(\mathcal{R}h\right)^p w'^{-\frac{1}{p'}}(z)\right)^{\frac{1}{p}} = S_2(\mathcal{R}h)(z) \leq S(\mathcal{R}h) \leq 2\|S\|_q\mathcal{R}h(z). \tag{8.3}$$

Now, set $w_1 = (\mathcal{R}h)^p w'^{-\frac{1}{p'}}$. By (8.3), we have $w_1 \in B_1$. Moreover, $w_1 w_2^{1-p} = w \in B_p$. This finishes the proof of Theorem 8.1. □

Lemma 8.3. *Let $p(\cdot) \in \mathcal{P}^{\log}_{\pm}(\mathbb{B})$ and $w \in B_{p(\cdot)}$. We define the operator R on $L^{p(\cdot)}(w)$ by*

$$Rh(x) = \sum_{k=0}^{\infty} \frac{m_\alpha^k h(x)}{2^k\|m_\alpha\|^k_{L^{p(\cdot)}(w)}}$$

where for $k \geq 1, m_\alpha^k = \underbrace{m_\alpha \circ m_\alpha \circ \cdots \circ m_\alpha}_{k-times}$ and $m_\alpha^0 h = |h|$. Then R satisfies the following properties:

a) $|h| \leq Rh$;
b) *R is bounded on $L^{p(\cdot)}(w)$ and $\|Rh\|_{p(\cdot),w} \leq 2\|h\|_{p(\cdot),w}$;*
c) *$Rh \in B_1$ and $[Rh]_{B_1} \leq 2\|m_\alpha\|_{L^{p(\cdot)}(w)}$.*

Proof. The proof of assertions a) and b) are the same as for assertions a) and b) of Lemma 8.2. Here, we use the sublinearity of m_α.

Finally, by the definition of Rh, we have

$$\begin{aligned} m_\alpha(Rh)(x) &\leq \sum_{k=0}^{\infty} \frac{m_\alpha^{k+1}h(x)}{2^k\|m_\alpha\|^k_{L^{p(\cdot)}(w)}} \\ &\leq 2\|m_\alpha\|_{L^{p(\cdot)}(w)} \sum_{k=0}^{\infty} \frac{m_\alpha^{k+1}h(x)}{2^{k+1}\|m_\alpha\|^{k+1}_{L^{p(\cdot)}(w)}} \\ &\leq 2\|m_\alpha\|_{L^{p(\cdot)}(w)} Rh(x). \end{aligned}$$

Thus $Rh \in B_1$ and $[Rh]_{B_1} \leq 2\|m_\alpha\|_{L^{p(\cdot)}(w)}$. □

Lemma 8.4. *Let $p(\cdot) \in \mathcal{P}^{\log}_{\pm}(\mathbb{B})$ and $w \in B_{p(\cdot)}$. Define the operator H on $L^{p'(\cdot)}$ by*

$$Hh = \mathcal{R}'\left(hw^{\frac{1}{p(\cdot)}}\right)w^{-\frac{1}{p(\cdot)}}$$

. where

$$\mathcal{R}'g(x) = \sum_{k=0}^{\infty} \frac{m_\alpha^k g(x)}{2^k\|m_\alpha\|^k_{L^{p'(\cdot)}(w')}}.$$

Then

a) $|h| \leq Hh$;
b) *H is bounded on $L^{p'(\cdot)}$ and $\|Hh\|_{p'(\cdot)} \leq 2\|h\|_{p'(\cdot)}$;*
c) *$Hh \cdot w^{\frac{1}{p(\cdot)}} \in B_1$ and $[Hh \cdot w^{\frac{1}{p(\cdot)}}]_{B_1} \leq 2\|m_\alpha\|_{L^{p'(\cdot)}(w')}$.*

Proof. The proof is the same as for Lemma 8.3. We replace $p(\cdot)$ by $p'(\cdot)$ and $w \in B_{p(\cdot)}$ by $w' \in B_{p'(\cdot)}$. The property $p'(\cdot) \in \mathcal{P}^{\log}_{\pm}(\mathbb{B})$ comes from Remark 2.8. □

8.2. **A weighted extrapolation theorem.** We denote by $\mathcal{F}$ a family of couples of non-negative measurable functions. We are now ready to state and prove the following weighted variable extrapolation theorem.

Theorem 8.5. *Suppose that for some constant exponent $p_0 > 1$, there exists a function $C : (0, \infty) \to (0, \infty)$ such that for all $v \in B_{p_0}$ and $(F, G) \in \mathcal{F}$, we have*

$$\int_{\mathbb{B}} F(x)^{p_0} v(x) d\mu_\alpha(x) \leq C([v]_{B_{p_0}}) \int_{\mathbb{B}} G(x)^{p_0} v(x) d\mu_\alpha(x). \tag{8.4}$$

Then given $p(\cdot) \in \mathcal{P}_{\pm}^{\log}(\mathbb{B})$ and $w \in B_{p(\cdot)}$, we have

$$\|F\|_{p(\cdot),w} \leq 16 \times 4^{-\frac{1}{p_0}} \left(C([v]_{B_{p_0}})\right)^{\frac{1}{p_0}} \|G\|_{p(\cdot),w}$$

for all $(F, G) \in \mathcal{F}$ and $F \in L^{p(\cdot)}(w)$.

Proof. We use the technique of Cruz–Uribe and Wang in [10, Theorem 2.6]. Let $(F, G) \in \mathcal{F}$. If $\|F\|_{p(\cdot),w} = 0$, we have the result. Otherwise, $\|F\|_{p(\cdot),w} > 0$ and hence $\|G\|_{p(\cdot),w} > 0$, because if $\|G\|_{p(\cdot),w} = 0$, then $G = 0$ a.e. and by (8.4) we will have $F = 0$ a.e. Henceforth, we assume $0 < \|F\|_{p(\cdot),w} < \infty$ and $0 < \|G\|_{p(\cdot),w} < \infty$. Define

$$h_1 = \frac{F}{\|F\|_{p(\cdot),w}} + \frac{G}{\|G\|_{p(\cdot),w}},$$

then $\|h_1\|_{p(\cdot),w} \leq 2$ and so, $h_1 \in L^{p(\cdot)}(w)$.

Since $F \in L^{p(\cdot)}(w)$, by the duality (Proposition 3.1), there exists $h_2 \in L^{p'(\cdot)}$ such that $\|h_2\|_{p'(\cdot)} = 1$ and

$$\|F\|_{p(\cdot),w} \leq 2 \int_{\mathbb{B}} F w^{\frac{1}{p(\cdot)}} h_2 d\mu_\alpha \leq 2 \int_{\mathbb{B}} F(Hh_2) w^{\frac{1}{p(\cdot)}} d\mu_\alpha, \tag{8.5}$$

where the latter inequality comes from assertion a) of Lemma 8.4.

Set $\gamma = \frac{1}{p_0'}$. By the usual Hölder inequality, we have

$$\begin{aligned}\int_{\mathbb{B}} F(Hh_2) w^{\frac{1}{p(\cdot)}} d\mu_\alpha &= \int_{\mathbb{B}} F(Rh_1)^{-\gamma}(Rh_1)^{\gamma}(Hh_2) w^{\frac{1}{p(\cdot)}} d\mu_\alpha \\ &\leq I_1^{\frac{1}{p_0}} I_2^{\frac{1}{p_0'}},\end{aligned} \tag{8.6}$$

where

$$I_1 := \int_{\mathbb{B}} F^{p_0}(Rh_1)^{1-p_0}(Hh_2) w^{\frac{1}{p(\cdot)}} d\mu_\alpha$$

and

$$I_2 := \int_{\mathbb{B}} (Rh_1)(Hh_2) w^{\frac{1}{p(\cdot)}} d\mu_\alpha.$$

In addition, from Lemma 8.3 and Lemma 8.4, respectively, R is bounded on $L^{p(\cdot)}(w)$ and H is bounded on $L^{p'(\cdot)}$. Thus by the Hölder inequality, assertions b) of Lemma 8.3 and Lemma 8.4, we have

$$I_2 \leq 2\|Rh_1\|_{p(\cdot),w}\|Hh_2\|_{p'(\cdot)} \leq 8\|h_1\|_{p(\cdot),w}\|h_2\|_{p'(\cdot)} \leq 16.$$

By the definition of h_1 and assertion a) of Lemma 8.3, we have

$$\frac{\varphi}{\|\varphi\|_{p(\cdot),w}} \leq h_1 \leq Rh_1$$

for $\varphi \in \{F,\ G\}$. Next, by the Hölder inequality and assertion b) of Lemma 8.4, we have

$$I_1 \leq \int_{\mathbb{B}} F^{p_0}(\zeta) \left(\frac{F(\zeta)}{\|F\|_{p(\cdot),w}}\right)^{1-p_0} H(\zeta)h_2(\zeta) w^{\frac{1}{p(\zeta)}} d\mu_\alpha(\zeta)$$

$$
\begin{aligned}
&= \|F\|_{p(\cdot),w}^{p_0-1} \int_{\mathbb{B}} F(\zeta)H(\zeta)h_2(\zeta)w^{\frac{1}{p(\zeta)}}\,d\mu_\alpha(\zeta)\\
&\le 2\|F\|_{p(\cdot),w}^{p_0-1}\|F\|_{p(\cdot),w}\|Hh_2\|_{p'(\cdot)}\\
&\le 4\|F\|_{p(\cdot),w}^{p_0}\\
&< \infty.
\end{aligned}
$$

Since $Rh_1 \in B_1$ and $(Hh_2)w^{\frac{1}{p(\cdot)}} \in B_1$ by Lemma 8.3 and Lemma 8.4, respectively, it follows from Theorem 8.1 that $v := (Rh_1)^{1-p_0}\big(Hh_2 w^{\frac{1}{p(\cdot)}}\big) \in B_{p_0}$. Hence by (8.4) and the same argument as above, we have

$$
\begin{aligned}
I_1 &= \int_{\mathbb{B}} F^{p_0}(Rh_1)^{1-p_0}(Hh_2)w^{\frac{1}{p(\cdot)}}\,d\mu_\alpha\\
&\le C([v]_{B_{p_0}})\int_{\mathbb{B}} G^{p_0}(Rh_1)^{1-p_0}(Hh_2)w^{\frac{1}{p(\cdot)}}\,d\mu_\alpha\\
&\le C([v]_{B_{p_0}})\int_{\mathbb{B}} G^{p_0}\left(\frac{G}{\|G\|_{p(\cdot),w}}\right)^{1-p_0}(Hh_2)w^{\frac{1}{p(\cdot)}}\,d\mu_\alpha\\
&= C([v]_{B_{p_0}})\|G\|_{p(\cdot),w}^{p_0-1}\int_{\mathbb{B}} G(Hh_2)w^{\frac{1}{p(\cdot)}}\,d\mu_\alpha\\
&\le 2C([v]_{B_{p_0}})\|G\|_{p(\cdot),w}^{p_0-1}\|G\|_{p(\cdot),w}\|Hh_2\|_{p'(\cdot)}\\
&\le 4C([v]_{B_{p_0}})\|G\|_{p(\cdot),w}^{p_0}.
\end{aligned}
$$

Thus from (8.5) and (8.6), we have the result. □

8.3. **The end of the proof of the sufficient condition in Theorem 1.6.** We prove the following

Proposition 8.6. *Let $p(\cdot) \in \mathcal{P}_{\pm}^{\log}(\mathbb{B})$ and $w \in B_{p(\cdot)}$. Then P_α^+ is a continuous operator on $L^{p(\cdot)}(w)$. Consequently, the Bergman projector P_α extends to a continuous operator on $L^{p(\cdot)}(w)$.*

Proof. We call again $\mathcal{C}_c(\mathbb{B})$ the space of continuous functions of compact support in $\mathbb{B}$ and take $\mathcal{F} = \{(P_\alpha^+ f, |f|) : \ f \in \mathcal{C}_c(\mathbb{B})\}$. We recall from Proposition 2.6 that $\mathcal{C}_c(\mathbb{B})$ is a dense subspace in $L^{p(\cdot)}(w)$.

Let p_0 be an arbitrary constant exponent greater than 1. Let $v \in B_{p_0}$. By Theorem 1.5, for every $f \in \mathcal{C}_c(\mathbb{B})$, we have

$$\int_{\mathbb{B}} (P_\alpha^+ f)^{p_0} v\,d\mu_\alpha \le C\left([v]_{B_{p_0}}\right)\int_{\mathbb{B}} |f|^{p_0} v\,d\mu_\alpha.$$

Thus by Theorem 8.5, for all $f \in \mathcal{C}_c(\mathbb{B})$, we have

$$\|P_\alpha^+ f\|_{p(\cdot),w} \le 16 \times 4^{-\frac{1}{p_0}}\left(C([v]_{B_{p_0}})\right)^{\frac{1}{p_0}}\|f\|_{p(\cdot),w}.$$

We conclude by density. □

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