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ρ-STRONG CONVERGENCE IN NEUTROSOPHIC NORMED SPACES

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Abstract. In this paper, we introduce the concept of ρ-strong convergence in the neutrosophic normed spaces. We investigate several fundamental properties of this new concept.

1. Introduction

The neutrosophic set (NS) was investigated by Smarandache [18], who defined the degree of indeterminacy (i) as an independent component. The neutrosophic logic was first examined in [19]. It is a logic in which each proposition is determined as having a degree of truth (T), falsity (F), and indeterminacy (I). Neutrosophic set and neutrosophic logic are frequently used in many branches of applied and theoretical sciences such as decision making, robotics, summability theory, and many others.

Quite recently, Kirişçi and Şimşek [6] introduced a new concept of a neutrosophic metric space (NMS). They investigated neutrosophic normed space (NNS) and statistical convergence in NNS [7]. Various convergence properties of the sequences on this space have been investigated since NNS was defined. Lacunary statistical convergence and lacunary ideal convergence of sequences in NNS were introduced by Kişi [8, 9]. Some related works can be found in [10–13, 17].

A sequence $x = (x_k)$ is said to be ρ-statistically convergent to ℓ if

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \leq n : |x_k - \ell| \geq \varepsilon\}| = 0$$

for each $\varepsilon > 0$, where $\rho = (\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n \frac{\rho_n}{n} < \infty$, $\Delta\rho_n = O(1)$, and $\Delta\rho_n = \rho_{n+1} - \rho_n$ for each positive integer n.

Later, the concepts of ρ-statistical convergence in topological groups and different types of ρ-convergence were investigated in [1–5, 15, 16]. In this paper, we introduce the ρ-strongly convergence with respect to NN (ρSC-NN) in NNS and investigate some properties and some inclusion theorems related to this concept.

2. Preliminaries

Now, we give the definition of triangular norms (TN) and their dual operations known as triangular conorms (TC) which are important for fuzzy operations.

Definition 2.1 ([14])**.** Let $* : [0, 1] \times [0, 1] \to [0, 1]$ be an operation. Then the operation $*$ is called continuous TN, if the following conditions are satisfied:
(i) $r_1 * 1 = r_1$,
(ii) if $r_1 \leq r_2$ and $r_3 \leq r_4$, then $r_1 * r_3 \leq r_2 * r_4$, for all $r_1, r_2, r_3, r_4 \in [0, 1]$,
(iii) $*$ is continuous,
(iv) $*$ is associative and commutative.

Definition 2.2 ([14])**.** Let $\Diamond : [0, 1] \times [0, 1] \to [0, 1]$ be an operation. Then the operation $\Diamond$ is said to be continuous TC (Triangular conorms (t-conorms)) if the following conditions are satisfied:
(i) $r_1 \Diamond 0 = r_1$,
(ii) if $r_1 \leq r_2$ and $r_3 \leq r_4$, then $r_1 \Diamond r_3 \leq r_2 \Diamond r_4$,

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(iii) $\Diamond$ is continuous,
(iv) $\Diamond$ is associative and commutative.

The concepts of the neutrosophic norm and the neutrosophic normed space were defined as follows:

Definition 2.3 ([7])**.** Let $\mathscr{F}$ be a vector space and $\mathscr{N} : \mathscr{F} \times \mathbb{R}^+ \to [0,1]$ such that $\mathscr{N} = \{\langle a, \mathscr{G}(a), \mathscr{B}(a), \mathscr{Y}(a)\rangle : a \in \mathscr{F}\}$ be a normed space (NS). If the following conditions hold, $\mathscr{V} = (\mathscr{F}, \mathscr{N}, *, \Diamond)$ is called to be NNS. For each $a, b \in \mathscr{F}$ and $r, t > 0$ and for all $\sigma \neq 0$:
(i) $0 \leq \mathscr{G}(a,t) \leq 1$, $0 \leq \mathscr{B}(a,t) \leq 1$, $0 \leq \mathscr{Y}(a,t) \leq 1$, $\forall t \in \mathbb{R}^+$,
(ii) $\mathscr{G}(a,t) + \mathscr{B}(a,t) + \mathscr{Y}(a,t) \leq 3$, $\forall t \in \mathbb{R}^+$,
(iii) $\mathscr{G}(a,t) = 1$ (for $t > 0$) iff $a = 0$,
(iv) $\mathscr{G}(\sigma a, t) = \mathscr{G}\left(a, \frac{t}{|\sigma|}\right)$,
(v) $\mathscr{G}(a,r) * \mathscr{G}(b,t) \leq \mathscr{G}(a+b, r+t)$,
(vi) $\mathscr{G}(a,.)$ is non-decreasing continuous function,
(vii) $\lim_{t\to\infty} \mathscr{G}(a,t) = 1$,
(viii) $\mathscr{B}(a,t) = 0$ (for $t > 0$) iff $a = 0$,
(ix) $\mathscr{B}(\sigma a, t) = \mathscr{B}\left(a, \frac{t}{|\sigma|}\right)$,
(x) $\mathscr{B}(a,r) \Diamond \mathscr{B}(b,t) \geq \mathscr{B}(a+b, r+t)$,
(xi) $\mathscr{B}(a,.)$ is non-increasing continuous function,
(xii) $\lim_{t\to\infty} \mathscr{B}(a,t) = 0$,
(xiii) $\mathscr{Y}(a,t) = 0$ (for $t > 0$) iff $a = 0$,
(xiv) $\mathscr{Y}(\sigma a, t) = \mathscr{Y}\left(a, \frac{t}{|\sigma|}\right)$,
(xv) $\mathscr{Y}(a,r) \Diamond \mathscr{Y}(b,t) \geq \mathscr{Y}(a+b, r+t)$,
(xvi) $\mathscr{Y}(a,.)$ is a non-increasing continuous function,
(xvii) $\lim_{t\to\infty} \mathscr{Y}(a,t) = 0$,
(xviii) If $t \leq 0$, then $\mathscr{G}(a,t) = 0, \mathscr{B}(a,t) = 1$ and $\mathscr{Y}(a,t) = 1$.
Then $\mathscr{N} = (\mathscr{G}, \mathscr{B}, \mathscr{Y})$ is called Neutrosophic norm (NN).

Definition 2.4 ([7])**.** Let $\mathscr{V}$ be an NNS, the sequence (x_k) in $\mathscr{V}$, $\varepsilon \in (0,1)$ and $t > 0$. Then the sequence (x_k) converges to ζ if there is $N \in \mathbb{N}$ such that $\mathscr{G}(x_k - \zeta, t) > 1 - \varepsilon, \mathscr{B}(x_k - \zeta, t) < \varepsilon, \mathscr{Y}(x_k - \zeta, t) < \varepsilon$. That is, $\lim_{k\to\infty} \mathscr{G}(x_k - \zeta, t) = 1, \lim_{k\to\infty} \mathscr{B}(x_k - \zeta, t) = 0$ and $\lim_{k\to\infty} \mathscr{Y}(x_k - \zeta, \lambda) = 0$ as $\lambda > 0$. In this case, the sequence (x_k) is called a convergent sequence in $\mathscr{V}$. The convergence in NNS is indicated by $N - \lim x_k = \zeta$.

Definition 2.5 ([7])**.** Let $\mathscr{V}$ be an NNS. For $t > 0$, $w \in \mathscr{F}$ and $\varepsilon \in (0,1)$,

$$OB(w, \varepsilon, t) = \{a \in \mathscr{F} : \mathscr{G}(w - a, t) > 1 - \varepsilon,\ \mathscr{B}(w - a, t) < \varepsilon,\ \mathscr{Y}(w - a, t) < \varepsilon\}$$

is called an open ball with center w and radius ε.

Definition 2.6 ([7])**.** The set $\mathscr{A} \subset \mathscr{F}$ is called neutrosophic-bounded (NB) in NNS $\mathscr{V}$, if there exist $t > 0$and $\varepsilon \in (0,1)$ such that $\mathscr{G}(a,t) > 1 - \varepsilon$, $\mathscr{B}(a,t) < \varepsilon$ and $\mathscr{Y}(a,t) < \varepsilon$ for each $a \in \mathscr{A}$.

3. Main Results

In this section we give the main results of this article.

Definition 3.1. Take an NNS $\mathscr{V}$. Let $\rho = (\rho_n)$ be a non-decreasing sequence of positive real numbers as above. The sequence $x = (x_k)$ is called to be ρ-strongly convergent to $\zeta \in \mathscr{F}$ with respect to NN, if for every $t > 0$ and $\varepsilon \in (0,1)$, there is $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\rho_n} \sum_{k=1}^{n} \mathscr{G}(x_k - \zeta, t) > 1 - \varepsilon \text{ and}$$

$$\frac{1}{\rho_n} \sum_{k=1}^{n} \mathscr{B}(x_k - \zeta, t) < \varepsilon, \frac{1}{\rho_n} \sum_{k=1}^{n} \mathscr{Y}(x_k - \zeta, t) < \varepsilon$$

for all $n \geq n_0$. We indicate $(\mathscr{G},\mathscr{B},\mathscr{Y})_\rho-\lim x=\zeta$. In case $\rho=(\rho_n)=n$, we have $(\mathscr{G},\mathscr{B},\mathscr{Y})-\lim x=\zeta$.

Theorem 3.1. *Let $\mathscr{V}$ be an NNS. If x is ρ-strongly convergent with respect to NN, then $(\mathscr{G},\mathscr{B},\mathscr{Y})_\rho-\lim x=\zeta$ is unique.*

Proof. Suppose that $(\mathscr{G},\mathscr{B},\mathscr{Y})_\rho-\lim x=\zeta_1$, $(\mathscr{G},\mathscr{B},\mathscr{Y})_\rho-\lim x=\zeta_2$ and $\zeta_1\neq\zeta_2$. Given $\varepsilon>0$, select $\tau\in(0,1)$ such that $(1-\tau)*(1-\tau)>1-\varepsilon$ and $\tau\Diamond\tau<\varepsilon$. For each $t>0$, there is $n_1\in\mathbb{N}$ such that

$$\frac{1}{\rho_n}\sum_{k=1}^{n}\mathscr{G}(x_k-\zeta_2,t)>1-\tau \text{ and}$$
$$\frac{1}{\rho_n}\sum_{k=1}^{n}\mathscr{B}(x_k-\zeta_2,t)<\tau,\ \ \frac{1}{\rho_n}\sum_{k=1}^{n}\mathscr{Y}(x_k-\zeta_2,t)<\tau,$$

for all $n\geq n_2$. Assume that $n_0=\max\{n_1,n_2\}$. Then for $n\geq n_0$, we can find an $m\in\mathbb{N}$ such that

$$\begin{aligned}\mathscr{G}(\zeta_1-\zeta_2,t)&\geq\mathscr{G}\Big(x_m-\zeta_1,\frac{t}{2}\Big)*\mathscr{G}\Big(x_m-\zeta_2,\frac{t}{2}\Big)\\&>(1-\tau)*(1-\tau)>1-\varepsilon,\end{aligned}$$
$$\mathscr{B}(\zeta_1-\zeta_2,t)\leq\mathscr{B}\Big(x_m-\zeta_1,\frac{t}{2}\Big)\Diamond\mathscr{B}\Big(x_m-\zeta_2,\frac{t}{2}\Big)<\tau\Diamond\tau<\varepsilon,$$

and

$$\mathscr{Y}(\zeta_1-\zeta_2,t)\leq\mathscr{Y}\Big(x_m-\zeta_1,\frac{t}{2}\Big)\Diamond\mathscr{Y}\Big(x_m-\zeta_2,\frac{t}{2}\Big)<\tau\Diamond\tau<\varepsilon.$$

Since $\varepsilon>0$ is abritrary, we get $\mathscr{G}(\zeta_1-\zeta_2,t)=1$, $\mathscr{B}(\zeta_1-\zeta_2,t)=0$ and $\mathscr{Y}(\zeta_1-\zeta_2,t)=0$ for all $t>0$, which gives $\zeta_1=\zeta_2$. □

We give an example to denote the sequence of strongly lacunary $(H,1)-$convergence in an NNS.

$$x_k=\begin{cases}1, & \text{if } \ k=u^2 (u\in\mathbb{N})\\ 0, & \text{otherwise.}\end{cases}$$

Consider

$$A=\{k\leq n:\mathscr{G}(x,t)>1-\varepsilon \ \text{ and } \ \mathscr{B}(x,t)<\varepsilon,\ \mathscr{Y}(x,t)<\varepsilon\}.$$

Then for any $t>0$ and $\varepsilon\in(0,1)$, the following set:

$$\begin{aligned}A&=\left\{k\leq n:\frac{t}{t+||x_k||}>1-\varepsilon,\ \text{ and } \ \frac{||x_k||}{t+||x_k||}<\varepsilon,\ \frac{||x_k||}{t}<\varepsilon\right\}\\&=\left\{k\leq n:||x_k||\leq\frac{t\varepsilon}{1-\varepsilon}\,,\ \text{ and } \ ||x_k||<t\varepsilon\right\}\\&\subset\{k\leq n:||x_k||=1\}=\left\{k\leq n:k=u^2\right\}\end{aligned}$$

i.e.,

$$A_n(\varepsilon,t)=\left\{n\in\mathbb{N}:\frac{1}{\rho_n}\sum_{k\leq n}\mathscr{G}(x_k,t)>1-\varepsilon\ \text{ and } \ \frac{1}{\rho_n}\sum_{k\leq n}\mathscr{B}(x_k,t)<\varepsilon,\ \frac{1}{\rho_n}\sum_{k\leq n}\mathscr{Y}(x_k,t)<\varepsilon\right\},$$

is a finite set.

Theorem 3.2. *Let $\mathscr{V}$ be an NNS. If $(\mathscr{G},\mathscr{B},\mathscr{Y})_\rho-\lim x=\zeta_1$ and $(\mathscr{G},\mathscr{B},\mathscr{Y})_\rho-\lim y=\zeta_2$, then $(\mathscr{G},\mathscr{B},\mathscr{Y})_\rho-\lim(x+y)=\zeta_1+\zeta_2$ and $c\in\mathscr{F}$, $(\mathscr{G},\mathscr{B},\mathscr{Y})_\rho-\lim cx=c\zeta$.*

Proof. For every $t>0$ and $\varepsilon\in(0,1)$, there is $n_0\in\mathbb{N}$ such that

$$\frac{1}{\rho_n}\sum_{k\leq n}\mathscr{G}(x_k-\zeta_1,t)>1-\tau\ \text{ and } \ \frac{1}{\rho_n}\sum_{k\leq n}\mathscr{B}(x_k-\zeta_1,t)<\tau,\ \frac{1}{\rho_n}\sum_{k\leq n}\mathscr{Y}(x_k-\zeta_1,t)<\tau,$$

for all $n \geq n_1$. Also, there is $n_2 \in \mathbb{N}$ such that

$$\frac{1}{\rho_n}\sum_{k\leq n}\mathscr{G}(y_k-\zeta_2,t)>1-\tau \text{ and } \frac{1}{\rho_n}\sum_{k\leq n}\mathscr{B}(y_k-\zeta_2,t)<\tau,\ \frac{1}{\rho_n}\sum_{k\leq n}\mathscr{Y}(y_k-\zeta_2,t)<\tau,$$

for all $n \geq n_2$. Assume that $n_0 = \max\{n_1, n_2\}$. Now, for $n \geq n_0$, we get

$$\begin{aligned}\frac{1}{\rho_n}\sum_{k\leq n}\mathscr{G}((x_k+y_k)-(\zeta_1+\zeta_2),t) &= \frac{1}{\rho_n}\sum_{k\leq n}\mathscr{G}(x_k-\zeta_1+y_k-\zeta_2,t)\\ &\geqslant \frac{1}{\rho_n}\sum_{k\leq n}\mathscr{G}\Big(x_k-\zeta_1,\frac{t}{2}\Big)*\mathscr{G}\Big(y_k-\zeta_2,\frac{t}{2}\Big)\\ &> (1-\tau)*(1-\tau)>1-\varepsilon\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\rho_n}\sum_{k\leq n}\mathscr{B}((x_k+y_k)-(\zeta_1+\zeta_2),t) &= \frac{1}{\rho_n}\sum_{k\leq n}\mathscr{B}(x_k-\zeta_1+y_k-\zeta_2,t)\\ &\leqslant \frac{1}{\rho_n}\sum_{k\leq n}\mathscr{B}\Big((x_k-\zeta_1),\frac{t}{2}\Big)\Diamond\mathscr{B}\Big((y_k-\zeta_2),\frac{t}{2}\Big)<\tau\Diamond\tau<\varepsilon.\end{aligned}$$

Further,

$$\begin{aligned}\frac{1}{\rho_n}\sum_{k\leq n}\mathscr{Y}((x_k+y_k)-(\zeta_1+\zeta_2),t) &= \frac{1}{\rho_n}\sum_{k\leq n}\mathscr{Y}(x_k-\zeta_1+y_k-\zeta_2,t)\\ &\leqslant \frac{1}{\rho_n}\sum_{k\leq n}\mathscr{Y}\Big((x_k-\zeta_1),\frac{t}{2}\Big)\Diamond\mathscr{Y}\Big((y_k-\zeta_2),\frac{t}{2}\Big)<\tau\Diamond\tau<\varepsilon.\end{aligned}$$

Similarly, we can show that $(\mathscr{G},\mathscr{B},\mathscr{Y})_\rho-\lim cx=c\zeta$. □

Theorem 3.3. *If $(\mathscr{G},\mathscr{B},\mathscr{Y})_\rho-\lim x=\zeta$, then there is a subsequence (x_{τ_k}) of x such that $(\mathscr{G},\mathscr{B},\mathscr{Y})_\rho-\lim x_{\tau_k}=\zeta$.*

Proof. Take $(\mathscr{G},\mathscr{B},\mathscr{Y})_\rho-\lim x=\zeta$. Then for every $t>0$ and $\varepsilon\in(0,1)$, there is $n_0\in\mathbb{N}$ such that

$$\frac{1}{\rho_n}\sum_{k\leq n}\mathscr{G}(x_k-\zeta,t)>1-\varepsilon \text{ and } \frac{1}{\rho_n}\sum_{k\leq n}\mathscr{B}(x_k-\zeta,t)<\varepsilon,\ \frac{1}{\rho_n}\sum_{k\leq n}\mathscr{Y}(x_k-\zeta,t)<\varepsilon,$$

for all $n\geq n_0$. Obviously, for each $n\geq n_0$, we choose $\tau_k\leq n$ such that

$$\begin{aligned}\mathscr{G}(x_{\tau_k}-\zeta,t) &> \frac{1}{\rho_n}\sum_{k\leq n}\mathscr{G}(x_k-\zeta,t)>1-\varepsilon,\\ \mathscr{B}(x_{\tau_k}-\zeta,t) &< \frac{1}{\rho_n}\sum_{k\leq n}\mathscr{B}(x_k-\zeta,t)<\varepsilon,\\ \mathscr{Y}(x_{\tau_k}-\zeta,t) &< \frac{1}{\rho_n}\sum_{k\leq n}Y(x_k-\zeta,t)<\varepsilon.\end{aligned}$$

It follows that $(\mathscr{G},\mathscr{B},\mathscr{Y})_\rho-\lim x_{\tau_k}=\zeta$. □

Theorem 3.4. *If $\liminf_n \frac{n}{\rho_n}>1$, then $(\mathscr{G},\mathscr{B},\mathscr{Y})\subset(\mathscr{G},\mathscr{B},\mathscr{Y})_\rho$.*

Proof. Take $(\mathscr{G},\mathscr{B},\mathscr{Y})-\lim x=\zeta$. Since $\liminf_n \frac{n}{\rho_n}>1$, we can write

$$\begin{aligned}\frac{1}{\rho_n}\sum_{k\leq n}\mathscr{G}(x_k-\zeta,t) &= \frac{n}{\rho_n}\frac{1}{n}\sum_{k\leq n}\mathscr{G}(x_k-\zeta,t)\\ &\geqslant \frac{1}{n}\sum_{k\leq n}\mathscr{G}(x_k-\zeta,t)>1-\varepsilon.\end{aligned}$$

From here, we obtain $\frac{1}{\rho_n}\sum\limits_{k\leq n}\mathscr{B}(x_k-\zeta,t)<\varepsilon$ and $\frac{1}{\rho_n}\sum\limits_{k\leq n}\mathscr{Y}(x_k-\zeta,t)<\varepsilon$. Thus, $(\mathscr{G},\mathscr{B},\mathscr{Y})_\rho-\lim x=\ \zeta$. □

Theorem 3.5. *Let $\rho=(\rho_n)$ and $s=(s_n)$ be the given sequences and $\rho_n<s_n$, for all n. If*

$$\lim_{n\to\infty}\frac{s_n}{\rho_n}>0 \tag{3.1}$$

holds and $\mathscr{A}\subset\mathscr{F}$ is neutrosophic-bounded (NB) in $NNS\ \mathscr{V}$, then $(\mathscr{G},\mathscr{B},\mathscr{Y})_\rho\subset(\mathscr{G},\mathscr{B},\mathscr{Y})_s$.

Proof. Let $x\in(\mathscr{G},\mathscr{B},\mathscr{Y})_\rho$ and assume that (3.1) holds. Since $\mathscr{A}\subset\mathscr{F}$ is neutrosophic-bounded (NB) in $NNS\ \mathscr{V}$, then there exists some $t>0$ such that $\frac{1}{\rho_n}\sum\limits_{k\leq n}\mathscr{G}(x_k-\zeta,t)>1-\varepsilon$ and $\frac{1}{\rho_n}\sum\limits_{k\leq n}\mathscr{B}(x_k-\zeta,t)<\ \varepsilon$, $\frac{1}{\rho_n}\sum\limits_{k\in n}\mathscr{Y}(x_k-\zeta,t)<\varepsilon$ for each $(x_k-\zeta)\in\mathscr{A}$. Now, since $\rho_n\leq s_n$ for all $n\in\mathbb{N}$, we can write

$$\frac{1}{(\rho_n)}\sum_{k\leq n}\mathscr{G}(x_k-\zeta,t)=\frac{(s_n)}{(\rho_n)}\frac{1}{(s_n)}\sum_{k\leq n}\mathscr{G}(x_k-\zeta,t),$$

for all $n\in\mathbb{N}$. Therefore, we obtain $\frac{1}{(s_n)}\sum_{k\leq n}\mathscr{G}(x_k-\zeta,t)>1-\varepsilon$ and $\frac{1}{(s_n)}\sum_{k\leq n}\mathscr{B}(x_k-\zeta,t)<\varepsilon$. By similar operations, it can be shown that $\frac{1}{(s_n)}\sum\limits_{k\leq n}\mathscr{Y}(x_k-\zeta,t)<\varepsilon$, and as a result, we obtain $(\mathscr{G},\mathscr{B},\mathscr{Y})_\rho\subset(\mathscr{G},\mathscr{B},\mathscr{Y})_s$. □

Definition 3.2. Take an $NNS\ \mathscr{V}$. A sequence $x=(x_k)$ is called to be ρ-strongly Cauchy with respect to the $NN\ N$ $(Ca-NN)$ if, for every $\varepsilon\in(0,1)$ and $t>0$, there are $n_0,p\in\mathbb{N}$ satisfying

$$\frac{1}{\rho_n}\sum_{k\leq n}\mathscr{G}(x_k-x_p,t)>1-\varepsilon\ \text{ and }\ \frac{1}{\rho_n}\sum_{k\leq n}\mathscr{B}(x_k-x_p,t)<\varepsilon,$$

$$\frac{1}{\rho_n}\sum_{k\leq n}\mathscr{Y}(x_k-x_p,t)<\varepsilon,$$

for all $n\geq n_0$.

Theorem 3.6. *If a sequence $x=(x_k)$ in NNS is ρ-strongly convergent with regard to $NN\ N$, then it is strongly Cauchy with regard to $NN\ N$.*

Proof. Let $(\mathscr{G},\mathscr{B},\mathscr{Y})_\rho-\lim x=\zeta$. Select $\varepsilon>0$. Then for a given $\tau\in(0,1),(1-\tau)*(1-\tau)>1-\varepsilon$ and $\tau\Diamond\tau<\varepsilon$, we get

$$\frac{1}{\rho_n}\sum_{k\leq n}\mathscr{G}\Big(x_k-\zeta,\frac{t}{2}\Big)>1-\tau\ \text{ and }\ \frac{1}{\rho_n}\sum_{k\leq n}\mathscr{B}\Big(x_k-\zeta,\frac{t}{2}\Big)<\tau,$$

$$\frac{1}{\rho_n}\sum_{k\leq n}\mathscr{Y}\Big(x_k-\zeta,\frac{t}{2}\Big)<\tau.$$

We have to show that

$$\frac{1}{\rho_n}\sum_{k\leq n}\mathscr{G}(x_k-x_m,t)>1-\varepsilon\ \text{ and }\ \frac{1}{\rho_n}\sum_{k\leq n}\mathscr{B}(x_k-x_m,t)<\varepsilon,$$

$$\frac{1}{\rho_n}\sum_{k\leq n}\mathscr{Y}(x_k-x_m,t)<\varepsilon.$$

We have three possible cases.

Case (i) for $t>0$, we get

$$\mathscr{G}(x_k-x_m,t)\geq\mathscr{G}\Big(x_k-\zeta,\frac{t}{2}\Big)*\mathscr{G}\Big(x_m-\zeta,\frac{t}{2}\Big)>(1-\tau)*(1-\tau)>1-\varepsilon.$$

Case (ii) we obtain

$$\mathscr{B}(x_k-x_m,t)\leq\mathscr{B}\Big(x_k-\zeta,\frac{t}{2}\Big)\Diamond\mathscr{B}\Big(x_m-\zeta,\frac{t}{2}\Big)<\tau\Diamond\tau<\varepsilon.$$

Case (iii) we have

$$\mathscr{Y}(x_k - x_m, t) \leq \mathscr{Y}\Big(x_k - \zeta, \frac{t}{2}\Big) \Diamond \mathscr{Y}\Big(x_m - \zeta, \frac{t}{2}\Big) < \tau \Diamond \tau < \varepsilon.$$

This shows that (x_k) is strongly Cauchy with respect to $NN\ N$. □

Competing Interests

The authors declare that they have no conflict of interest.

Authors Contributions

All authors of the manuscript have read and agreed to its content and are accountable for all aspects of the accuracy and integrity of the manuscript.

Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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