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GLOBAL WELL-POSEDNESS, GEVREY CLASS REGULARITY FOR THE DEBYE–HÜCKEL SYSTEM IN VARIABLE FOURIER–BESOV–MORREY SPACES

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Abstract. This paper is devoted to studying the existence of solutions for the Cauchy problem of the Debye–Hückel system with low regularity initial data in variable Fourier–Besov–Morrey spaces. We show that there exists a unique global solution if the initial data are sufficiently small and belong to the variable Fourier–Besov space $\mathscr{F}\mathscr{N}^{-2+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q} \times \mathscr{F}\mathscr{N}^{-2+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}$. In addition, we study the analyticity of global solutions and prove that global solutions are Gevrey regular.

1. Introduction

In this paper, for the Debye–Hückel system in $\mathbb{R}^n \times \mathbb{R}^+$, we consider the following Cauchy problem:

$$\begin{cases} \partial_t v - \Delta v = -\nabla \cdot (v \nabla \phi) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \partial_t w - \Delta w = \nabla \cdot (w \nabla \phi) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \Delta \phi = v - w & \text{in } \mathbb{R}^n \times (0,\infty), \\ v(x,0) = v_0(x), \quad w(x,0) = w_0(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where the unknown functions $v = v(x,t)$ and $w = w(x,t)$ denote densities of the electron and the hole in electrolytes, respectively, $\phi = \phi(x,t)$ denotes the electric potential, $v_0(x)$ and $w_0(x)$ are the initial data. Throughout this paper, we assume that $n \geq 4$.

Note that the function ϕ is determined by the Poisson equation in the third equation (1.1) and has the form

$$\phi = (-\Delta)^{-1}(w - v) = \mathscr{F}^{-1}(|\xi|^{-2}\mathscr{F}(w - v)),$$

where $\mathscr{F}$ is the Fourier transform. So, system (1.1) can be reduced to the system

$$\begin{cases} \partial_t v - \Delta v = -\nabla \cdot \big(v \nabla (-\Delta)^{-1}(w - v)\big) & \text{in } \mathbb{R}^n \times (0,\infty) \\ \partial_t w - \Delta w = \nabla \cdot \big(w \nabla (-\Delta)^{-1}(w - v)\big) & \text{in } \mathbb{R}^n \times (0,\infty) \\ v(x,0) = v_0(x), \quad w(x,0) = w_0(x) & \text{in } \mathbb{R}^n. \end{cases}$$

The Debye–Hückel system (1.1) is scaling invariant in the following sense: if (v,w) solves (1.1) with the initial data (v_0,w_0) (ϕ can be determined by (v,w)), then (v_γ, w_γ) with $(v_\gamma, w_\gamma)(x,t) := \big(\gamma^2 v, \gamma^2 w\big)\big(\gamma x, \gamma^2 t\big)$ is also a solution to (1.1) with the initial data

$$(v_{0,\gamma}, w_{0,\gamma})(x) := \big(\gamma^2 v_0, \gamma^2 w_0\big)(\gamma x) \tag{1.2}$$

(ϕ_γ can be determined by (v_γ, w_γ)).

Definition 1.1. A critical space for the initial data of system (1.1) is any Banach space $E \subset \mathscr{S}'(\mathbb{R}^n)$ whose norm is invariant under the scaling (1.2) for all $\gamma > 0$, i.e.,

$$\|(v_{0,\gamma}, w_{0,\gamma})(x)\|_E \approx \|(v_0, w_0)(x)\|_E .$$

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In accordance with these scales, we can show that the space pairs $\mathscr{F}\mathscr{N}_{p(\cdot),\lambda(\cdot),q}^{-2+\frac{n}{p'(\cdot)}} \times \mathscr{F}\mathscr{N}_{p(\cdot),\lambda(\cdot),q}^{-2+\frac{n}{p'(\cdot)}}$ are critical for (1.1).

System (1.1) has been studied extensively in various function spaces. Karch in [20] established the proof of the existence and uniqueness of global solutions of system (1.1) for initial data in the Besov spaces $B_{p,\infty}^{s}$ with the conditions $-1 < s < 0$ and $p = \frac{n}{s+2}$. Later, Zhao et al. [27] established the global and local well-posedness for system (1.1) in the critical Besov space $\dot{\mathscr{B}}_{p,r}^{-2+\frac{n}{p}}(\mathbb{R}^n)$ with $2 \leq p < 2n$ and $1 \leq r \leq \infty$ (which improved the corresponding results of Karch obtained in [20]). Kurokiba and Ogawa in [21] obtained similar results for the initial data in critical Lebesgue and Sobolev spaces. Very recently, Azanzal, Abbassi and Allalou [4] proved that small data global existence and large data local existence of mild solutions of system (1.1) in critical Fourier–Morrey–Besov space $\mathscr{F}\mathscr{N}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}} \times \mathscr{F}\mathscr{N}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}}$ (more related research can be found in [5–10, 13, 16, 17, 22–25]). It should be noted that for the Navier–Stokes equations, there is no existence result for initial data in a space with regularity index $s < -1$. In fact, the nonlinear term of (1.1) appears to be more closely related to the quadratic nonlinear heat equation ($\sim u^2$) than to the Navier–Stokes equations ($\sim u \cdot \nabla u$).Thus, the Debye–Hückel system has a better property than the Navier–Stokes equations in regard to the existence of solutions.

Our first aim in this paper is to show the existence of global solutions of system (1.1). The second aim is to prove the analyticity of global solutions to system (1.1) by using the method of Gevrey estimate, which was first introduced by Foias and Temam [18]. Since then, the Gevrey class technique has become an important approach in the study of the space analyticity of solutions, which was later developed by several researchers, particularly with regard to the Navier–Stokes equations (NSE). In 2017, Zhao [26] proved that the global mild solutions to system (1.1) are Gevrey regular for all $2 \leq p < 2n$ and $1 \leq r \leq \infty$. Inspired by this, we will establish the Gevrey class regularity for system (1.1) in the variable Fourier–Besov–Morrey spaces $\mathscr{F}\mathscr{N}_{p(\cdot),\lambda(\cdot),q}^{-2+\frac{n}{p'(\cdot)}}$. The Gevrey class technique enables us to avoid cumbersome recursive estimation of higher-order derivatives.

Throughout this paper, let X, Y be Banach spaces, we use $(v,w) \in X$ to denote $(v,w) \in X \times X$ and

$$\|v\|_{X\cap Y} := \|v\|_X + \|v\|_Y; \qquad \|(v,w)\|_X := \|v\|_X + \|w\|_X,$$

C will denote the constants which may be different at different places, $A \sim B$ means that there are two constants $C_1, C_2 > 0$ such that

$$C_1 B \leq A \leq C_2 B,$$

$V \lesssim W$ denotes the estimate $V \leq CW$ for some constant $C \geq 1$, and p' is the conjugate of p satisfying $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 \leq p \leq \infty$.

1.1. **Preliminaries.** The proofs of the results discussed in this work are based on a dyadic partition of unity in the Fourier variables, known as the homogeneous Littlewood–Paley decomposition. We present briefly this construction below. For more detail, we refer the reader to [11].

Let $f \in S'(\mathbb{R}^n)$. Define the Fourier transform as

$$\hat{f}(\xi) = \mathscr{F}f(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx$$

and its inverse Fourier transform as

$$\check{f}(x) = \mathscr{F}^{-1}f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)d\xi.$$

Let $\varphi \in S(\mathbb{R}^d)$ be such that $0 \leq \varphi \leq 1, \quad \operatorname{supp}(\varphi) \subset \left\{\xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\right\}$ and

$$\sum_{j\in\mathbb{Z}} \varphi\left(2^{-j}\xi\right) = 1, \quad \text{for all} \quad \xi \neq 0.$$

We denote

$$\varphi_j(\xi)=\varphi\left(2^{-j}\xi\right), \quad \psi_j(\xi)=\sum_{k\leq j-1}\varphi_k(\xi)$$

and

$$h(x)=\mathscr{F}^{-1}\varphi(x), \quad g(x)=\mathscr{F}^{-1}\psi(x).$$

We now present some frequency localization operators:

$$\Delta_j f:=\varphi_j(D)f=2^{nj}\int\limits_{\mathbb{R}^d} h\left(2^j y\right)f(x-y)dy,$$

$$S_j f:=\sum_{k\leq j-1}\Delta_k f=\psi_j(D)f=2^{nj}\int\limits_{\mathbb{R}^d} g\left(2^j y\right)f(x-y)dy,$$

where $\Delta_j=S_j-S_{j-1}$ is a frequency projection to the annulus $\{|\xi|\sim 2^j\}$ and S_j is a frequency to the ball $\{|\xi|\lesssim 2^j\}$.
From the definition of Δ_j and S_j, one easily derives that

$$\Delta_j\Delta_k f=0, \quad \text{if} \quad |j-k|\geq 2$$

$$\Delta_j\left(S_{k-1}f\Delta_k f\right)=0, \quad \text{if} \quad |j-k|\geq 5.$$

We define the Morrey space with a variable exponent $M^{\lambda(\cdot)}_{p(\cdot)}$.

Definition 1.2 ([2])**.** Let $\mathscr{P}_0$ denote the set of all measurable functions $p(\cdot):\mathbb{R}^n\to(0,\infty)$ such that

$$0<p_-=\operatorname{ess}\inf_{\mathrm{x}\in\mathbb{R}^n}p(x), \quad \operatorname{ess}\sup_{\mathrm{x}\in\mathbb{R}^n}p(x)\ =p_+<\infty.$$

The Lebesgue space with a variable exponent is defined by

$$L^{p(\cdot)}(\mathbb{R}^n)=\left\{f:\mathbb{R}^n\to\mathbb{R}\text{ is measurable},\int\limits_{\mathbb{R}^n}|f(x)|^{p(x)}dx<\infty\right\},$$

with the Luxemburg–Nakano norm

$$\|f\|_{L^{p(\cdot)}}=\inf\left\{\lambda>0:\int\limits_{\mathbb{R}^n}(\frac{|f(x)|}{\lambda})^{p(x)}dx\leq 1\right\}.$$

The space $L^{p(\cdot)}(\mathbb{R}^n)$ equipped with the norm $\|\cdot\|_{L^{p(\cdot)}}$ is a Banach space, since the $L^{p(\cdot)}$ does not have the same desired properties as L^p. So, we assume the following standard conditions to ensure that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$:

(1) (Locally log-Hölder continuous). There exists a constant $C_{\log}(p)$ such that

$$|p(x)-p(y)|\leq\frac{C_{\log}(p)}{\log(e+|x-y|^{-1})}, \text{ for any } x,\ y\in\mathbb{R}^n \text{ and } x\neq y.$$

(2) (Locally log-Hölder continuous). There exist a constant $C_{\log}(p)$ and some constant independent of x such that

$$|p(x)-p_\infty|\leq\frac{C_{\log}(p)}{\log(e+|x|)}, \text{ for all } x\in\mathbb{R}^n.$$

$C^{\log}(\mathbb{R}^n)$ denote the set of all functions $p(\cdot):\mathbb{R}^n\to\mathbb{R}$ satisfying (1) and (2).

Definition 1.3 ([2])**.** Let $p(\cdot)$, $\lambda(\cdot)\in\mathscr{P}_0(\mathbb{R}^n)$ with $0<p_-\leq p(x)\leq\lambda(x)\leq\infty$, the Morrey space with a variable exponent $M^{\lambda(\cdot)}_{p(\cdot)}:=M^{\lambda(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ is defined as the set of all measurable functions on $\mathbb{R}^n$ such that

$$\|f\|_{M^{\lambda(\cdot)}_{p(\cdot)}}:=\sup_{x_0\in\mathbb{R}^n,r>0}\|r^{\frac{n}{\lambda(x)}-\frac{n}{p(x)}}f\chi_{B(x_0,r)}\|_{L^{p(\cdot)}}<\infty.$$

According to the definition of the $L^{p(\cdot)}$-norm, $\|f\|_{M^{\lambda(\cdot)}_{p(\cdot)}}$ also has the following form:

$$\|f\|_{M^{\lambda(\cdot)}_{p(\cdot)}}:=\sup_{x_0\in\mathbb{R}^n,r>0}\inf\left\{\lambda>0:\rho_{p(\cdot)}(r^{\frac{n}{\lambda(x)}-\frac{n}{p(x)}}\frac{f}{\lambda}\chi_{B(x_0,r)})\leq 1\right\}.$$

We now present some important lemmas from [2].

Lemma 1.1. *Let $p(\cdot) \in \mathscr{P}_0(\mathbb{R}^n)$. For any measurable function f,*

$$\sup_{x\in\mathbb{R}^n, r>0} \rho_{p(\cdot)}(f\chi_{B(x,r)}) = \rho_{p(\cdot)}(f) \ .$$

Lemma 1.2. *If $p(\cdot) \in \mathscr{P}_0(\mathbb{R}^n)$, then $\|f\|_{M^{p(\cdot)}_{p(\cdot)}} = \|f\|_{L^{p(\cdot)}}$.*

Definition 1.4 ([2])**.** Let $p(\cdot)$, $q(\cdot)$, $\lambda(\cdot) \in \mathscr{P}_0(\mathbb{R}^n)$ with $p(\cdot) \le \lambda(\cdot)$, the mixed Morrey-sequence space $l^{q(\cdot)}(M^{\lambda(\cdot)}_{p(\cdot)})$ includes all sequences $\{h_j\}_{j\in\mathbb{Z}}$ of measurable functions in $\mathbb{R}^n$ such that $\rho_{l^{q(\cdot)}(M^{\lambda(\cdot)}_{p(\cdot)})}(\lambda\{h_j\}_{j\in\mathbb{Z}}) < \infty$ for some $\lambda > 0$. For $\{h_j\}_j \in l^{q(\cdot)}(M^{\lambda(\cdot)}_{p(\cdot)})$, we define

$$\|\{h_j\}_{j\in\mathbb{Z}}\|_{l^{q(\cdot)}(M^{\lambda(\cdot)}_{p(\cdot)})} := \inf\left\{\lambda > 0,\ \rho_{l^{q(\cdot)}(M^{\lambda(\cdot)}_{p(\cdot)})}(\{\frac{h_j}{\lambda}\}_{j\in\mathbb{Z}}) \le 1\right\} < \infty,$$

where:

$$\rho_{l^{q(\cdot)}(M^{\lambda(\cdot)}_{p(\cdot)})}(\{h_j\}_{j\in\mathbb{Z}}) := \sum_{j\in\mathbb{Z}} \inf\left\{\theta_j > 0,\ \int_{\mathbb{R}^n} (\frac{|r^{\frac{n}{\lambda(x)}-\frac{n}{p(x)}} h_j\chi_{B(x_0,r)}|}{\theta_j^{\frac{1}{q(x)}}})^{p(x)} dx \le 1\right\}.$$

Notice that if $q_+ < \infty$ and $p(x) \le q(x)$, then

$$\rho_{l^{q(\cdot)}(M^{\lambda(\cdot)}_{p(\cdot)})}(\{h_i\}_{i\in\mathbb{N}_0}) = \sum_{i\in\mathbb{N}_0} \sup_{x_0\in\mathbb{R}^n, r>0} \|(|r^{\frac{n}{\lambda(x)}-\frac{n}{p(x)}} f_i\chi_{B(x_0,r)}|)^{q(\cdot)}\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}.$$

Definition 1.5 ([3])**.** Let $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $p(\cdot),\ q(\cdot) \in \mathscr{P}_0(\mathbb{R}^n)\cap C^{\log}(\mathbb{R}^n)$ with $0 < p_- \le p(\cdot) \le \infty$.

- The homogeneous variable Besov space

  $$\dot{B}^{s(\cdot)}_{p(\cdot),q(\cdot)} := \left\{u \in \mathscr{S}'/\mathrm{P} : \|u\|_{\dot{B}^{s(\cdot)}_{p(\cdot),q(\cdot)}} < +\infty\right\}$$

  is a Banach space with the norm

  $$\|u\|_{\dot{B}^{s(\cdot)}_{p(\cdot),q(\cdot)}} := \left\|\left\{2^{js(\cdot)}\Delta_j u\right\}_{j\in\mathbb{Z}}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)},$$

  where $\Delta_j u = \check{\varphi}_j(\cdot) * u$ and P denotes the set of all polynomials.
- The homogeneous variable Fourier–Besov space

  $$\mathscr{F}\dot{B}^{s(\cdot)}_{p(\cdot),q(\cdot)} := \left\{u \in \mathscr{S}'/\mathrm{P} : \|u\|_{\mathscr{F}\dot{B}^{s(\cdot)}_{p(\cdot),q(\cdot)}} < +\infty\right\}$$

  is a Banach space with the norm

  $$\|u\|_{\mathscr{F}\dot{B}^{s(\cdot)}_{p(\cdot),q(\cdot)}} := \left\|\left\{2^{js(\cdot)}\varphi_j\hat{u}\right\}_{j\in\mathbb{Z}}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}.$$

Definition 1.6 ([2])**.** Let $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $p(\cdot),\ q(\cdot),\ \lambda(\cdot) \in \mathscr{P}_0(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$ with $0 < p_- \le p(x) \le \lambda(x) \le \infty$. The homogeneous Besov–Morrey space with a variable exponent $\mathscr{N}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q(\cdot)}$ is defined by the set of all $f \in \mathscr{D}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathscr{N}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q(\cdot)}} := \left\{2^{js(\cdot)}\Delta_j f\right\}_{j\in\mathbb{Z}}\|_{l^{q(\cdot)}(M^{\lambda(\cdot)}_{p(\cdot)})} < \infty.$$

The space $\mathscr{D}'(\mathbb{R}^n)$ is the dual space of

$$\mathscr{D}(\mathbb{R}^n) = \{f \in S(\mathbb{R}^n) : (D^\alpha f)(0) = 0,\ \ \forall\alpha \in \mathbb{N}^n\}.$$

Definition 1.7 ([1])**.** Let $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $p(\cdot),\ q(\cdot),\ \lambda(\cdot) \in \mathscr{P}_0(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$ with $0 < p_- \le p(\cdot) \le \lambda(\cdot) \le \infty$. The homogeneous Fourier–Besov–Morrey space with a variable exponent $\mathscr{F}\mathscr{N}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q(\cdot)}$ is defined by the set of all $f \in \mathscr{D}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathscr{F}\mathscr{N}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q(\cdot)}} := \|\{2^{js(\cdot)}\varphi_j\hat{f}\}^{\infty}_{-\infty}\|_{l^{q(\cdot)}(M^{\lambda(\cdot)}_{p(\cdot)})} < \infty.$$

Remark 1.1.
- Notice that if $\lambda(\cdot) = p(\cdot)$, then from Lemma (1.2), we have $M^{p(\cdot)}_{p(\cdot)} = L^{p(\cdot)}$. As a result, $\mathscr{F}\mathscr{N}^{s(\cdot)}_{p(\cdot),p(\cdot),q(\cdot)} = \mathscr{F}B^{s(\cdot)}_{p(\cdot),q(\cdot)}$.
- In case $p = \lambda = 2$, we have $\mathscr{F}\mathscr{N}^{s}_{2,2,q} = \mathscr{F}B^{s}_{2,2,q} = B^{s}_{2,q} = \mathscr{N}^{s}_{2,2,q}$.

Definition 1.8 ([1])**.** Let $s(\cdot) \in C^{\log}(\mathbb{R}^n)$, $p(\cdot),\ q(\cdot),\ \lambda(\cdot) \in \mathscr{P}_0(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$, $T \in (0,\infty)$ and $1 \le q,\ \theta \le \infty$. We define the Chemin–Lerner type homogeneous Fourier–Besov–Morrey space with a variable exponent $\mathscr{L}^{\theta}([0,T);\mathscr{F}\mathscr{N}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q})$ by

$$\mathscr{L}^{\theta}\ ([0,T);\mathscr{F}\mathscr{N}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q}) = \Big\{ f \in \mathscr{D}'(\mathbb{R}^n);\ \|f\|_{\mathscr{L}^{\theta}\ ([0,T);\mathscr{F}\mathscr{N}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q})} < \infty \Big\},$$

with the norm

$$\|f\|_{\mathscr{L}^{\theta}(0,T,\cdot\mathscr{F}\mathscr{N}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q})} = \bigg(\sum_{j\in\mathbb{Z}} \|2^{js(\cdot)}\varphi_j \hat{f}\|^{q}_{L^{\theta}([0,T);M^{\lambda(\cdot)}_{p(\cdot)})} \bigg)^{\frac{1}{q}}.$$

Lemma 1.3. *The derivation* $\partial^{\alpha}_{\xi} : \mathscr{F}\mathscr{N}^{s(\cdot)+|\alpha|}_{p(\cdot),\lambda(\cdot),q} \to \mathscr{F}\mathscr{N}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q}$ *is a bounded operator.*

Proof. We have

$$\begin{aligned}
\|\partial^{\alpha}_{\xi} f\|_{\mathscr{F}\mathscr{N}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q}} &= \|\{2^{js(\cdot)}\varphi_j \widehat{\partial^{\alpha}_{\xi} f}\}^{\infty}_{-\infty}\|_{l^q(M^{\lambda(\cdot)}_{p(\cdot)})} \\
&= \|\{2^{js(\cdot)}\varphi_j |\xi|^{\alpha} \widehat{f}\}^{\infty}_{-\infty}\|_{l^q(M^{\lambda(\cdot)}_{p(\cdot)})} \\
&\lesssim \|\{2^{js(\cdot)}2^{j|\alpha|}\varphi_j \widehat{f}\}^{\infty}_{-\infty}\|_{l^q(M^{\lambda(\cdot)}_{p(\cdot)})} \\
&\lesssim \|f\|_{\mathscr{F}\mathscr{N}^{s(\cdot)+|\alpha|}_{p(\cdot),\lambda(\cdot),q}}, \qquad (1.3)
\end{aligned}$$

where in (1.3) we used the fact that $|\xi| \sim 2^j\ \forall j \in \mathbb{Z}$. □

Remark 1.2. As a consequence of Lemma 1.3, we have the following estimates:

$$\begin{aligned}
\|\nabla \cdot f\|_{\mathscr{F}\mathscr{N}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q}} &\lesssim \|f\|_{\mathscr{F}\mathscr{N}^{s(\cdot)+1}_{p(\cdot),\lambda(\cdot),q}}, \\
\|\Delta f\|_{\mathscr{F}\mathscr{N}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q}} &\lesssim \|f\|_{\mathscr{F}\mathscr{N}^{s(\cdot)+2}_{p(\cdot),\lambda(\cdot),q}}.
\end{aligned}$$

Proposition 1.1. *Let g be a smooth function on $\mathbb{R}^n \backslash \{0\}$ which is homogeneous of degree k. The operator $g(D)$ is continuous from $\mathscr{F}\mathscr{N}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q}$ to $\mathscr{F}\mathscr{N}^{s(\cdot)-k}_{p(\cdot),\lambda(\cdot),q}$.*

Proof. Let $u \in \mathscr{F}\mathscr{N}^{s(\cdot)-k}_{p(\cdot),\lambda(\cdot),q}$, we obtain

$$\begin{aligned}
\|g(D)u\|_{\mathscr{F}\mathscr{N}^{s(\cdot)-k}_{p(\cdot),\lambda(\cdot),q}} &= \|\{2^{j(s(\cdot)-k)}\varphi_j(\xi)\widehat{g(D)u}(\xi)\}^{\infty}_{-\infty}\|_{l^q(M^{\lambda(\cdot)}_{p(\cdot)})} \\
&= \|\{2^{j(s(\cdot)-k)}\varphi_j(\xi) g(\xi)\widehat{u}\}^{\infty}_{-\infty}\|_{l^q(M^{\lambda(\cdot)}_{p(\cdot)})} \\
&= \|\{2^{j(s(\cdot)-k)}\varphi_j(\xi)|\xi|^k g\Big(\frac{\xi}{|\xi|}\Big)\widehat{u}\}^{\infty}_{-\infty}\|_{l^q(M^{\lambda(\cdot)}_{p(\cdot)})} \\
&\lesssim \|\{2^{j(s(\cdot)-k)}\varphi_j(\xi)2^{jk}\widehat{u}\}^{\infty}_{-\infty}\|_{l^q(M^{\lambda(\cdot)}_{p(\cdot)})} \\
&\le C\|u\|_{\mathscr{F}\mathscr{N}^{s(\cdot)}_{p(\cdot),\lambda(\cdot),q}}.
\end{aligned}$$
□

We have the following

Proposition 1.2. *For the Morrey spaces with variable exponents, the following inclusions are established.*

(1) (*Hölder inequality*) ([2]) *Let* $p(\cdot)$, $p_1(\cdot)$, $p_2(\cdot)$, $\lambda(\cdot)$, $\lambda_1(\cdot)$, $\lambda_2(\cdot) \in \mathscr{P}_0(\mathbb{R}^n)$ *such that* $p(x) \le \lambda(x)$, $p_1(x) \le \lambda_1(x)$, $p_2(x) \le \lambda_2(x)$, $\dfrac{1}{p(x)} = \dfrac{1}{p_1(x)} + \dfrac{1}{p_2(x)}$

and $\dfrac{1}{\lambda(x)}=\dfrac{1}{\lambda_1(x)}+\dfrac{1}{\lambda_2(x)}$. *Then there exists a constant C depending only on p_- and p_+ such that*

$$\|fg\|_{M^{\lambda(\cdot)}_{p(\cdot)}}\leq C\|f\|_{M^{\lambda_1(\cdot)}_{p_1(\cdot)}}\|g\|_{M^{\lambda_2(\cdot)}_{p_2(\cdot)}}$$

holds for every $f\in M^{\lambda_1(\cdot)}_{p_1(\cdot)}$ and $g\in M^{\lambda_2(\cdot)}_{p_2(\cdot)}$.

(2) ([2]) *Let $p_0(\cdot),p_1(\cdot),\lambda_0(\cdot),\lambda_1(\cdot),q(\cdot)\in\mathscr{P}_0$, and $s_0(\cdot)$, $s_1(\cdot)\in L^\infty\cap C^{\log}(\mathbb{R}^n)$ with $s_0(\cdot)>s_1(\cdot)$. If $\dfrac{1}{q(\cdot)}$ and $s_0(x)-\frac{n}{p_0(x)}=s_1(x)-\frac{n}{p_1(x)}$ are locally log-Hölder continuous, then*

$$\mathscr{N}^{s_0(\cdot)}_{p_0(\cdot),\lambda_0(\cdot),q(\cdot)}\hookrightarrow\mathscr{N}^{s_1(\cdot)}_{p_1(\cdot),\lambda_1(\cdot),q(\cdot)}. \tag{1.4}$$

(3) ([2]) *For $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ and $\psi\in L^1(\mathbb{R}^n)$, assume $\Psi(x)=\sup\limits_{y\notin B(0,|x|)}|\psi(y)|$ is integrable. Then*

$$\|f*\psi_\epsilon\|_{M^{\lambda(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}\leq C\|f\|_{M^{\lambda(\cdot)}_{p(\cdot)}(\mathbb{R}^n)}\|\Psi\|_{L^1(\mathbb{R}^n)},$$

for all $f\in M^{\lambda(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$, where $\psi_\epsilon=\dfrac{1}{\epsilon^n}\psi(\dfrac{1}{\epsilon})$ and C depends only on n.

In our discussion, we will use the following result related to the Chemin–Lerner space, which deals with the product of two functions in this space.

Proposition 1.3 ([28])**.** *Let $1\leq p,q,r,r_1,r_2\leq\infty$ and $s_1,s_2\in\mathbb{R}$ such that $s_1<\frac{n}{p'}$, $s_2<\frac{n}{p'}$ and $s_1+s_2>\max\left\{\frac{n}{p'}-\frac{n}{p},0\right\}$, where $1/p+1/p'=1$. Then for $u\in\mathscr{F}B^{s_1}_{p,q},v\in\mathscr{F}B^{s_2}_{p,q}$, one has*

$$\|uv\|_{\mathscr{L}^r(0,T;\mathscr{F}B^{s_1+s_2-n/p'}_{p,q})}\leq C\|u\|_{\mathscr{L}^{r_1}(0,T;\mathscr{F}B^{s_1}_{p,q})}\|v\|_{\mathscr{L}^{r_2}(0,T;\mathscr{F}B^{s_2}_{p,q})}, \tag{1.5}$$

where $\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}$.

Remark 1.3. If we take $p=2$, then (1.5) becomes

$$\|uv\|_{\mathscr{L}^r(0,T;\mathscr{F}\mathscr{N}^{s_1+s_2-n/p'}_{2,2,q})}\leq C\|u\|_{\mathscr{L}^{r_1}(0,T;\mathscr{F}\mathscr{N}^{s_1}_{2,2,q})}\|v\|_{\mathscr{L}^{r_2}(0,T;\mathscr{F}\mathscr{N}^{s_2}_{2,2,q})}.$$

2. Well-Posedness

In this section, we use contraction mapping in the critical Banach spaces to obtain the global well-posedness of the Debye–Hückel system.

Lemma 2.1 ([3])**.** *Let X be a Banach space with the norm $\|\cdot\|$ and $B:X\times X\to X$ a bilinear operator such that for any $x_1,x_2\in X$, $\|B(x_1,x_2)\|\leq\eta\|x_1\|\|x_2\|$, then for any $y\in X$ such that $4\eta\|y\|<1$, the equation $x=y+B(x,x)$ has a solution x in X. In particular, the solution is such that $\|x\|\leq 2\|y\|$ and it is the only one such that $\|x\|<\frac{1}{2\eta}$.*

Now, we consider the Cauchy problem of the dissipative equation

$$\begin{cases}u_t-\Delta u=f(x,t) & \text{in } \mathbb{R}^n\times\mathbb{R}^+,\\ u(x,0)=u_0 & \text{in } \mathbb{R}^n,\end{cases} \tag{2.1}$$

for which we have the following

Lemma 2.2 ([28])**.** *Let $I=[0,T)$, $T\in(0,\infty]$, $1\leq r$, $q\leq\infty$, $p(\cdot)$, $p_1(\cdot),\lambda(\cdot)\in\mathscr{P}_0(\mathbb{R}^n)$, $p_1(\cdot)\leq p(\cdot)$, $p(\cdot)\leq\lambda(\cdot)<\infty$ and $s(\cdot)\in C^{\log}(\mathbb{R}^n)$. Assume that $u_0\in\mathscr{F}\mathscr{N}^{s(\cdot)+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}$ and $f\in\mathscr{L}^r\big(I,\mathscr{F}\mathscr{N}^{s(\cdot)+\frac{n}{p'(\cdot)}+\frac{2}{r}-2}_{p(\cdot),\lambda(\cdot),q}\big)$. Then the Cauchy problem (1.1) has a unique solution $u\in\mathscr{L}^\infty\big(I;\mathscr{F}\mathscr{N}^{s(\cdot)+\frac{n}{p_1'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}\big)\cap\mathscr{L}^r\big(I;\mathscr{F}\mathscr{N}^{s(\cdot)+\frac{n}{p_1'(\cdot)}+\frac{2}{r}-2}_{p(\cdot),\lambda(\cdot),q}\big)$ such that for all $r_1\in[r,\infty]$,*

$$\|u\|_{\mathscr{L}^{r_1}(I,\mathscr{F}\mathscr{N}^{s(\cdot)+\frac{n}{p_1'(\cdot)}+\frac{2}{r_1}}_{p_1(\cdot),\lambda(\cdot),q})}\lesssim\|u_0\|_{\mathscr{F}\mathscr{N}^{s(\cdot)+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}}+\|f\|_{\mathscr{L}^r(I,\mathscr{F}\mathscr{N}^{s(\cdot)+\frac{n}{p'(\cdot)}+\frac{2}{r}-2}_{p(\cdot),\lambda(\cdot),q})}.$$

Moreover, if $q<\infty$, then $u\in\mathscr{C}\big(I,\mathscr{F}\mathscr{N}^{s(\cdot)+\frac{n}{p_1'(\cdot)}}_{p_1(\cdot),\lambda(\cdot),q}\big)$.

We are now in the position to present our first main result.

Theorem 2.1. *Let $p(\cdot),\lambda(\cdot)\in\mathscr{P}_0(\mathbb{R}^n)\cap C^{\log}(\mathbb{R}^n)$, $2<r\le+\infty$, $2\le p(\cdot)\le 6$, $p(\cdot)\le\lambda(\cdot)<\infty$ and $1\le q<3$. Then there exists a positive constant σ_0 such that for any initial data $(v_0,w_0)\in \mathscr{F}\mathscr{N}^{-2+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}$ with*

$$\|(v_0,w_0)\|_{\mathscr{F}\mathscr{N}^{-2+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}}\le\sigma_0,$$

system (1.1) has a unique global solution $(v,w)\in\mathscr{Y}$, where

$$\mathscr{Y}:=\mathscr{L}^r\Big(\mathbb{R}^+;\mathscr{F}\mathscr{N}^{-2+\frac{n}{p'(\cdot)}+\frac{2}{r}}_{p(\cdot),\lambda(\cdot),q}\Big)\cap\mathscr{L}^r\Big(\mathbb{R}^+;\mathscr{F}\mathscr{N}^{-2+\frac{n}{2}+\frac{2}{r}}_{2,2,q}\Big)\cap\mathscr{L}^\infty\Big(\mathbb{R}^+;\mathscr{F}\mathscr{N}^{-2+\frac{n}{2}}_{2,2,q}\Big).$$

In addition,

$$\|(v,w)\|_{\mathscr{Y}}\lesssim\|(v_0,w_0)\|_{\mathscr{F}\mathscr{N}^{-2+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}}.$$

Remark 2.1.
- Recently, Cui and Xiao [15] established the global existence of (1.1) in the Fourier–Besov space $\mathscr{F}B^s_{p,q}$ with $s=-2+\frac{n}{p'}$. Consequently, Theorem 2.1 extends and complements his result.
- Theorem 2.1 can be seen as a meaningful complement to the corresponding results of the Debye–Hückel system in usual Fourier–Besov–Morrey spaces.

Proof of Theorem 2.1. According to Duhamel's principle, the mild solution (v,w) for system (1.1) can be represented as

$$v=\mathscr{K}(t)v_0-\int_0^t\mathscr{K}(t-\tau)\nabla\cdot(v\nabla\phi)(\cdot,\tau)d\tau:=\mathscr{Q}_1(v,w),$$

$$w=\mathscr{K}(t)w_0-\int_0^t\mathscr{K}(t-\tau)\nabla\cdot(w\nabla\phi)(\cdot,\tau)d\tau:=\mathscr{Q}_2(v,w),$$

where $\mathscr{K}(t)u:=e^{t\Delta}u=\mathscr{F}^{-1}(e^{-t|\xi|^2}\mathscr{F}(u))$.

It is worth noting that the space $\mathscr{Y}$ defined in Theorem 2.1 is a Banach space equipped with the norm

$$\|u\|_{\mathscr{Y}}=\|u\|_{\mathscr{L}^r(\mathbb{R}^+;\mathscr{F}\mathscr{N}^{-2+\frac{n}{p'(\cdot)}+\frac{2}{r}}_{p(\cdot),\lambda(\cdot),q})}+\|u\|_{\mathscr{L}^r(\mathbb{R}^+;\mathscr{F}\mathscr{N}^{-2+\frac{n}{2}+\frac{2}{r}}_{2,2,q})}+\|u\|_{\mathscr{L}^\infty(\mathbb{R}^+;\mathscr{F}\mathscr{N}^{-2+\frac{n}{2}}_{2,2,q})}.$$

Let

$$\mathscr{B}(v,\phi):=\int_0^t\mathscr{K}(t-\tau)\nabla\cdot(v\nabla\phi)(\tau,x)d\tau.$$

We define the mapping ψ as: $\psi(v,w):=(\mathscr{Q}_1(v,w),\mathscr{Q}_2(v,w))=(v,w)$. Notice that $\mathscr{K}(t)v_0$ can be regarded as the solution to equation (2.1) with $f=0$. According to Lemma 2.2 and considering the assumption $p(\cdot)\ge 2$, we obtain

$$\|\mathscr{K}(t)v_0\|_{\mathscr{L}^r\Big(\mathbb{R}^+;\mathscr{F}\mathscr{N}^{-2+\frac{n}{p'(\cdot)}+\frac{2}{r}}_{p(\cdot),\lambda(\cdot),q}\Big)}\lesssim\|v_0\|_{\mathscr{F}\mathscr{N}^{-2+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}},$$

$$\|\mathscr{K}(t)v_0\|_{\mathscr{L}^r\Big(\mathbb{R}^+;\mathscr{F}\mathscr{N}^{-2+\frac{n}{2}+\frac{2}{r}}_{2,2,q}\Big)}\lesssim\|v_0\|_{\mathscr{F}\mathscr{N}^{-2+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}},$$

$$\|\mathscr{K}(t)v_0\|_{\mathscr{L}^\infty\Big(\mathbb{R}^+;\mathscr{F}\mathscr{N}^{-2+\frac{n}{2}}_{2,2,q}\Big)}\lesssim\|v_0\|_{\mathscr{F}\mathscr{N}^{-2+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}}.$$

Then $\|\mathscr{K}(t)v_0\|_{\mathscr{Y}}\lesssim\|v_0\|_{\mathscr{F}\mathscr{N}^{-2+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}}$.

Similary, $\|\mathscr{K}(t)w_0\|_{\mathscr{Y}}\lesssim\|w_0\|_{\mathscr{F}\mathscr{N}^{-2+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}}$. Thus

$$\|(\mathscr{K}(t)v_0,\mathscr{K}(t)w_0)\|_{\mathscr{Y}}\le C_1\|(v_0,w_0)\|_{\mathscr{F}\mathscr{N}^{-2+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}}.\tag{2.2}$$

We have the following bilinear estimate:

$$\|\mathscr{B}(v,\phi)\|_{\mathscr{Y}} \leq C_2\|(v,w)\|^2_{\mathscr{Y}}. \tag{2.3}$$

Indeed, let $p^{\star}(\cdot)=\frac{6p(\cdot)}{6-p(\cdot)}$ and $s(\cdot)=-2+\frac{n}{p'(\cdot)}+\frac{2}{r}$, using Hölder's inequality, Hausdorf–Young's inequality, Young's inequality, embedding (1.4), and Proposition 1.3, we get

$$\begin{aligned}
&\|\mathscr{B}(v,\phi)\|_{\mathscr{L}^r\left(\mathbb{R}^+;\mathscr{F}\mathscr{N}^{-2+\frac{n}{p'(\cdot)}+\frac{2}{r}}_{p(\cdot),\lambda(\cdot),q}\right)}\\
&=\left\|\int_0^t \mathscr{K}(t-\tau)\nabla\cdot(v\nabla\phi)d\tau\right\|_{\mathscr{L}^r\left(\mathbb{R}^+;\mathscr{F}\mathscr{N}^{-2+\frac{n}{p'(\cdot)}+\frac{2}{r}}_{p(\cdot),\lambda(\cdot),q}\right)}\\
&\lesssim\left\|\left\|\int_0^t 2^{js(\cdot)}\varphi_j e^{-(t-\tau)|\xi|^2}\mathscr{F}\big(\nabla\cdot(v\nabla\phi)\big)d\tau\right\|_{L^r\left(\mathbb{R}^+;M^{\lambda(\cdot)}_{p(\cdot)}\right)}\right\|_{l^q}\\
&\lesssim\left\|\left\|\int_0^t \left\|r^{\frac{-n}{p^{\star}(\cdot)}}2^{j(s(\cdot)+1)}\varphi_j e^{-(t-\tau)|\xi|^2}\right\|_{L^{p^{\star}(\cdot)}}\|\mathscr{F}(v\nabla\phi)\|_{M^{\lambda(\cdot)}_{6}}\,d\tau\right\|_{L^r(\mathbb{R}^+)}\right\|_{l^q}\\
&\lesssim\left\|\left\|\int_0^t \left\|r^{\frac{-n}{p^{\star}(\cdot)}}2^{j(s(\cdot)+1)}\varphi_j e^{-(t-\tau)|\xi|^2}\right\|_{L^{p^{\star}(\cdot)}}\|v\nabla\phi\|_{M^{\lambda(\cdot)}_{\frac{6}{5}}}\,d\tau\right\|_{L^r(\mathbb{R}^+)}\right\|_{l^q}\\
&\lesssim\left\|\left\|\int_0^t 2^{j\left(-1+\frac{5n}{6}+\frac{2}{r}\right)}e^{-(t-\tau)2^{2j}}\left\|r^{\frac{-n}{p^{\star}(\cdot)}}2^{-nj\frac{1}{p^{\star}(\cdot)}}\varphi_j\right\|_{L^{p^{\star}(\cdot)}}\left\|\dot{\Delta}_j(v\nabla\phi)\right\|_{M^{\lambda(\cdot)}_{\frac{6}{5}}}\,d\tau\right\|_{L^r(\mathbb{R}^+)}\right\|_{l^q}\\
&\lesssim\left\|\left\|2^{j\left(-3+\frac{5n}{6}+\frac{2}{r}\right)}\left\|\dot{\Delta}_j(v\nabla\phi)\right\|_{M^{\lambda(\cdot)}_{\frac{6}{5}}}\right\|_{L^r(\mathbb{R}^+)}\left\|e^{-t2^{2j}}2^{2j}\right\|_{L^1(\mathbb{R}^+)}\right\|_{l^q}\\
&\lesssim\left\|\left\|2^{j\left(-3+\frac{5n}{6}+\frac{2}{r}\right)}\left\|\dot{\Delta}_j(v\nabla\phi)\right\|_{L^{\frac{6}{5}}}\right\|_{L^r(\mathbb{R}^+)}\right\|_{l^q}\\
&\lesssim\|v\nabla\phi\|_{\mathscr{L}^r(\mathbb{R}^+;\mathscr{N}^{-3+\frac{5n}{6}+\frac{2}{r}}_{\frac{6}{5},q})}\\
&\lesssim\|v\nabla\phi\|_{\mathscr{L}^r(\mathbb{R}^+;\mathscr{F}\mathscr{N}^{-3+\frac{n}{2}+\frac{2}{r}}_{2,2,q})}\\
&\lesssim\|v\|_{\mathscr{L}^\infty(\mathbb{R}^+;\mathscr{F}\mathscr{N}^{-2+\frac{n}{2}}_{2,2,q})}\|\nabla\phi\|_{\mathscr{L}^r(\mathbb{R}^+;\mathscr{F}\mathscr{N}^{-1+\frac{n}{2}+\frac{2}{r}}_{2,2,q})}
\end{aligned}\tag{2.4}$$

where in inequality (2.4) the following fact is used:

$$\begin{aligned}
&\|2^{\frac{-n}{p^{\star}(\cdot)}j}\varphi_j\|_{L^{p^{\star}(\cdot)}}\\
&=\inf\{\lambda>0:\int_{\mathbb{R}^3}\left|\frac{2^{\frac{-n}{p^{\star}(\cdot)}j}\varphi_j}{\lambda}\right|^{p^{\star}(\cdot)}\mathrm{d}x\leq 1\}\\
&=\inf\{\lambda>0\ :\ \int_{\mathbb{R}^3}\left|\frac{\varphi_j}{\lambda}\right|^{p^{\star}(\cdot)}2^{-nj}\mathrm{d}x\leq 1\}\\
&=\inf\{\lambda>0\ :\ \int_{\mathbb{R}^3}\left|\frac{\varphi}{\lambda}\right|^{\frac{6p(2^j)}{6-p(2^j)}}\mathrm{d}x\leq 1\}\\
&\leq C.
\end{aligned}$$

Since $\phi = (-\Delta)^{-1}(w-v)$, Lemma 1.3 implies that

$$\|\nabla\phi\|_{\mathscr{L}^r(\mathbb{R}^+;\mathscr{F}\mathscr{N}^{-1+\frac{n}{2}+\frac{2}{r}}_{2,2,q})} \lesssim \|(v,w)\|_{\mathscr{L}^r(\mathbb{R}^+;\mathscr{F}\mathscr{N}^{-2+\frac{n}{2}+\frac{2}{r}}_{2,2,q})}.$$

Thus we have

$$\begin{aligned}&\|\mathscr{B}(v,\phi)\|_{\mathscr{L}^r\left(\mathbb{R}^+;\mathscr{F}\mathscr{N}^{-2+\frac{n}{p'(\cdot)}+\frac{2}{r}}_{p(\cdot),\lambda(\cdot),q}\right)}\\ &\lesssim \|v\|_{\mathscr{L}^\infty(\mathbb{R}^+;\mathscr{F}\mathscr{N}^{-2+\frac{n}{2}}_{2,2,q})}\|(v,w)\|_{\mathscr{Y}}\\ &\lesssim \|(v,w)\|^2_{\mathscr{Y}}.\end{aligned}$$

Similary, we obtain

$$\begin{aligned}&\|\mathscr{B}(v,\phi)\|_{\mathscr{L}^r\left(\mathbb{R}^+;\mathscr{F}\mathscr{N}^{-2+\frac{n}{2}+\frac{2}{r}}_{2,2,q}\right)\cap\mathscr{L}^\infty\left(\mathbb{R}^+;\mathscr{F}\mathscr{N}^{-2+\frac{n}{2}}_{2,2,q}\right)}\\ &\lesssim \|(v,w)\|^2_{\mathscr{Y}}.\end{aligned}$$

Consequently, the desired estimate (2.3) is established.

By using the estimate (2.3), we get

$$\|\mathscr{Q}_1(v,w)\|_{\mathscr{Y}} \le \|\mathscr{K}(t)v_0\|_{\mathscr{Y}} + C_2\|(v,w)\|^2_{\mathscr{Y}}.$$

Similary, we obtain

$$\|\mathscr{Q}_2(v,w)\|_{\mathscr{Y}} \le \|\mathscr{K}(t)w_0\|_{\mathscr{Y}} + C_2\|(v,w)\|^2_{\mathscr{Y}}.$$

$$\|\psi(v,w)\|_{\mathscr{Y}} \le \|(\mathscr{K}(t)v_0,\mathscr{K}(t)w_0)\|_{\mathscr{Y}} + C_2\|(v,w)\|^2_{\mathscr{Y}}.$$

By Lemma 2.1, we know that if $\|(\mathscr{K}(t)v_0,\mathscr{K}(t)w_0)\|_{\mathscr{Y}} < \kappa$ with $\kappa = \frac{1}{4C_2}$, then ψ has a fixed point in the closed ball $B(0,2\kappa) := \{x\in\mathscr{Y} : \|x\|_{\mathscr{Y}} \le 2\kappa\}$. From the estimate (2.2), there exists a positive constant C_1 depending only on n such that

$$\|(\mathscr{K}(t)v_0,\mathscr{K}(t)w_0)\|_{\mathscr{Y}} \le C_1\|(v_0,w_0)\|_{\mathscr{F}\mathscr{N}^{-2+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}}.$$

Thus, if $\|(v_0,w_0)\|_{\mathscr{F}\mathscr{N}^{-2+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}} < \sigma$ with $\sigma = \frac{\kappa}{C_1}$, then we have $\|(\mathscr{K}(t)v_0,\mathscr{K}(t)w_0)\|_{\mathscr{Y}} < \kappa$. This proves the global existence for small initial data.

3. Gevrey Class Regularity

In this section, we show the analyticity of the solution obtained in Theorem 2.1 (in the sense of Gevray class).

Our second main result of this paper is given below.

Theorem 3.1. *Assume that* $p(\cdot),\lambda(\cdot)\in\mathscr{P}_0(\mathbb{R}^n)\cap C^{\log}(\mathbb{R}^n)$, $1\le r\le+\infty$, $2\le p(\cdot)\le 6$, $p(\cdot)\le\lambda(\cdot)<\infty$ *and* $1\le q<3$. *Then there exists a positive constant* σ_0' *such that for any initial data* (v_0,w_0) *in* $\mathscr{F}\mathscr{N}^{-2+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}$ *with*

$$\|(v_0,w_0)\|_{\mathscr{F}\mathscr{N}^{-2+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}} \le \sigma_0',$$

system (1.1) *has a unique analytic solution such that*

$$\left\|\left(e^{\sqrt{t}|D|}v, e^{\sqrt{t}|D|}w\right)\right\|_{\mathscr{Y}} \lesssim \|(v_0,w_0)\|_{\mathscr{F}\mathscr{N}^{-2+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}},$$

where $e^{\sqrt{t}|D|}u = \mathscr{F}^{-1}(e^{\sqrt{t}|\xi|}\hat{u})$.

Proof. Let us sketch the proof of Theorem 1.2. Setting $V(t)=e^{\sqrt{t}|D|}v(t), W(t)=e^{\sqrt{t}|D|}w(t)$ and $\Phi(t)=e^{\sqrt{t}|D|}\phi(t)=W(t)-V(t)$. Then we see that $(V(t),W(t))$ satisfies the following integral system:

$$\begin{cases} V(t)=e^{\sqrt{t}|D|}\mathscr{K}(t)v_0-\int\limits_0^t e^{\left[(\sqrt{t}-\sqrt{s})|D|+(t-s)\Delta\right]}\nabla\cdot e^{\sqrt{s}|D|}\left(e^{-\sqrt{s}|D|}V(s)e^{-\sqrt{s}|D|}\nabla\Phi(s)\right)ds \\ \quad := e^{\sqrt{t}|D|}\mathscr{K}(t)v_0-\mathscr{B}(V,\Phi), \\ W(t)=e^{\sqrt{t}|D|}\mathscr{K}(t)w_0+\int\limits_0^t e^{\left[(\sqrt{t}-\sqrt{s})|D|+(t-s)\Delta\right]}\nabla\cdot e^{\sqrt{s}|D|}\left(e^{-\sqrt{s}|D|}W(s)e^{-\sqrt{s}|D|}\nabla\Phi(s)\right)ds \\ \quad := e^{\sqrt{t}|D|}\mathscr{K}(t)w_0+\mathscr{B}(W,\Phi). \end{cases}$$

We recall an auxiliary lemma that will help us to prove that the global-in-time mild solutions of system (1.1) are Gevrey regular.

Lemma 3.1 ([3])**.** *Let $0<s\le t<\infty$. Then the following inequality*

$$t|a|-\frac{1}{2}(t^2-s^2)|a|^2-s|a-b|-s|b|\le\frac{1}{2}$$

holds for all $a,b\in\mathbb{R}^n$.

In order to obtain the Gevrey class regularity of the solution, we start with estimating the linear term $e^{\sqrt{t}|D|}\mathscr{K}(t)v_0$. Using the Fourier transform, multiplying by $2^{-2+\frac{n}{p'(\cdot)}+\frac{2}{r}}\varphi_j$ and taking the $L^r(\mathbb{R}^+;M^{\lambda(\cdot)}_{p(\cdot)})$-norm, we obtain

$$\begin{aligned} &\left\|2^{j(-2+\frac{n}{p'(\cdot)}+\frac{2}{r})}\varphi_j\widehat{e^{\sqrt{t}|D|}\mathscr{K}(t)v_0}\right\|_{L^r(\mathbb{R}^+;M^{\lambda(\cdot)}_{p(\cdot)})} \\ &\lesssim\left\|e^{\sqrt{t}|\xi|-t|\xi|^2}2^{j(-2+\frac{n}{p'(\cdot)}+\frac{2}{r})}\varphi_j\hat{v}_0\right\|_{L^r(\mathbb{R}^+;M^{\lambda(\cdot)}_{p(\cdot)})} \\ &\lesssim\left\|e^{-\frac{t}{2}|\xi|^2}2^{j(-2+\frac{n}{p'(\cdot)}+\frac{2}{r})}\varphi_j\hat{v}_0\right\|_{L^r(\mathbb{R}^+;M^{\lambda(\cdot)}_{p(\cdot)})}, \end{aligned}$$

where we have used the fact that $e^{\sqrt{t}|\xi|-\frac{1}{2}t|\xi|^2}=e^{-\frac{1}{2}(\sqrt{t}|\xi|-1)^2+\frac{1}{2}}\le e^{\frac{1}{2}}$.
Hence, by taking the l^q-norm, we conclude that

$$\left\|e^{\sqrt{t}|D|}\mathscr{K}(t)v_0\right\|_{\mathscr{L}^r(\mathbb{R}^+;\mathscr{FN}^{-2+\frac{n}{p'(\cdot)}+\frac{2}{r}}_{p(\cdot),\lambda(\cdot),q})}\lesssim\|v_0\|_{\mathscr{FN}^{-2+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}}.$$

Analogously, we have

$$\left\|e^{\sqrt{t}|D|}\mathscr{K}(t)v_0\right\|_{\mathscr{L}^r(\mathbb{R}^+;\mathscr{FN}^{-2+\frac{n}{2}+\frac{2}{r}}_{2,2,q})}\lesssim\|v_0\|_{\mathscr{FN}^{-2+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}}$$

and

$$\left\|e^{\sqrt{t}|D|}\mathscr{K}(t)v_0\right\|_{\mathscr{L}^\infty(\mathbb{R}^+;\mathscr{FN}^{-2+\frac{n}{2}}_{2,2,q})}\lesssim\|v_0\|_{\mathscr{FN}^{-2+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}}.$$

Then $\left\|e^{\sqrt{t}|D|}\mathscr{K}(t)v_0\right\|_{\mathscr{Y}}\lesssim\|v_0\|_{\mathscr{FN}^{-2+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}}$.
Similary, $\left\|e^{\sqrt{t}|D|}\mathscr{K}(t)w_0\right\|_{\mathscr{Y}}\lesssim\|w_0\|_{\mathscr{FN}^{-2+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}}$.
Thus

$$\left\|(e^{\sqrt{t}|D|}\mathscr{K}(t)v_0,e^{\sqrt{t}|D|}\mathscr{K}(t)w_0)\right\|_{\mathscr{Y}}\lesssim\|(v_0,w_0)\|_{\mathscr{FN}^{-2+\frac{n}{p'(\cdot)}}_{p(\cdot),\lambda(\cdot),q}}.$$

On the other hand, let

$$s(\cdot):=-2+\frac{n}{p'(\cdot)}+\frac{2}{r}.$$

Using Lemma 3.1, we get

$$\begin{aligned}
&\left\| 2^{js(\cdot)}\varphi_j \widehat{\overline{\mathscr{B}}(V,\Phi)} \right\|_{L^r(\mathbb{R}^+;M^{\lambda(\cdot)}_{p(\cdot)})} \\
&\lesssim \left\| 2^{j(s(\cdot)+1)}\varphi_j e^{\sqrt{t}|\xi|} \int_0^t e^{-(t-\tau)|\xi|^2} (\widehat{e^{-\sqrt{\tau}|D|}Ve^{-\sqrt{\tau}|D|}\nabla\Phi}) d\tau \right\|_{L^r(\mathbb{R}^+;M^{\lambda(\cdot)}_{p(\cdot)})} \\
&\lesssim \left\| 2^{j(s(\cdot)+1)}\varphi_j e^{\sqrt{t}|\xi|} \int_0^t e^{-(t-\tau)|\xi|^2} \int_{\mathbb{R}^n} (e^{-\sqrt{\tau}|\xi-y|}\widehat{V(\xi-y)}e^{-\sqrt{\tau}|y|}\widehat{\nabla\Phi(y)}) dy d\tau \right\|_{L^r(\mathbb{R}^+;M^{\lambda(\cdot)}_{p(\cdot)})} \\
&\lesssim \left\| 2^{(s(\cdot)+1)}\varphi_j \int_0^t e^{-\frac{1}{2}(t-\tau)|\xi|^2} \int_{\mathbb{R}^n} e^{\sqrt{t}|\xi|-\frac{1}{2}(t-\tau)|\xi|^2-\sqrt{\tau}(|\xi-y|+|y|)} (\widehat{V(\xi-y)}\widehat{\nabla\Phi(y)}) dy d\tau \right\|_{L^r(\mathbb{R}^+;M^{\lambda(\cdot)}_{p(\cdot)})} \\
&\lesssim \left\| 2^{j(s(\cdot)+1)}\varphi_j \int_0^t e^{-\frac{1}{2}(t-\tau)|\xi|^2} \int_{\mathbb{R}^n} (\widehat{V(\xi-y)}\widehat{\nabla\Phi(y)}) dy d\tau \right\|_{L^r(\mathbb{R}^+;M^{\lambda(\cdot)}_{p(\cdot)})} \\
&\lesssim \left\| 2^{j(s(\cdot)+1)}\varphi_j \int_0^t e^{-\frac{1}{2}(t-\tau)|\xi|^2} (\widehat{V\nabla\Phi}) d\tau \right\|_{L^r(\mathbb{R}^+;M^{\lambda(\cdot)}_{p(\cdot)})}.
\end{aligned}$$

By repeating the same arguments used to obtain the bilinear estimate (2.3), we get

$$\left\| \overline{\mathscr{B}}(V,\Phi) \right\|_{\mathscr{Y}} \lesssim \|(V,W)\|^2_{\mathscr{Y}}. \qquad \square$$

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