Transactions of A. Razmadze
Mathematical Institute
Vol. **179** (2025), issue 1, 147–155

SOLUTIONS OF SINGULAR INTEGRAL EQUATIONS OF GENERALIZED CONVOLUTION TYPE

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Abstract. This article presents a class of singular integral equations of convolution kind in the class $\{0\}$. Fourier transforms are used to transform these equations into Riemann boundary value problems (RBVPs) with discontinuous coefficients. For such problems, we proposed a novel method and found the solutions in the class $\{0\}$.

1. Introduction

In several fields, especially engineering mechanics, physics, elasticity theory and fracture mechanics, the boundary value problems for analytic functions have been widely used. Many boundary value problems for analytic functions and singular integral equations with Cauchy kernel have seen extensive study and are widely used by many authors (see [5,9,10,15]). Convolution type integral equations and singular integral equations are the two important classes of integral equations. These equations are transformed into novel, discontinuous RBVPs by using Fourier transforms, different from the classical ones. The aim of this paper is to develop an application of the theory to generalised convolution-type singular integral equations with a Cauchy kernel. We investigate the problem in both cases, normal type and non-normal type, by using the theory of integral equations and the generalized theory of resolvent kernel operator. In this paper several results are improved (see References [1,16,19].

2. Preliminaries

Definition 2.1. We say that $\emptyset(x)$ is an element of a space with Hölder continuous functions H on $[-N,N]$, if any positive real integer r exists such that for any $x_1,\ x_2 \in [-N,N]$, the condition $|\emptyset(x_2)-\emptyset(x_1)| \leq r\ \ |x_2-\ x_1|^{\alpha}$ $(0<\alpha\leq 1)$ holds.

Definition 2.2. Suppose $\emptyset(x)$ is a continuous function on the entire real domain. The function $\emptyset(x) \in \tilde{H}$, if the following conditions are satisfied:
(i) $\emptyset(x)\ \in H$ on $[-N,N]$ in the case of any large enough positive number N.
(ii) $|\emptyset(x_2)-\emptyset(x_1)| \leq k\left|\frac{1}{x_2}-\frac{1}{x_1}\right|$ for any $|x_i|>N$ $(i=1,2)$, $k>0$.

Definition 2.3. Let the function $\emptyset(x)$ satisfy the conditions:
(i) $\emptyset(x) \in \tilde{H}$.
(ii) $\emptyset(x) \in L^1(R)$, where $L^1(R)=\left\{\emptyset(x)\,|\int_R |\emptyset(x)|dx<\infty\right\}$.
The function $\emptyset(x) \in \{\{0\}\}$, if the Hölder criterion for a neighborhood N_∞ of ∞ is satisfied, we denote as $\emptyset(x) \in H(N_\infty)$.

Definition 2.4 (see [4,6,13,14,17,18])**.** (1) If the function $\emptyset(x)$ belongs to the class$\{0\}$, the Fourier transform of $\emptyset(x)$ is

$$\mathbb{F}[\emptyset(x)] = \frac{1}{\sqrt{2\pi}}\int_R \emptyset(x)\ e^{isx}\ dx = \Phi(s),$$

belongs to the class $\{\{0\}\}$.

2020 *Mathematics Subject Classification.* 30E20, 30E25, 44A15, 44A35.
Key words and phrases. Fourier transforms; Riemann boundary value problems; Singular integral equations.

(2) The convolution for any functions $f(x), g(x) \in \{0\}$, has the form

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_R f(x-t)\, g(t)\, dt, \quad t \in R.$$

(3) The operator T of the Cauchy principal integral is defined as follows:

$$Tf(x) = \frac{1}{\pi i} \int_R \frac{f(y)}{y-x} dy.$$

Lemma 2.1. *If $f(x) \in \{0\}$, then $\mathbb{F}[Tf(x)] = -\operatorname{sgn}(s) F(s)$, where $F(s) = \mathbb{F}[f(x)]$.*

Proof. We have

$$\begin{aligned} \mathbb{F}[Tf(x)] &= \frac{1}{\sqrt{2\pi}} \int_R \left[\frac{1}{\pi i} \int_R \frac{f(y)}{y-x} dy \right] e^{isx} dx \\ &= -\frac{1}{\sqrt{2\pi}} \int_R \left[\frac{1}{\pi i} \int_R \frac{e^{isx}}{x-y} dx \right] f(y) dy. \end{aligned} \tag{2.1}$$

Extended Residue Theorem provides us with

$$\frac{1}{\pi i} \int_R \frac{e^{isx}}{x-y} dx = \begin{cases} e^{isy} & s > 0, \\ 0 & s = 0, \\ -e^{isy} & s < 0. \end{cases} \tag{2.2}$$

Substituting (2.2) into (2.1), we obtain

$$\mathbb{F}[Tf(x)] = -\operatorname{sgn}(s) \frac{1}{\sqrt{2\pi}} \int_R f(x)\, e^{isx} dx = -\operatorname{sgn}(s) F(s).$$

Similarly, $\mathbb{F}[Tf(-x)] = -\operatorname{sgn}(s) F(-s)$. ☐

Lemma 2.2. *If $f(x) \in \{0\}$, then $\mathbb{F}[\operatorname{sgn}(x) f(x)] = TF(s)$.*

Proof. We have

$$\begin{aligned} \mathbb{F}[\operatorname{sgn}(x)\, f(x)] &= \frac{1}{\sqrt{2\pi}} \int_R \operatorname{sgn}(x) f(x)\, e^{isx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{R^+} f(x)\, e^{isx}\, dx + \frac{-1}{\sqrt{2\pi}} \int_{R^-} f(x)\, e^{isx}\, dx \\ &= F^+(s) + F^-(s), \end{aligned}$$

where $R^+ = [0, \infty)$, $R^- = (-\infty, 0]$.

Since $TF = F^+ + F^-$, therefore $\mathbb{F}[\operatorname{sgn}(x) f(x)] = TF(s)$. ☐

Lemma 2.3 (see [5])**.** *If the functions $f, g \in \{0\}$, then*

$$(f * g)(x) \in \{0\}, \quad \mathbb{F}[f * g(x)] = F(s)\, G(s),$$

where $F(s) = \mathbb{F}[f(x)]$, $G(s) = \mathbb{F}[g(x)]$.

3. Presentation of the Problem

We study the solutions of the following generalized singular integral equation (SIE) with a convolution kernel :

$$\begin{aligned} af(t) + bTf(t) + cT(\, d * f)(t) + (\alpha * f)(t) + (\beta * \operatorname{sgn}(y)\, f)(t) \\ + \operatorname{sgn}(-t)(r * f)(t) \\ +(r * (\operatorname{sgn}(y)\, f)(t) = g(t), \end{aligned} \tag{3.1}$$

where a, b, c are the constants, the functions $d(t)$, $\alpha(t)$, $\beta(t)$, $r(t)$, $g(t) \in \{0\}$.

In this article, we present a novel method [17, 18] for solving equation (3.1), this method differs from the classical methods [2,4,6,8]. Applying Fourier transforms to both sides of equation (3.1) and using Lemmas 2.1–2.3, we obtain

$$A(x) F(x) + B(x) TF(x) + \int_R K(y,x) F(y) \, dy = G(x), \quad x \in R, \tag{3.2}$$

where $A(x) = \mathbb{F}[\alpha(x)] - c \operatorname{sgn}(x) \mathbb{F}[d(x)] - b \operatorname{sgn}(x) + a$, $\quad B(x) = \mathbb{F}[\beta(x)]$,

$$K(y,x) = -\frac{1}{\pi i} \frac{r(y) - r(x)}{y - x}.$$

By the inverse Fourier transform, $f(x) = \mathbb{F}^{-1}[F(x)]$, the solution of equation (3.2) is the same as that of equation (3.1).

Section 3.1 The case of normal type. If $A(x) \pm B(x) \neq 0$, equation (3.2) is said to be of the normal type. We look for the general solution of the following characteristic equation of (3.2):

$$A(x) F(x) + B(x) TF(x) = G(x), \quad x \in R. \tag{3.3}$$

We define the holomorphic function

$$\Psi(z) = \frac{1}{2\pi i} \int_R \frac{F(y)}{y - z} \, dy, \quad z \notin R. \tag{3.4}$$

From the Plemelj formula [15], we have

$$F(x) = \Psi^+(x) - \Psi^-(x), \quad TF(x) = \Psi^+(x) + \Psi^-(x), \quad x \in R. \tag{3.5}$$

Substituting equation (3.4) into equation (3.3), equation (3.3) is related to the Riemann boundary value problem (RBVP)

$$\Psi^+(x) = \mathrm{N}(\mathrm{x}) \Psi^-(x) + M(x), \quad x \in R, \tag{3.6}$$

where

$$N(x) = \frac{\boldsymbol{A}(\boldsymbol{x}) - \boldsymbol{B}(\boldsymbol{x})}{\boldsymbol{A}(\boldsymbol{x}) + \boldsymbol{B}(\boldsymbol{x})}, \qquad M(x) = \frac{\mathrm{G}(x)}{\boldsymbol{A}(\boldsymbol{x}) + \boldsymbol{B}(\boldsymbol{x})}.$$

Let $\chi = \operatorname{Ind}_R N(x) = \frac{1}{2\pi}[\arg N(x)]_R$, then we called the value χ is the index of problem (3.6). Let the two points in the upper– and lower-half of the planes z_2, z_1, respectively, be fixed, then $\operatorname{Ind}\left(\frac{z-z_2}{z-z_1}\right)^{-\chi} N(x) = 0$. We take a continuous branch of $\log N(x)$ such that $\log N(\infty) = 0$, letting $\sigma = \propto + i\theta = \frac{1}{2\pi i}\{\log N(-0) - \log N(+0)\}$. If $\propto$ is an integer, then $x = 0$ is referred to as an ordinary node, otherwise when $x = 0$, is called as a special node. We define

$$\frac{X^+(x)}{X^-(x)} = \frac{\boldsymbol{A}(\boldsymbol{x}) - \boldsymbol{B}(\boldsymbol{x})}{\boldsymbol{A}(\boldsymbol{x}) + \boldsymbol{B}(\boldsymbol{x})}, \tag{3.7}$$

$$X(z) = \begin{cases} e^{\Gamma(z)} & \operatorname{Re} Z > 0 \\ \left(\frac{z-z_1}{z-z_2}\right)^{\chi} e^{\Gamma(z)} & \operatorname{Re} z < 0 \end{cases},$$

$$\Gamma(z) = \frac{1}{2\pi i} \int_R \frac{\log\left[\left(\frac{y-z_1}{y-z_2}\right)^{\chi} N(y)\right]}{y - z} \, dy.$$

If the order of $\Psi(z)$ in (3.6) are m at infinity, we define the solution obtained in R_m. From equation (3.5), we have $\Psi(\infty) = 0$, therefore we look for a solution of equation (3.6) in R_{-1}. From the method used in [3, 11, 12], we have the following:

When $\chi \geq 0$, equation (3.6) has the solution

$$\Psi(z) = \frac{X(z)}{2\pi i} \int_R \frac{G(y)}{[\boldsymbol{A}(\boldsymbol{y}) + \boldsymbol{B}(\boldsymbol{y})]X^+(y)(y-z)} dy + X(z) P_{\chi-1}(z), \quad z \notin R, \tag{3.8}$$

where $P_{\chi-1}(z)$ is an arbitrary polynomial of degree $\chi - 1$.

When $\chi < 0$, equation (3.6) has a unique solution if and only if the conditions

$$\int_R \frac{G(y)\, y^j}{[\boldsymbol{A}(\boldsymbol{y}) + \boldsymbol{B}(\boldsymbol{y})]X^+(y)} dy = 0, \quad j = 0, 1, 2, \ldots, -\chi - 1$$

are satisfied. From equation (3.7), we suppose

$$Z(x) = [\boldsymbol{A}(\boldsymbol{x}) + \boldsymbol{B}(\boldsymbol{x})]\, X^+(x) = [\boldsymbol{A}(\boldsymbol{x}) - \boldsymbol{B}(\boldsymbol{x})]\, X^-(x),$$

$$\boldsymbol{A_1}(\boldsymbol{x}) = \frac{\boldsymbol{A(x)}}{\boldsymbol{A^2}(\boldsymbol{x}) - \boldsymbol{B^2}(\boldsymbol{x})}, \quad \boldsymbol{B_1}(\boldsymbol{x}) = \frac{\boldsymbol{B(x)}}{\boldsymbol{A^2}(\boldsymbol{x}) - \boldsymbol{B^2}(\boldsymbol{x})}.$$

From equations (3.8) and (3.5), we obtain the following general solution $F_\circ(x)$ of (3.3):

$$F_\circ(x) = \Psi^+(x) - \Psi^-(x) = \rho G(x) + \boldsymbol{B_1}(x)\, Z(x)\, P_{\chi-1}(z),$$

where

$$\rho G(x) = \boldsymbol{A_1}(\boldsymbol{x})\; G(x) - \frac{\boldsymbol{B_1}(x)\, Z(x)}{\pi i} \int_R \frac{G(y)}{Z(y)(y-x)} dy.$$

When $\chi \geq 0$, by the Vekua regularization method [7] in equation (3.2), we get the Fredholm integral equation (FIE)

$$F(x) + \rho H(x) = F_\circ(x), \quad x \in R, \tag{3.9}$$

where

$$H(x) = \int_R K(y, x)\, F(y)\, dy, \quad x \in R.$$

Hence the solution $f(x)$ of equation (3.1) belongs to $\{0\}$ and the solution $F(x)$ of equation (3.2) belongs to $\{\{0\}\}$.

When $x = 0$ is an ordinary node, the condition

$$H(0) = G(0) \tag{3.10}$$

must be necessary for equation (3.9) solved in $\{\{0\}\}$.

If $x = 0$ is a special node, C_0 is a polynomial constant term of the polynomial $P_{\chi-1}$, the condition

$$C_0 + \frac{\boldsymbol{1}}{2\pi i} \int_R \frac{H(y) - G(y)}{Z(y)} dy = 0$$

holds. When $\chi < 0$, in addition to condition (3.10), the condition

$$\int_R \frac{H(y) - G(y)}{Z(y)} (y - z_1)^{-j} dy = 0, \quad j = 1, 2, \ldots, (-\chi) \tag{3.11}$$

must be satisfied when $x = 0$ is an ordinary node.

If $x = 0$ is a special node, the condition

$$\int_R \frac{H(y) - G(y)}{Z(y)} y^{-1} dy = 0$$

should be added to formula (3.11) for the solvability. The Fredholm integral equation

$$F(x) + \rho H(x) = \rho G(x), \quad x \in R \tag{3.12}$$

is obtained by using the regularised equation (3.2).

Applying the concepts of Fredholm integral equations to equations (3.9) and (3.12), we obtain the following results:

Theorem 3.1. *If $\chi \geq 0$, the solution of equation* (3.9) *exists if and only if the conditions*

$$\int_R F_\circ(x)\, u_j(x)\, dx = 0, \quad j = 1, 2, \ldots, n$$

are satisfied, where $u_1, u_2, \ldots, u_n$ are the set of all solutions of the corresponding homogeneous equation (3.9). *The solution of equation* (3.9) *is*

$$F(x) = \Delta(x) + \sum_{j=1}^{n} k_j u_j(x), \tag{3.13}$$

where k_j, $j : 1 \to n$ are the constants and

$$\Delta(x) = F_\circ(x) + \int_R \eta_1(y, x)\, F_\circ(y)\, dy,$$

where $\eta_1(y, x)$ is the resolvent kernel of equation (3.9) (*see* [2]).

From equation (3.13), the solutions of equation (3.1) in the class $\{0\}$ is expressed by

$$f(x) = \mathbb{F}^{-1}[F(x)].$$

If $\chi < 0$, the necessary and sufficient conditions for the solutions of equation (3.12) in the class $\{\{0\}\}$ has the form

$$\int_R G(x) u_j(x)\, dx = 0, \;\; j = 1, 2, \ldots, n,$$

and the general solution of equation (3.12) is

$$F(x) = \Delta^*(x) + \sum_{j=1}^{n} k_j u_j(x), \;\; x \in R, \tag{3.14}$$

where

$$\Delta^*(x) = G(x) + \int_R \eta_2(y, x)\, G(y)\, dy, \;\; x \in R,$$

and, $\eta_2(y, x)$ is the resolvent kernel of equation (3.12). From equation (3.14), hence the solutions of equation (3.1) in the class $\{0\}$ is expressed by

$$f(x) = \mathbb{F}^{-1}[F(x)].$$

Section 3.2 The case of non-normal type. Consider the functions $A(x) + B(x)$ and $A(x) - B(x)$ having common and equal order zero points $r_1, r_2, \ldots, r_m$ $(r_j \in R)$ with the orders $a_1, a_2, \ldots, a_m$, respectively, and $A(x) + B(x)$ contains a number of zero points $r'_1, r'_2, \ldots, r'_n$ $\left(r_j \neq r'_k,\;\; r'_k \in R\right)$ with the orders $b_1, b_2, \ldots, b_n$, respectively, $A(x) - B(x)$ contains a number of zero points $r''_1, r''_2, \ldots, r''_q$, $(r_j \neq r''_i, r''_i \in R)$ with the orders $c_1, c_2, \ldots c_q$, respectively.

Let

$$\omega_1(x) = \prod_{j=1}^{n} (x - r'_j)^{b_j}, \quad \omega_2(x) = \prod_{j=1}^{q} (x - r''_j)^{c_j}, \quad \sum_{j=1}^{n} b_n = Q_1, \quad \sum_{j=1}^{q} c_j = Q_2,$$

where $a_j \geq 0$, $b_j \geq 0$, $c_j \geq 0$. Equation (3.6) can be written in the form

$$\Psi^+(x) = \frac{\omega_2(x)(x - z_1)^{Q_1}}{\omega_1(x)(x - z_2)^{Q_2}} \widetilde{\mathrm{N}}(\mathrm{x})\, \Psi^-(x) + M(x), \;\; x \in R, \tag{3.15}$$

where $\widetilde{\mathrm{N}}(\mathrm{x}) \neq 0$ $x \in R$.

By the generalized Liouville theorem, the general solution of the homogeneous equation (3.15) is given by

$$\widetilde{\Psi}(z) = \begin{cases} X(z)\, \omega_2(z)\, (z - z_1)^{Q_1 - \chi} P_{\chi - Q_1 - Q_2 - 1}(z) & \mathrm{Re}\, z > 0, \\ X(z)\, \omega_1(z)\, \frac{(z - z_2)^{Q_2}}{(z - z_1)^{\chi}} P_{\chi - Q_1 - Q_2 - 1}(z) & \mathrm{Re}\, z < 0, \end{cases}$$

where $P_{\chi-Q_1-Q_2-1}(z)$ is a polynomial of degree $(\chi-Q_1-Q_2-1)\geq 0$, if $(\chi-Q_1-Q_2-1)<0$ then $P_{\chi-Q_1-Q_2-1}(z)=0$.

Using the generalized Liouville theorem, the solution of the non-homogeneous equation (3.15) has singularity at ${r'}_j$, ${r''}_j$. We use the Hermite interpolation polynomial $H_l(z)$ of degree Q_1+Q_2-1 with zero-points of order b_j, c_j at ${r'}_j$, ${r''}_j$, respectively to find a solution to equation (3.15) in the class $\{0\}$, then the solution is given by

$$\mathrm{E}(z)=\begin{cases} X(z)\left[V(z)-\frac{H_l(z)}{(z-z_1)^{\chi}}\right]\frac{(z-z_1)^{Q_1}}{\omega_1(z)} & \operatorname{Re} z>0, \\ X(z)\left[V(z)-\frac{H_l(z)}{(z-z_1)^{\chi}}\right]\frac{(z-z_2)^{Q_2}}{\omega_2(z)} & \operatorname{Re} z<0, \end{cases} \tag{3.16}$$

where

$$V(z)=\frac{1}{2\pi i}\int\limits_R \frac{G(y)}{\omega_2(y)(y-z_1)^{Q_1}Z(y)(y-z)}dy, \quad z\notin R.$$

The solution in (3.16) is a particular solution of (3.15), hence the general solution of (3.15) is given by

$$\Psi(\mathrm{z})=\mathrm{E}(z)+\widetilde{\Psi}(z).$$

Similarly to the normal type case (Section 3.1), we have:

1. If $\chi\geq 0$, equation (3.2) corresponds to FIE,

$$F(x)+\widetilde{\rho}H(x)=\tilde{G}(x), \quad x\in R, \tag{3.17}$$

which satisfies the condition

$$\left.[H(x)-G(x)]^{(l)}\right|_{x=r_j}=0, \quad j=1,2,\ldots,m; \;\; l=0,1,2,\ldots,a_j-1, \tag{3.18}$$

where

$$\widetilde{\rho}\,H(x)=\frac{A_1(x)U(x)H(x)}{\omega_2(x)(x-z_1)^{Q_1}}-\frac{B_1(x)U(x)Z(x)}{\pi i}\int\limits_R\frac{H(y)}{\omega_2(y)(y-z_1)^{Q_1}Z(y)(y-x)}dy,$$

$$\tilde{G}(x)=\widetilde{\rho}G(x)+\frac{2B_1(x)R(x)Z(x)}{(x-z_1)^{\chi}}.$$

When $x\geq 0$, we have

$$U(x)=\frac{(x-z_1)^{Q_1}}{\omega_1(x)}, \quad R(x)=(x-z_1)^{Q_1}\left[\frac{H_l(x)}{\omega_1(z)}-\omega_2(z)P_{\chi-Q_1-Q_2-1}\right].$$

When $x<0$, we have

$$U(x)=\frac{(x-z_2)^{Q_2}}{\omega_2(x)}, \quad R(x)=(x-z_2)^{Q_2}\left[\frac{H_l(x)}{\omega_2(z)}-\omega_1(z)P_{\chi-Q_1-Q_2-1}(z)\right].$$

2. If $\chi<0$, equation (3.2) corresponds to FIE,

$$F(x)+\widetilde{\rho}H(x)=G_0(x), \quad x\in R, \tag{3.19}$$

which satisfies condition (3.18). If $x\geq 0$, we get

$$G_0(x)=\widetilde{\rho}G(x)+\frac{2B_1(x)H_l(x)Z(x)}{\omega_1(z)},$$

and if $x<0$, we get

$$G_0(x)=\widetilde{\rho}G(x)+\frac{2B_1(x)H_l(x)Z(x)(x-z_2)^{Q_2}}{\omega_2(z)(x-z_1)^{Q_1}}.$$

Theorem 3.2. *If $\chi \geq 0$, the solution of equation* (3.17) *exists if and only if the conditions*

$$\int_R \tilde{G}(x)\, {u^*}_j(x)\, dx = 0, \quad j = 1, 2, \ldots, n,$$

are satisfied, where ${u^*}_1, {u^*}_2, \ldots, {u^*}_n$ are the set of all solutions of the corresponding homogeneous equation (3.17). *The solution of equation* (3.17) *is*

$$F(x) = \Delta^*(x) + \sum_{j=1}^{n} e_j {u^*}_j(x), \tag{3.20}$$

where e_j $(j = 1, 2, \ldots, n)$ are the constants and

$$\Delta^*(x) = \tilde{G}(x) + \int_R \eta_3(y, x)\, \tilde{G}(y)\, dy, \quad x \in R,$$

where $\eta_3(y, x)$ is the resolvent kernel of equation (3.17) (*see* [2]).

From equation (3.20), the solutions of equation (3.1) in the class $\{0\}$ is expressed by

$$f(x) = \mathbb{F}^{-1}[F(x)].$$

If $\chi < 0$, the necessary and sufficient conditions for the solutions of equation (3.19) in the class $\{\{0\}\}$ are

$$\int_R G_0(x)\, {u^*}_j(x)\, dx = 0, \quad j = 1, 2, \ldots, n,$$

where ${u^*}_1, {u^*}_2, \ldots, {u^*}_n$ are the set of all solutions of the corresponding homogeneous equation (3.19). The solution of equation (3.19) is

$$F(x) = \Delta_1^*(x) + \sum_{j=1}^{n} e_j {u^*}_j(x), \quad x \in R, \tag{3.21}$$

where

$$\Delta_1^*(x) = G_0(x) + \int_R \eta_4(y, x)\, G_0(y)\, dy, \quad x \in R,$$

and $\eta_4(y, x)$ is the resolvent kernel of equation (3.19). From equation (3.21), hence the solutions of equation (3.1) in the class $\{0\}$ is expressed by

$$f(x) = \mathbb{F}^{-1}[F(x)].$$

4. Example

To illustrate how the method succeeds, we give an example.

In equation (3.1) we assume

$$a = c = 0, \quad b = 1, \quad \alpha(y) = r(y) = 0, \quad \beta(y) = 4\left(\sqrt{2\pi}\right)^{-1}\left(1 + y^2\right),$$

$$g(y) = \begin{cases} 2\left(\sqrt{2\pi}\right)^{-1}\left(1 + y^2\right) - 2\left(2i\sqrt{2\pi}\right)^{-1}, & y \neq 0, \\ 0, & y = 0, \end{cases}$$

the functions $\alpha(y)$, $r(y)$, $\beta(y)$, $g(y)$ belong to the class $\{0\}$, hence equation (3.1) can be written in the form

$$Tf(t) + (\ \beta * \operatorname{sgn}(y) f)(t) = g(t). \tag{4.1}$$

Let $l_1(y) = l_2(y) = \frac{\beta(y)}{2}$, then equation (4.1) has the form

$$Tf(t) + \left(l_1 * f^+\right)(t) + \left(l_2 * f^+\right)(t) = g(t).$$

By the Fourier transformation to equation (4.1), we obtain

$$-\operatorname{sgn}(t)\, F(t) + e^{-|t|} F^+(t) - e^{-|t|} F^-(t) = e^{-|t|} - \operatorname{sgn}(t). \tag{4.2}$$

Then equation (4.2) has the form

$$F^{+}(t) = C(t)\, F^{-}(t) + Q(t), \tag{4.3}$$

where $C(t) = 1$, $Q(t) = 1$.

Since

$$\Gamma(z) = \frac{1}{2\pi i} \int_{R} \frac{\log C(y)}{y - z}\, dy = 0,$$

and from [4], the index $\chi = 0$, $X(z) = 1$.

Hence, from the method used in Section 3, the solution of equation (4.3) is

$$F(z) = \frac{X(z)}{2\pi i} \int_{R} \frac{Q(y)}{X^{+}(y)(y - z)} dy + eX(z), \quad z \notin R,$$

and then

$$F(z) = \frac{1}{2\pi i} \int_{R} \frac{1}{(y - z)} dy + e, \quad z \notin R.$$

From [19], we have

$$\frac{1}{2\pi i} \int_{R} \frac{1}{(y - z)} dy = \begin{cases} \frac{1}{2} & \operatorname{Im} z > 0 \\ -\frac{1}{2} & \operatorname{Im} z < 0, \end{cases}$$

hence the solution of equation (4.3) is given by

$$F(z) = \begin{cases} e + \frac{1}{2} & \operatorname{Im} z > 0, \\ e - \frac{1}{2} & \operatorname{Im} z < 0. \end{cases}$$

Thus equation (4.1) has the solution $f(x) = \mathbb{F}^{-1}[F(x)]$.

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(Received 28.01.2023)

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