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ON THE EXISTENCE OF BOUNDED SOLUTIONS ON THE REAL AXIS $\mathbb{R}$ FOR SYSTEMS OF LINEAR IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. Effective sufficient conditions for the existence of bounded solutions satisfying the Nicoletti condition for systems of linear impulsive differential equations on the real axis are established. The method for constructing such solutions is given. The sufficient conditions for the existence of a unique solution and its positivity are established, as well. As a particular case, the problem of the existence of bounded solutions is studied.

1. Statement of the Problem. Basic Notation and Definitions

For the linear system of impulsive differential equations

$$\frac{dx}{dt} = P(t)x + q(t), \ t \in \mathbb{R}\backslash T, \tag{1.1}$$

$$x(\tau_l+) - x(\tau_l-) = G(\tau_l)x(\tau_l) + u(\tau_l) \ (l = 1, 2, \ldots), \tag{1.2}$$

consider the problem of the bounded on $\mathbb{R}$ solution

$$\sup\{\|x(t)\| : t \in \mathbb{R}\} < +\infty, \tag{1.3}$$

where $P \in L_{\rm loc}(\mathbb{R};\mathbb{R}^{n\times n})$, $q \in L_{\rm loc}(\mathbb{R};\mathbb{R}^n)$, $G \in B_{\rm loc}(T;\mathbb{R}^{n\times n})$, $u \in B_{\rm loc}(T;\mathbb{R}^n)$, $T = \{\tau_1, \tau_2, \ldots\}$, $\tau_l \in \mathbb{R}$ $(l = 1, 2, \ldots)$, $\tau_l \neq \tau_k$ if $l \neq k$ $(l, k = 1, 2, \ldots)$.

In this paper, the effective sufficient conditions are established for the existence of solutions of problem (1.1), (1.2), (1.3). Analogous results for the problem involving the systems of ordinary differential equations can be found in [4, 5] (see also references therein).

Quite a number of issues on the theory of linear systems of differential equations with impulsive effect have been studied sufficiently well (for a survey of the results in impulsive systems see, e.g., [1–3, 6], and the references therein).

In the present paper, the use will be made of the following notation and definitions:

$\mathbb{R} =]-\infty; +\infty[$; $[a; b]$ and $]a; b[$ $(a; b \in \mathbb{R})$ are, respectively, closed and open intervals.

I is an arbitrary finite or infinite interval from $\mathbb{R}$.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\ldots,m} \sum_{i=1}^{n} |x_{ij}|.$$

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1}, $\det(X)$ and $r(X)$ are, respectively, the matrices, inverse to X, the determinant of X and the spectral radius of X.

I_n is the identity $n \times n$ - matrix; δ_{ij} is the Kroneker symbol, i.e. $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$ $(i, j = 1, 2, \ldots)$.

The inequalities between the real matrices are understood componentwise.

We say that some property holds in the set I if it holds on every closed interval from I.

A matrix-function is said to be continuous, integrable, non-decreasing, etc., if each of its components is such.

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$\bigvee\limits_a^b(X)$ is the sum of total variations of components x_{ij} $(i = 1,\dots,n;\ j = 1,\dots,m)$ of the matrix-function $X : [a,b] \to \mathbb{R}^{n\times m}$.

If $X : I \to \mathbb{R}^{n\times m}$ is a matrix-function, then $\bigvee\limits_I(X)$ is the sum of total variations on I of its components x_{ij} $(i = 1,\dots,n;\ j = 1,\dots,m)$; $\bigvee(X)(t) = (v(x_{ij})(t))_{i,j=1}^{n,m}$ for $t \in I$; where $v(x_{ij})(a) = 0$, $v(x_{ij})(t) \equiv \bigvee\limits_a^t(x_{ij})$ and $a \in I$ is some fixed point.

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of the matrix-function $X : I \to \mathbb{R}^{n\times m}$.

$\|X\|_\infty = \sup\{\|X(t)\| : t \in I\}$.

$BV(I;\mathbb{R}^{n\times m})$ is the normed space of all bounded variation matrix-functions $X : I \to \mathbb{R}^{n\times m}$ (i.e., such that $\bigvee\limits_I(x) < \infty$) with norm $\|X\|_S$.

$BV(I;D)$, where $D \subset \mathbb{R}^{n\times m}$, is the set of all bounded variation matrix-functions $X : I \to D$.

$BV_{\rm loc}(I;D)$ is the set of all $X : I \to D$ for which the restriction on $[a,b]$ belongs to $BV([a,b]\,;D)$ for every closed interval $[a,b]$ from I.

$AC([a,b]\,;D)$ is the set of all absolutely continuous matrix-functions $X : [a,b] \to D$.

$AC_{\rm loc}(I;D)$ is the set of all matrix-functions $X : I \to D$, whose restrictions to an arbitrary closed interval $[a,b]$ from I belong to $AC([a,b]\,;D)$.

$AC_{\rm loc}(I\backslash T;D)$, where $T = \{\tau_1,\tau_2,\dots\}$, $\tau_l \in I$ $(l = 1,2,\dots)$, $\tau_l \neq \tau_k$ $(l \neq k)$, is the set of all matrix-functions $X : I \to D$, whose restrictions to an arbitrary closed interval $[a,b]$ from $I\backslash T$ belong to $AC([a,b]\,;D)$.

$ACV(I,T;D) = AC(I\backslash T;D) \cap BV(I;D)$.

$ACV_{\rm loc}(I,T,D) = AC_{\rm loc}(I\backslash T;D) \cap BV_{\rm loc}(I;D)$.

$T_j = T \cap J$ for every interval $J \subset I$.

$B(T;\mathbb{R}^{n\times m})$ is the set of all matrix-functions $G : T \to \mathbb{R}^{n\times m}$ such that

$$\sum_{l=1}^{+\infty} \|G(\tau_l)\| < +\infty.$$

$B_{\rm loc}(T;\mathbb{R}^{n\times m})$ is the set of all matrix-functions $G : T \to \mathbb{R}^{n\times m}$ such that

$$\sum_{\tau_l \in T_{[a,b]}} \|G(\tau_l)\| < +\infty \text{ for every } [a,b] \subset I.$$

$L([a,b]\,;\mathbb{R}^{n\times m})$ is the set of all Lebesgue integrable matrix-functions $X : [a,b] \to \mathbb{R}^{n\times m}$.

$L_{\rm loc}(I;\mathbb{R}^{n\times m})$ is the set of all the Lebesgue integrable matrix-functions $X : [a,b] \to \mathbb{R}^{n\times m}$ whose restrictions to an arbitrary closed interval $[a,b]$ from I belong to $L([a,b]\,;\mathbb{R}^{n\times m})$.

We say that the pair (X,Y) consisting of the matrix-functions $X \in L(I;\mathbb{R}^{n\times n})$ and $Y \in B(T;\mathbb{R}^{n\times n})$ satisfies the Lappo-Danilevskiĭ condition at the point a if

$$X(t)\int_a^t X(\tau)\,d\tau = \int_a^t X(\tau)\,d\tau \cdot X(t)$$

$$\int_a^t X(\tau)\,d\tau \cdot \sum_{a\leqslant\tau_l<t} Y(\tau_l) = \sum_{a\leqslant\tau_l<t} Y(\tau_l)\cdot \int_a^t X(\tau)\,d\tau$$

for $t \in I$.

Definition 1.1. Under a solution of the impulsive differential system (1.1), (1.2) we understand a continuous from the left vector-function $x \in ACV_{\rm loc}(R,T;\mathbb{R}^n)$ satisfying both the system

$$x'(t) = p(t)x(t) + q(t) \text{ for a.a. } t \in \mathbb{R}\backslash T$$

and relation (1.2) for every $l = \{1,2,\dots\}$.

We assume that the condition

$$\det(I_n + G(\tau_l)) \neq 0 \ (l = 1, 2, \dots) \tag{1.4}$$

holds.

Remark 1.1. By Definition 1.1, under a solution of the impulsive system (1.1), (1.2) we understand the continuous from the left vector-function. If under a solution we understand the continuous from the right vector-function, then we have to require the condition

$$\det(I_n - G(\tau_l)) \neq 0 \ (l = 1, 2, \dots)$$

instead of (1.4).

The results corresponding to this case are analogous to those corresponding to the first case given in this paper, if we replace the expressions of type $I_n + G(\tau_l)$ by $I_n - G(\tau_l)$ the intervals $[s, t[$ by $]s, t]$, and the right limits by the left ones.

If $\alpha \in L_{\text{loc}}(\mathbb{R}; \mathbb{R})$ and $\beta \in B(T; \mathbb{R})$ are such that $1 + \beta(\tau_l) \neq 0$ $(l = 1, 2, \dots)$, then the problem

$$\frac{dx}{dt} = \alpha(t)x \text{ for a.a. } t \in \mathbb{R} \backslash T,$$
$$x(\tau_l+) - x(\tau_l-) = \beta(\tau_l)x(\tau_l) \ (l = 1, 2, \dots);$$
$$x(0) = 1$$

has the unique solution $\xi_{(\alpha,\beta)}$ and it is defined by

$$\xi_{(\alpha,\beta)}(t) = \begin{cases} \exp\left(\int\limits_0^t \alpha(\tau)\, d\tau\right) \prod\limits_{0 \leqslant \tau_l < t} (1 + \beta(\tau_l)), & t > 0, \\ \exp\left(\int\limits_0^t \alpha(\tau)\, d\tau\right) \prod\limits_{t \leqslant \tau_l < 0} (1 + \beta(\tau_l))^{-1}, & t < 0. \end{cases}$$

Let $\gamma_{(\alpha,\beta)}(t, s) \equiv \xi_{(\alpha,\beta)}(t)\xi^{-1}_{(\alpha,\beta)}(s)$ be the Cauchy function of the problem. Then

$$\gamma_{(\alpha,\beta)}(t, s) = \exp\left(\int\limits_s^t \alpha(\tau)\, d\tau\right) \prod_{s \leqslant \tau_l < t} \operatorname{sgn}(1 + \beta(\tau_l)) \text{ for } t > s,$$
$$\gamma_{(\alpha,\beta)}(t, s) = \gamma^{-1}_{(\alpha,\beta)}(s, t) \text{ for } t < s.$$

Note that the equalities

$$\frac{d\xi^{-1}_{(\alpha,\beta)}(t, t_0)}{dt} = -\alpha(t)\xi^{-1}_{(\alpha,\beta)}(t, t_0) \text{ for a.a. } t \in \mathbb{R} \backslash T, \tag{1.5}$$

$$\xi^{-1}(\tau_l+) - \xi^{-1}(\tau_l-) = \xi^{-1}(\tau_l)\beta(\tau)(1 + \beta(\tau_l)) \ (l = 1, 2, \dots) \tag{1.6}$$

hold (see [2, 3]).

Remark 1.2. Let $\alpha \in L([a, b]; \mathbb{R})$, $\beta \in B(T; \mathbb{R})$ be such that $1 + \beta(\tau_l) > 0$ $(l = 1, 2, \dots)$ and one of the function $\int\limits_a^t \alpha(\tau)\, d\tau + \sum\limits_{a \leqslant \tau_l < t} \beta(\tau_l)$, $\int\limits_a^t \alpha(\tau)\, d\tau + \sum\limits_{a \leqslant \tau_l < t} \ln|1 + \beta(\tau_l)|$ and $\int\limits_a^t \alpha(\tau)\, d\tau + \sum\limits_{a \leqslant \tau_l < t} (1 + \beta(\tau_l))^{-1}\beta(\tau_l)$ be non-decreasing (non-increasing). Then the other two functions will be non-decreasing (non-increasing), as well.

We introduce the operator

$$v(\zeta)(t) = \sup\{\tau \geqslant t : \zeta(\tau) \leqslant \zeta(t+) + 1\},$$

if $\zeta : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function, and

$$v(\zeta)(t) = \inf\{\tau \leqslant t : \zeta(\tau) \leqslant \zeta(t-) + 1\},$$

if $\zeta : \mathbb{R} \to \mathbb{R}$ is a non-increasing function.

2. Formulation of the Results

For every $t_i \in \mathbb{R} \cup \{-\infty, +\infty\}$ $(i = 1, \dots, n)$, we put $N_0(t_1, \dots, t_n) = \{i : t_i \in \mathbb{R}\}$. It is evident that $N_0(t_1, \dots, t_n) = \{1, \dots, n\}$ if $t_i \in \mathbb{R}$ $(i = 1, \dots, n)$, and $N_0(t_1, \dots, t_n) = \varnothing$ if $t_i \in \{-\infty, +\infty\}$ $(i = 1, \dots, n)$.

In the case, where $t_i = -\infty$ $(t_i = +\infty)$, we assume $\operatorname{sgn}(t - t_i) = 1$ for $t \in \mathbb{R}$ $(\operatorname{sgn}(t - t_i) = -1$ for $t \in \mathbb{R})$.

Theorem 2.1. *Let*

$$1 + g_{ii}(\tau_l) \neq 0 \ (i = 1, \dots, n; \ l = 1, 2, \dots) \tag{2.1}$$

and let there exist $t_i \in \mathbb{R} \cup \{-\infty, +\infty\}$ $(i = 1, \dots, n)$ such that

$$s_{ik} = \sup\left\{ \left| \int_{t_i}^{t} |\gamma_i(t,\tau)||p_{ik}(\tau)|\, d\tau + \sum_{t_i \leqslant \tau_l < t} |\gamma_i(t,\tau_l)||(1 + g_{ii}(\tau_l))^{-1} g_{ik}(\tau_l)| \right| : t \in \mathbb{R} \right\} < +\infty$$
$$(i \neq k; \ \ i,k = 1, \dots, n), \tag{2.2}$$

$$\sup\left\{ \left| \int_{t_i}^{t} |\gamma_i(t,\tau)||q_{ik}(\tau)|\, d\tau + \sum_{t_i \leqslant \tau_l < t} |\gamma_i(t,\tau_l)||(1 + g_{ii}(\tau_l))^{-1} u_i(\tau_l)| \right| : t \in \mathbb{R} \right\} < +\infty$$
$$(i \neq k; \ \ i,k = 1, \dots, n) \tag{2.3}$$

and

$$\sup\{|\gamma_i(t,t_i)| : t \in \mathbb{R}\} < +\infty \ \ \textit{for} \ \ i \in N_0(t_1, \dots, t_n), \tag{2.4}$$

where $\gamma_i(t,\tau) \equiv \gamma_{(p_{ii},g_{ii})}(t,\tau)$ $(i = 1, \dots, n)$.

Let, moreover, the matrix $S = (s_{ik})_{i,k=1}^{n}$, where $s_{ii} = 0$ $(i = 1, \dots, n)$, be such that

$$r(S) < 1. \tag{2.5}$$

Then for every $c_i \in \mathbb{R}$ $(i \in N_0(t_1, \dots, t_n))$, system (1.1), (1.2) has at last one bounded on $\mathbb{R}$ solution satisfying the condition

$$x_i(t_i) = c_i \ \ \textit{for} \ \ i \in N_0(t_1, \dots, t_n). \tag{2.6}$$

In the case, if $N_0(t_1, \dots, t_n) = \varnothing$, conditions (2.4) and (2.6) are eliminated and the theorem takes the following form.

Theorem 2.1′. *Let conditions (2.1), (2.2) and (2.3) hold for some $t_i \in \{-\infty, +\infty\}$ $(i = 1, \dots, n)$, where $\gamma_i(t,\tau) \equiv \gamma_{(p_{ii},g_{ii})}(t,\tau)$ $(i = 1, \dots, n)$, and the matrix $S = (s_{ik})_{i,k=1}^{n}$, where $s_{ii} = 0$ $(i = 1, \dots, n)$, satisfy condition (2.5). System (1.1), (1.2) has at last one solution, bounded on $\mathbb{R}$.*

Corollary 2.1. *Let*

$$1 + g_{ii}(\tau_l) > 0 \ (i = 1, \dots, n; \ l = 1, 2, \dots) \tag{2.7}$$

and let there exist $t_i \in \mathbb{R} \cup \{-\infty; +\infty\}$ $(i = 1, \dots, n)$ such that conditions (2.2), (2.3), (2.4) and (2.5) hold, where $S = (s_{ik})_{i,k=1}^{n}$, $s_{ii} = 0$ $(i = 1, \dots, n)$ and $\gamma_i(t,\tau) \equiv \gamma_{(p_{ii},g_{ii})}(t,\tau)$ $(i = 1, \dots, n)$. Let, moreover, the functions

$$\operatorname{sgn}(t - t_i)\Big[\int_{t_i}^{t} p_{ik}(\tau)\, d\tau + \sum_{t_i \leqslant \tau_l < t} (1 + g_{ii}(\tau_l))^{-1} g_{ik}(\tau_l) \Big],$$

$$\operatorname{sgn}(t - t_i)\Big[\int_{t_i}^{t} q_i(\tau)\, d\tau + \sum_{t_i \leqslant \tau_l < t} (1 + g_{ii}(\tau_l))^{-1} u_i(\tau_l) \Big], \ \ (i \neq k; \ i,k = 1, \dots, n) \tag{2.8}$$

be non-decreasing on $\mathbb{R}$.

Then for every $c_i \in \mathbb{R}_+$ $(i \in N_0(t_1, \dots, t_n))$, system (1.1), (1.2) has at last one nonnegative and bounded on $\mathbb{R}$ solution satisfying condition (2.6).

If $N_0(t_1,\dots,t_n)=\emptyset$, then Corollary 2.1 has the following form.

Corollary 2.1′. *Let conditions* (2.7) *and* (2.8) *hold and let there exist* $t_i\in\{-\infty,+\infty\}$ $(i=1,\dots,n)$ *such that condition* (2.2), (2.3) *and* (2.5) *hold, where* $S=(s_{ik})_{i,k=1}^n$, $s_{ii}=0$ $(i=1,\dots,n)$ *and* $\gamma_i(t,\tau)\equiv\gamma_{(p_{ii},g_{ii})}(t,\tau)$ $(i=1,\dots,n)$. *Then system* (1.1), (1.2) *has at last one nonnegative and bounded on* $\mathbb{R}$ *solution.*

Theorem 2.2. *Let* (2.1) *hold and let there exist* $t_i\in\mathbb{R}\cup\{-\infty,+\infty\}$ $(i=1,\dots,n)$ *such that conditions* (2.2), (2.3), (2.4) *and* (2.5) *hold, where* $S=(s_{ik})_{i,k=1}^n$, $s_{ii}=0$ $(i=1,\dots,n)$ *and* $\gamma_i(t,\tau)\equiv\gamma_{(p_{ii},g_{ii})}(t,\tau)$ $(i=1,\dots,n)$. *Let moreover,*

$$\liminf_{t\to t_i}\gamma_i(0,t)=0 \ \text{ for } \ i\in\{1,\dots,n\}\backslash N_0(t_1,\dots,t_n). \tag{2.9}$$

Then for every $c_i\in\mathbb{R}$ $(i\in N_0(t_1,\dots,t_n))$, *system* (1.1), (1.2) *has the unique and bounded on* $\mathbb{R}$ *solution* $(x_i)_{i=1}^n$ *satisfying condition* (2.6) *and*

$$\sum_{i=1}^n |x_i(t)-x_{im}(t)|\leqslant \rho_0\alpha^m \ \text{ for } \ t\in\mathbb{R} \ (m=1,2,\dots), \tag{2.10}$$

where ρ_0 *and* α *are the positive numbers independent of* m, $(x_{im})_{i=1}^n$ $(m=0,1,\dots)$ *is the sequence of the vector-functions of the components which are defined by*

$$x_{i0}(t)\equiv 0;\ x_{im}(t)=y_i(t)$$
$$+\sum_{k=1;k\neq i}^n\Bigg[\int_{t_i}^t \gamma_i(t,\tau)p_{ik}(\tau)x_{km-1}(\tau)\,d\tau+\sum_{t_i\leqslant\tau_l<t}\gamma_i(t,\tau_l)(1+g_{ii}(\tau_l))^{-1}g_{ik}(\tau_l)x_{xm-1}(\tau_l)\Bigg]$$
$$(i=1,\dots,n;\ l,m=1,2,\dots) \tag{2.11}$$

and the functions y_i $(i=1,\dots,n)$ *are defined due to*

$$y_i(t)\equiv c_i\gamma_i(t,t_i)+\int_{t_i}^t\gamma_i(t,\tau)q_i(\tau)\,d\tau$$
$$+\sum_{t_i\leqslant\tau_l<t}\gamma_i(t,\tau_l)(1+g_{ii}(\tau_l))^{-1}u_i(\tau_l) \ \text{ for } \ i\in N_0(t_1,\dots,t_n), \tag{2.12}$$

$$y_i(t)\equiv \int_{t_i}^t\gamma_i(t,\tau)q_i(\tau)\,d\tau$$
$$+\sum_{t_i\leqslant\tau_l<t}\gamma_i(t,\tau_l)(1+g_{ii}(\tau_l))^{-1}u_i(\tau_l) \ \text{ for } \ i\in\{1,\dots,n\}\backslash N_0(t_1,\dots,t_n), \tag{2.13}$$

Corollary 2.2. *Let* (2.7) *hold and let there exist* $t_i\in\mathbb{R}\cup\{-\infty,+\infty\}$ $(i=1,\dots,n)$ *such that the functions* $\operatorname{sgn}(t-t_i)\Big(\int_{t_i}^t p_{ii}(\tau)\,d\tau+\sum_{t_i\leqslant\tau_l<t}g_{ii}(\tau_l)\Big)$ $(i=1,\dots,n)$ *are non-increasing on* $\mathbb{R}$,

$$\liminf_{t\to t_i}\Bigg[\int_0^t p_{ii}(\tau)\,d\tau+\sum_{0\leqslant\tau_l<t}g_{ii}(\tau_l)\Bigg]=+\infty \ \text{for } i\in\{1,\dots,n\}\backslash N_0(t_1,\dots,t_n), \tag{2.14}$$

$$\int_{t_i}^t|p_{ik}(\tau)|\,d\tau+\sum_{t_i\leqslant\tau_l<t}|(1+g_{ii}(\tau))^{-1}g_{ik}(\tau_l)|$$
$$\leqslant h_{ik}\operatorname{sgn}(t-t_i)\Bigg[\int_{t_i}^t p_{ii}(\tau)\,d\tau+\sum_{t_i\leqslant\tau_l<t}(1+g_{ii}(\tau_l))^{-1}g_{ii}(\tau_l)\Bigg] \tag{2.15}$$

for $t \in \mathbb{R}$ $(i \neq k;\ i,k = 1,\dots,n)$ *and*

$$r(H) < 1, \tag{2.16}$$

where h_{ik} $(i,k = 1,\dots,n)$ *are such that* $H = ((1-\delta_{ik})h_{ik})_{i,k=1}^n$.

Let, moreover,

$$\rho_i = \sup\Big\{\Big|\int\limits_t^{v(\zeta_i)(t)} |q_i(\tau)|\,d\tau + \sum_{t\leqslant\tau_l<v(\zeta_i)(t)} |(1+g_{ii}(\tau_l))^{-1}u_i(\tau_l)|\Big| : t\in\mathbb{R}\Big\} < +\infty\ (i=1,\dots,n), \tag{2.17}$$

where $\zeta_i(t) \equiv \xi_{(p_{ii},g_{ii})}(t)\operatorname{sgn}(t-t_i)$ $(i=1,\dots,n)$. *Then the condition of Theorem* 2.2 *is true.*

Corollary 2.3. *Let there exist the points* $t_i \in \mathbb{R}\cup\{-\infty,+\infty\}$ $(i=1,\dots,n)$ *and the functions* $\alpha_i : \mathbb{R}\to\mathbb{R}$, $\beta_i : T\to\mathbb{R}$ $(i=1,\dots,n)$ *such that* $\operatorname{sgn}(t-t_i)\Big(\int\limits_{t_i}^t \alpha_i(\tau)\,d\tau + \sum\limits_{t_i\leqslant\tau_l<t}\beta_i(\tau_l)\Big)$ $(i=1,\dots,n)$ *are non-decreasing on* $\mathbb{R}$ *and the conditions*

$$p_{ii}(t)\operatorname{sgn}(t-t_i) \leqslant \eta_{ii}\alpha_i(t)\ (i=1,\dots,n), \tag{2.18}$$

$$-1 < g_{ii}(\tau_l) \leqslant \eta_{ii}\beta_i(\tau_l)\ (i=1,\dots,n;\ l=1,2,\dots), \tag{2.19}$$

$$|p_{ik}(t)| < \eta_{ik}\alpha_i(t)\ (i\neq k;\ i,k=1,\dots,n) \tag{2.20}$$

and

$$|g_{ik}(\tau_l)| \leqslant \eta_{ik}\beta_i(\tau_l)\ \ (i\neq k;\ i,k=1,\dots,n;\ l=1,2,\dots) \tag{2.21}$$

hold on $\mathbb{R}$, *where* $H = ((1-\delta_{ik})\eta_{ik}|\eta_{ii}|^{-1})_{i,k=1}^n$.

Let, moreover,

$$\rho_i = \sup\Big\{\Big|\int\limits_t^{v(\vartheta_i)(t)} |q_i(\tau)|\,d\tau + \sum_{t\leqslant\tau_l<v(\vartheta_i)(t)} |(1+\eta_i\beta_i(\tau_l))^{-1}u_i(\tau_l)|\Big| : t\in\mathbb{R}\Big\} < +\infty$$
$$(i=1,\dots,n), \tag{2.22}$$

where $\vartheta_i(t) \equiv \xi_{(\eta_i\alpha_i,\beta_i)}(t)\operatorname{sgn}(t-t_i)$ $(i=1,\dots,n)$. *Then Theorem* 2.2 *is true.*

Theorem 2.2′. *Let* (2.1) *hold and let there exist* $t_i \in \{-\infty,+\infty\}$ $(i=1,\dots,n)$ *such that conditions* (2.2), (2.3), (2.4) *and* (2.5) *hold, where* $S = (s_{ik})_{i,k=1}^n$, $s_{ii} = 0$ $(i=1,\dots,n)$ *and* $\gamma_i(t,\tau) \equiv \gamma_{(p_{ii},g_{ii})}(t,\tau)$ $(i=1,\dots,n)$.

Let, moreover,

$$\liminf_{t\to t_i}\gamma_i(0,t) = 0 \ \ \text{for}\ \ i\in\{1,\dots,n\}.$$

Then system (1.1), (1.2) *has the unique and bounded on* $\mathbb{R}$ *solution* $(x_i)_{i=1}^n$, *and*

$$\sum_{i=1}^n |x_i(t) - x_{im}(t)| \leqslant \rho_0\alpha^m \ \ \text{for}\ \ t\in\mathbb{R}\ (m=1,2,\dots),$$

where ρ_0 *and* α *are the positive numbers independent of* m, $(x_{im})_{i=1}^n$ $(m=0,1,\dots)$ *is the sequence of vector-functions the components which are defined by*

$$x_{i0}(t) \equiv 0,$$

$$x_{im}(t) \equiv \int\limits_{t_i}^t \gamma_i(t,\tau)q_i(\tau)\,d\tau + \sum_{t_i\leqslant\tau_l<t}\gamma_i(t,\tau_l)(1+q_{ii}(\tau_l))^{-1}u_i(\tau_l)$$

$$+\sum_{k=1,k\neq i}^n\Big[\int\limits_{t_i}^t \gamma_i(t,\tau)p_{ik}(\tau)x_{km-1}(\tau)\,d\tau + \sum_{t_i\leqslant\tau_l<t}\gamma_i(t,\tau_l)(1+g_{ii}(\tau_l))^{-1}g_{ik}(\tau_l)x_{km-1}(\tau_l)\Big]$$

$$(i=1,\dots,n;\ m=1,2,\dots).$$

Corollary 2.2$'$. *Let* (2.7) *hold and let there exist* $t_i \in \{-\infty, +\infty\}$ $(i = 1, \dots, n)$ *such that the functions* $\operatorname{sgn}(t - t_i)\Big(\int\limits_{t_i}^{t} p_{ii}(\tau)\,d\tau + \sum\limits_{t_i \leqslant \tau_l < t} g_{ii}(\tau_l)\Big)$ $(i = 1, \dots, n)$ *are non-increasing on* $\mathbb{R}$, *conditions* (2.15), (2.16), (2.17) *and*

$$\liminf_{t \to t_i} \Big[\int\limits_0^t p_{ii}(\tau)\,d\tau + \sum_{0 \leqslant \tau_l < t} g_{ii}(\tau_l)\Big] = +\infty \ \text{ for } i \in \{1, \dots, n\} \tag{2.23}$$

hold, where $\zeta_i(t) \equiv \xi_{(p_{ii}, g_{ii})}(t)\operatorname{sgn}(t - t_i)$ $(i = 1, \dots, n)$, *and the numbers* h_{ik} $(i, k = 1, \dots, n)$ *are such that* $H = ((1 - \delta_{ik})h_{ik})_{i,k=1}^n$. *Then conditions of Theorem* 2.2$'$ *are true.*

Corollary 2.3$'$. *Let there exist the point* $t_i \in \{-\infty, +\infty\}$ $(i = 1, \dots, n)$ *and the functions* $\alpha_i : \mathbb{R} \to \mathbb{R}$, $\beta_i : T \to \mathbb{R}$ $(i = 1, \dots, n)$ *such that* $\operatorname{sgn}(t - t_i)\Big(\int\limits_{t_i}^{t} \alpha_i(\tau)\,d\tau + \sum\limits_{0 \leqslant \tau_l < t} \beta_i(\tau)\Big)$ $(i = 1, \dots, n)$ *are non-decreasing on* $\mathbb{R}$ *and condition* (2.16), (2.18)–(2.22) *hold on* $\mathbb{R}$, *where* $\vartheta_i(t) \equiv \xi_{(\eta_i\alpha_i, \beta_i)}(t)\operatorname{sgn}(t - t_i)$ $(i = 1, \dots, n)$, *and the numbers* η_{ik}, $\eta_{ii} < 0$ $(i, k = 1, \dots, n)$ *are such that* $H = ((1 - \delta_{ik})\eta_{ik}|\eta_{ii}|)_{i,k=1}^n$. *Then the condition of Theorem* 2.2$'$ *is true.*

Corollary 2.4. *Let the conditions of Theorem* 2.2, *or Corollary* 2.2, *or Corollary* 2.3 *be fulfilled. Let, in addition, condition* (2.8) *hold. Then for every* $c_i \in \mathbb{R}_+$ $(i \in N_0(t_1, \dots, t_n))$, *system* (1.1), (1.2) *has the unique and bounded on* $\mathbb{R}$ *solution satisfying condition* (2.6) *and it is non-negative.*

Corollary 2.4$'$. *Let the conditions of Theorem* 2.2$'$, *or Corollary* 2.2$'$, *or Corollary* 2.3$'$ *be fulfilled. Let, in addition, condition* (2.8) *hold. Then system* (1.1), (1.2) *has the unique and bounded on* $\mathbb{R}$ *solution and it is non-negative.*

3. Proof of the Results

Proof of Theorem 2.1. Let $c_i \in \mathbb{R}$ $(i \in N_0(t_1, \dots, t_n))$ be an arbitrary fixed numbers. Consider the initial problems

$$\frac{dy}{dt} = p_{ii}(t)y + q_i(t) \ \text{ for } \ t \in \mathbb{R}\backslash T,$$
$$y(\tau_l+) - y(\tau_l-) = g_{ii}(\tau_l)y(\tau_l) + u_i(\tau_l) \quad (i = 1, 2, \dots);$$
$$y(t_i) = c_i,$$

$(i \in N_0(t_1, \dots, t_n))$. By (2.1), the problem has the unique solution $y_i \in ACV(\mathbb{R}, T; \mathbb{R})$ and, according to modified variation of constant formulas (see [4]), it has the form of (2.12).

Consider the system of integral equations

$$x_i(t) = y_i(t)$$
$$+ \sum_{k=1, k\neq i}^{n} \Big[\int\limits_{t_i}^{t} \gamma_i(t, \tau)p_{ik}(\tau)x_k(\tau)\,d\tau + \sum_{t_i \leqslant \tau_l < t} \gamma_i(t, \tau_l)(1 + g_{ii}(\tau_l))^{-1}g_{ik}(\tau_l)x_k(\tau_l)\Big]$$
$$\text{for } t \in \mathbb{R} \ (i = 1, \dots, n). \tag{3.1}$$

Due to the modified variation of constant formulas (see [2]), we conclude that the vector function $(x_i)_{i=1}^n$ is a solution of system one. Moreover, it is evident that the vector-function $(x_i)_{i=1}^n$ satisfies condition (2.6).

The solution of the last integral system will be found in $ACV(\mathbb{R}, T; \mathbb{R}^n)$.

Consider the sequence of vector-functions $(x_{im})_{i=1}^n$ $(m = 0, 1, \dots)$ defined by

$$x_{i0}(t) = 0, \quad x_{im}(t) = y_i(t)$$
$$+ \sum_{k=1, k\neq i}^{n} \Big[\int\limits_{t_i}^{t} \gamma_i(t, \tau)p_{ik}(\tau)x_{km-1}(\tau)\,d\tau + \sum_{t_i \leqslant \tau_l < t} \gamma_i(t, \tau_l)(1 + g_{ik}(\tau_l))^{-1}g_{ik}(\tau_l)x_{km-1}(\tau_l)\Big]$$

$$\text{for } t \in \mathbb{R} \ (i = 1, \dots, n).$$

In view of conditions (2.2) and (2.3), from (2.12) and (2.13), we get

$$(y_i)_{i=1}^n \in ACV(\mathbb{R}, T; \mathbb{R}^n). \tag{3.2}$$

It is clear that $(x_i)_{i=1}^n \in ACV(\mathbb{R}, T; \mathbb{R}^n)$. Now, if we assume that

$$(x_{im-1})_{i=1}^n \in ACV(\mathbb{R}, T; \mathbb{R}^n) \tag{3.3}$$

for some m, then due (2.2) and (3.2), from (3.2), we get $(x_{im})_{i=1}^n \in ACV(\mathbb{R}, T; \mathbb{R}^n)$ and $\|x_{im}\|_\infty \leqslant \|y_i\|_\infty + \sum\limits_{k=1, k\neq i}^n s_{ik}\|x_{km-1}\|_\infty < +\infty$ $(i = 1, \dots, n)$.

Therefore, condition (3.3) holds for every natural m.

Let us show that the sequence $(x_{im})_{i=1}^n$ $(m = 1, 2, \dots)$ converges uniformly on $\mathbb{R}$.

Towards this end, it suffices to show that the functional series

$$\sum_{m=1}^{\infty} |x_{im}(t) - x_{im-1}(t)| \ \ (i = 1, \dots, n) \tag{3.4}$$

converges uniformly on $\mathbb{R}$.

According to (2.2) and (2.3), from (2.11) it follows that

$$(\|x_{im} - x_{im-1}\|_\infty)_{i=1}^n \leqslant S(\|x_{im-1} - x_{im-2}\|_\infty)_{i=1}^n \ \ (m = 2, 3, \dots)$$

and, therefore,

$$(\|x_{im} - x_{im-1}\|_\infty)_{i=1}^n \leqslant S^{m-1}(\|y_i\|_\infty)_{i=1}^n \ \ (m = 1, 2, \dots).$$

Due to (2.5) there exist numbers $\alpha \in]r(S), 1[$ and $\beta > 0$ such that

$$\|S^{m-1}\| \leqslant \beta\alpha^{m-1} \ \ (m = 1, 2 \dots).$$

Hence

$$\|x_{im} - x_{im-1}\|_\infty \leqslant \beta_0\alpha^m \ \ (i = 1, \dots, n; \ m = 1, 2, \dots),$$

where $\beta_0 = \beta\alpha^{-1}\sum\limits_{i=1}^n \|y_i\|_\infty$.

So,

$$\sum_{m=0}^{+\infty} \beta_0\alpha^m$$

is the convergence major numerical series for the functional series (3.4) on $\mathbb{R}$. From this, due to the Weierstrass theorem the sequence $(x_{im})_{i=1}^n$ $(m = 0, 1, \dots)$ converges uniformly on $\mathbb{R}$.

Let

$$\lim_{m\to+\infty} x_{im}(t) = x_i(t) \ \text{ for } \ t \in \mathbb{R} \ \ (i = 1, \dots, n). \tag{3.5}$$

Then $(x_i)_{i=1}^n$ will be a solution of system (3.1). Moreover, it is evident that $\|x_i\|_\infty < +\infty$ $(i = 1, \dots, n)$ and by equality (3.1) and estimates (2.2)–(2.4), we have $(x_i)_{i=1}^n \in ACV(\mathbb{R}, T; \mathbb{R}^n)$. □

Proof of Corollary 2.1. As we have proved above, system (1.1),(1.2) under the conditions of Theorem 2.1, has the bounded solution on $\mathbb{R}$ satisfying condition (2.6) and it is obtained as the uniformly limits $\mathbb{R}$ of the sequence of vector-functions $(x_{im-1})_{i=1}^n \in ACV(\mathbb{R}, T; \mathbb{R}^n)$ $(m = 0, 1, \dots)$ whose components are defined by (2.11), and y_i $(i = 1, \dots, n)$ are defined by (2.12) and (2.13).

In view of (2.8), because $c_i \in \mathbb{R}_+$ $(i \in N_0(t_1, \dots, t_n))$, it follows from (2.11)–(2.13) that $x_{im}(t) \geqslant 0$ and $x_i(t) \geqslant 0$ for $t \in \mathbb{R}$ $(i = 1, \dots, n)$. □

Proof of Theorem 2.2. First, we show that every bounded on $\mathbb{R}$ solution $(x_i)_{i=1}^n$ of system (1.1), (1.2), satisfying condition (2.6) will be the solution of the system of integral equations (3.1).

By (2.9), there exist the sequences t_{im} $(i = 1, \dots, n; \ m = 1, 2, \dots)$ such that

$$\lim_{m\to+\infty} t_{im} = t_i, \quad \lim_{m\to+\infty} \gamma_i(0, t_{im}) = 0 \ \ (i = 1, \dots, n). \tag{3.6}$$

We assume

$$t_{im} = t_i \text{ for } t \in N(t_1, \dots, t_n) \ \ (m = 1, 2, \dots). \tag{3.7}$$

By the modified variation of constant formula and equations (2.12), (3.7), we have

$$x_i(t) = y_{im}(t) + \sum_{k=1,k\neq i}^{n} \Big[\int_{t_{im}}^{t} \gamma_i(t,\tau)p_{ik}(\tau)x_k(\tau)\,d\tau + \sum_{t_{im}\leqslant\tau_l<t} \gamma_i(t,\tau_l)(1+g_{ii}(\tau_l))^{-1}g_{ik}(\tau_l)x_k(\tau_l)\Big]$$
$$(i=1,\dots,n;\ m=1,2,\dots) \tag{3.8}$$

on $\mathbb{R}$, where

$$y_{im}(t) \equiv y_i(t) \text{ for } i \in N_0(t_1,\dots,t_n)\ (m=1,2,\dots) \tag{3.9}$$

$$y_{im}(t) \equiv x_{im}(t_{im})\gamma_i(t,t_{im}) + \int_{t_{im}}^{t} \gamma_i(t,\tau)q_i(\tau)\,d\tau$$
$$+ \sum_{t_{im}\leqslant\tau_l<t} \gamma_i(t,\tau_l)(1+g_{ii}(\tau_l))^{-1}u_i(\tau_l) \text{ for } i \in \{1,\dots,n\}\backslash N_0(t_1,\dots,t_n)\ (m=1,2,\dots). \tag{3.10}$$

Let $i \in \{1,\dots,n\}\backslash N_0(t_1,\dots,t_n)$. Then because x_i is bounded, by (2.3) and (3.6), from (2.13) and (3.10), we find

$$\lim_{m\to+\infty} y_{im}(t) = y_i(t) \text{ for } t \in \mathbb{R}.$$

On the other hand, due to (2.2), we get

$$\lim_{m\to+\infty} \Big[\int_{t_{im}}^{t} \gamma_i(t,\tau)p_{ik}(\tau)x_k(\tau)\,d\tau + \sum_{t_{im}\leqslant\tau_l<t} \gamma_i(t,\tau_l)(1+g_{ii}(\tau_l))^{-1}g_{ik}(\tau_l)x_k(\tau_l)\Big]$$
$$= \int_{t_i}^{t} \gamma_i(t,\tau)p_{ik}(\tau)x_k(\tau)\,d\tau + \sum_{t_i\leqslant\tau_l<t} \gamma_i(t,\tau_l)(1+g_{ii}(\tau_l))^{-1}g_{ik}(\tau_l)x_k(\tau_l) \text{ for } t \in \mathbb{R}.$$

Therefore, from (3.8), we have

$$x_i(t) = y_i(t) + \sum_{k=1,k\neq i}^{n} \Big[\int_{t_i}^{t} \gamma_i(t,\tau)p_{ik}(\tau)x_k(\tau)\,d\tau + \sum_{t_i\leqslant\tau_l<t} \gamma_i(t,\tau_l)(1+g_{ii}(\tau_l))^{-1}g_{ik}(\tau_l)x_k(\tau_l)\Big]$$
$$\text{for } t \in \mathbb{R}.$$

Due to (3.7)–(3.9), the last equality is true for the case where $i \in N_0(t_1,\dots,t_n)$, as well. So, it is proved that the vector-function $(x_i)_{i=1}^n$ is the solution of system (3.1).

In the proof of Theorem 2.1, we have shown that system (3.1) has the solution $(x_i)_{i=1}^n$ and

$$\lim_{m\to+\infty} \|x_i - x_{im}\|_\infty = 0\ \ (i=1,\dots,n).$$

In addition,

$$\|x_{im} - x_{im-1}\|_\infty \leqslant \beta_0\alpha^m\ \ (i=1,\dots,n;\ m=1,2,\dots),$$

where β_0 and $\alpha \in]0,1[$ are the numbers independent of m, whence we get

$$\|x_{im+j} - x_{im}\|_\infty \leqslant \sum_{k=m+1}^{m+j} \|x_{ik} - x_{ik-1}\|_\infty \leqslant \beta_0 \sum_{k=m+1}^{m+j} \alpha^k < \beta_0 \frac{\alpha}{1-\alpha}\alpha^m$$

and

$$\|x_i - x_{im}\|_\infty \leqslant \beta_0 \frac{\alpha}{1-\alpha}\alpha^m\ \ (j=1,2;\ i=1,\dots,n;\ m=1,2,\dots).$$

So, estimate (2.10) holds for $\rho_0 = n\beta_0\alpha(1-\alpha)^{-1}$.

Finally, we have to verify that system (3.1) has the unique solution $(x_i)_{i=1}^n$. Let $(\overline{x}_i)_{i=1}^n \in ACV(\mathbb{R},T;\mathbb{R}^n)$ be an arbitrary solution of the system and let $z_i(t) \equiv \overline{x}_i(t) - x_i(t)$ $(i = 1,\dots,n)$. Then

$$z_i(t) = \sum_{k=1,k\neq i}^{n} \Big[\int\limits_{t_i}^{t} \gamma_i(t,\tau)p_{ik}(\tau)z_k(\tau)\,d\tau + \sum_{t_i\leqslant\tau_l<t} \gamma_i(t,\tau_l)(1+g_{ii}(\tau_l))^{-1}g_{ik}(\tau_l)z_k(\tau_l)\Big] \text{ for } t\in\mathbb{R}.$$

Owing to this fact, in view of (2.2), $(\|z_i\|_\infty)_{i=1}^n \leqslant S(\|z_i\|_\infty)_{i=1}^n$, i.e., $(I_n - S)(\|z_i\|_\infty)_{i=1}^n \leqslant 0_n$. So, because S is non-negative, by condition (2.5), we have $(\|z_i\|_\infty)_{i=1}^n \leqslant 0_n$ and $\|z_i\|_\infty = 0$ $(i = 1,\dots,n)$. Consequently, $\overline{x}_i(t) \equiv x_i(t)$ $(i = 1,\dots,n)$. □

Proof of Corollary 2.2. Let $\xi_i(t) \equiv \xi_{(p_{ii},g_{ii})}(t)$, $\gamma_i(t,\tau) \equiv \xi_i(t)\xi_i^{-1}(\tau)$ and $v_i(t) \equiv v(\zeta_i)(t)$ $(i = 1,\dots,n)$. Due to condition (2.7), we have $\gamma_i(t,t_i) > 0$ $(i = 1,\dots,n)$.

Let $i \in \{1,\dots,n\}$ be fixed. First, consider the case $t \geqslant t_i$. Then by (2.15) and equalities (1.5), (1.6), we find that

$$\frac{d\xi_i^{-1}(t)}{dt} = -\xi_i^{-1}(t)p_{ii}(t) \text{ for } t\in\mathbb{R}\backslash T,$$
$$\xi_i^{-1}(\tau_l+) - \xi_i^{-1}(\tau_l-) = -\xi_i^{-1}(\tau_l)(1+g_{ii}(\tau_l))^{-1}g_{ii}(\tau_l) \ \ (l = 1,2,\dots)$$

and

$$\Big| \int\limits_{t_i}^{t} |\gamma_i(t,\tau)||p_{ik}(\tau)|\,d\tau + \sum_{t_i\leqslant t_l<t} |\gamma_i(t,\tau_l)||(1+g_{ii}(\tau_l))^{-1}g_{ik}(\tau_l)|\Big|$$
$$\leqslant h_{ik}\xi_i(t)\Big[\int\limits_{t_i}^{t} \xi_i^{-1}(\tau)p_{ii}(\tau)\,d\tau + \sum_{t_i\leqslant t_l<t} \xi_i^{-1}(\tau_l)(1+g_{ii}(\tau_l))^{-1}g_{ii}(\tau_l)\Big]$$
$$= h_{ik}\xi_i(t)\int\limits_{t_i}^{t} d\xi_i^{-1}(\tau) \leqslant h_{ik} \ (i\neq k;\ i,k = 1,\dots,n).$$

So, condition (2.2) holds. Beside, we have $s_{ik} \leqslant h_{ik}$ $(i,k = 1,\dots,n)$ and, therefore, by (2.16), condition (2.5) holds.

If $i \in N_0(t_1,\dots,t_n)$, then due to Remark 1.2 the functions ξ_i $(i = 1,\dots,n)$ are non-increasing. Hence the functions $\gamma_i(t,t_i) \equiv \gamma_{(p_{ii},g_{ii})}(t,t_i)$ $(i = 1,\dots,n)$ are non-increasing and so, estimate (2.4) holds for $t \geqslant t_i$.

Let now $i \in \{1,\dots,n\}\backslash N_0(t_1,\dots,t_n)$ be such that $t_i = -\infty$. Due to (2.7), we find that $\gamma_i(0,t) = \exp(1-\xi_i(t))$ for $t < 0$. Therefore, owing to (2.23), we can conclude that

$$\liminf_{t\to t_i} \gamma_i(0,t) = 0.$$

Consequently, condition (2.9) holds for the case.

Let us verify (2.3). Let $i \in \{1,\dots,n\}$, $t_i \in \mathbb{R}\cup\{-\infty,+\infty\}$ and $t > t_i$ be fixed. Then $\zeta_i(\tau) = -\xi_i(\tau)$ for $\tau \geqslant t_i, \zeta_i(t_i) = 0$. Then due to the conditions of the theorem, the function ζ_i is non-decreasing on the interval $[t_i,+\infty[$.

$$T_{ij} = \{\tau\in[t_i,t] \ :\ j\leqslant\zeta_i(\tau)<(j+1)\} \ \ (j = 0,\dots,k_i(t)+1),$$

where $k_i(t) \equiv [\zeta_i(t)]$ (the integer part) and let

$$\tau_{i0} = t_i;\ \tau_{ij} = \begin{cases} \tau_{ij-1} & \text{if } \ T_{ij-1} = \emptyset, \\ \sup T_{ij-1} & \text{if } \ T_{ij-1}\neq\emptyset \ (j = 1,\dots,k_i(t)+1). \end{cases}$$

Let us show that

$$T_{ij+1} \leqslant v_i(\tau_{ij}) \ \ (j = 0,\dots,k_i(t)). \tag{3.11}$$

Let $j \in \{0, \dots, k_i(t)\}$ be fixed. If $T_{ij} = \emptyset$, then (3.11) is evident. Let now $T_{ij} \neq \emptyset$. It suffices to show that

$$T_{ij} \subset Q_{ij},$$

where $Q_{ij} = \{\tau \in [t_i, t] \ : \ \zeta_i(\tau) < \zeta_i(\tau_{ij-1}+) + 1\}$. It is easy to verify that

$$\zeta_i(\tau_{ij-1}+) \geqslant j. \tag{3.12}$$

Indeed, otherwise there exists $\delta > 0$ such that $\zeta_i(\tau_{ij-1}+s) < \zeta_i(\tau_{i0})+j$ for $0 \leqslant s \leqslant \delta$. Next, by the definition of τ_{ij-1}, we have $\zeta_i(\tau_{i0}) + (j-1) \leqslant \zeta_i(\tau_{ij-1}-)$ and, therefore, $(j-1) \leqslant \zeta_i(\tau_{ij-1}+s) < j$ for $0 \leqslant s \leqslant \delta$. But this contradicts the definition of τ_{ij-1}. So, if $\tau \in T_{ij}$, then from (3.12) and the inequality $\zeta_i(\tau) < (j+1)$, we get $\zeta_i(\tau) < \zeta_i(\tau_{ij-1}+)+1$ and hence $\tau \in Q_{ij}$. Therefore (3.11) is proved.

Due to (3.11), we find

$$\begin{aligned} v_i(t) &\leqslant \int_{t_i}^{t} \exp(\xi_i(t) - \xi_i(\tau)) q_i(\tau)\, d\tau \\ &\quad + \sum_{t_i \leqslant \tau_l < t} \exp(\xi_i(t) - \xi_i(\tau_l))(1 + g_{ii}(\tau_l))^{-1} u_i(\tau_l) \\ &\leqslant \exp(\xi_i(t)) \sum_{j=1}^{k_i(t)+1} \Bigg[\int_{\tau_{ij-1}}^{\tau_{ij}} \exp(\zeta_i(\tau)) q_i(\tau)\, d\tau + \sum_{\tau_{ij-1} \leqslant \tau_l < \tau_{ij}} \exp(\zeta_i(\tau_l))(1 + g_{ii}(\tau_l))^{-1} u_i(\tau_l) \Bigg], \end{aligned} \tag{3.13}$$

where

$$v_i(t) \equiv \Bigg| \int_{t_i}^{t} |\gamma_i(t,\tau)||q_i(\tau)|\, d\tau + \sum_{t_i \leqslant \tau_l < t} |\gamma_i(t,\tau_l)||(1 + g_{ii}(\tau_l))^{-1} u_i(\tau_l)| \Bigg|.$$

On the other hand,

$$\begin{aligned} &\int_{\tau_{ij-1}}^{t} \exp(\zeta_i(\tau)) q_i(\tau)\, d\tau + \sum_{\tau_{ij-1} \leqslant \tau_l < \tau_{ij}} \exp(\zeta_i(\tau_l))(1 + g_{ii}(\tau_l))^{-1} u_i(\tau_l) \\ &= \lim_{\epsilon \to 0+} \Bigg[\int_{\tau_{ij-1}}^{\tau_{ij}-\epsilon} \exp(\zeta_i(\tau)) q_i(\tau)\, d\tau + \sum_{\tau_{ij-1} \leqslant \tau_l < \tau_{ij}-\epsilon} \exp(\zeta_i(\tau_l))(1 + g_{ii}(\tau_l))^{-1} u_i(\tau_l) \Bigg] \\ &\leqslant \exp(\zeta_i(\tau_{ij})) \Bigg[\int_{\tau_{ij-1}}^{\tau_{ij}} q_i(\tau)\, d\tau + \sum_{\tau_{ij-1} \leqslant \tau_l < \tau_{ij}} (1 + g_{ii}(\tau_l))^{-1} u_i(\tau_l) \Bigg] \\ &\leqslant \exp(j) \Bigg[\int_{\tau_{ij-1}}^{\tau_{ij}} q_i(\tau)\, d\tau + \sum_{\tau_{ij-1} \leqslant \tau_l < \tau_{ij}} (1 + g_{ii}(\tau_l))^{-1} u_i(\tau_l) \Bigg]. \end{aligned}$$

Similarly, we verify that

$$\begin{aligned} &\int_{\tau_{ik_i(t)}}^{t} \exp(\zeta_i(\tau)) q_i(\tau)\, d\tau + \sum_{\tau_{ik_i(t)} \leqslant \tau_l < t} \exp(\zeta_i(\tau_l))(1 + g_{ii}(\tau_l))^{-1} u_i(\tau_l) \\ &\leqslant \exp(\zeta_i(t)) \Bigg[\int_{\tau_{ik_i(t)}}^{t} q_i(\tau)\, d\tau + \sum_{\tau_{ik_i(t)} \leqslant \tau_l < t} (1 + g_{ii}(\tau_l))^{-1} u_i(\tau_l) \Bigg] \\ &\leqslant \exp(k_i(t) + 1) \Bigg[\int_{\tau_{ik_i(t)}}^{t} q_i(\tau)\, d\tau + \sum_{\tau_{ik_i(t)} \leqslant \tau_l < t} (1 + g_{ii}(\tau_l))^{-1} u_i(\tau_l) \Bigg]. \end{aligned}$$

Taking into account the last two estimate, by (3.13), we find

$$v_i(t) \leqslant \exp(-k_i(t)) \sum_{j=1}^{k_i(t)+1} \exp(j)\Bigg[\int\limits_{\tau_{ij-1}}^{\tau_{ij}} q_i(\tau)\,d\tau + \sum_{\tau_{ij-1}\leqslant\tau_l<\tau_{ij}} (1+g_{ii}(\tau_l))^{-1}u_i(\tau_l)\Bigg]$$

$$\leqslant \exp(-k_i(t))\Bigg[\exp(1)\bigg(\int\limits_{\tau_{i0}}^{\tau_{i1}} q_i(\tau)\,d\tau + \sum_{\tau_{i0}\leqslant\tau_l<\tau_{i1}} (1+g_{ii}(\tau_l))^{-1}u_i(\tau_l)\bigg)$$

$$+\sum_{j=2}^{k_i(t)+1} \exp(j)\bigg(\int\limits_{\tau_{ij-1}}^{\tau_{ij}} q_i(\tau)\,d\tau + \sum_{\tau_{ij-1}\leqslant\tau_l<\tau_{ij}} (1+g_{ii}(\tau_l))^{-1}u_i(\tau_l)\bigg)\Bigg].$$

Whence, due to (2.17) and (3.11), we have

$$v_i(t) \leqslant 2\rho_i \exp(-k_i(t)) \sum_{j=1}^{k_i(t)+2} \exp(j)$$

$$= 2\rho_i \exp(-k_i(t))\exp(k_i(t)+2)\exp(1)(\exp(1)-1)^{-1}.$$

Consequently,

$$v_i(t) \leqslant \eta\rho \ \text{ for } \ t \geqslant t_i, \tag{3.14}$$

where $\eta = 2\exp(3)(\exp(1)-1)^{-1}$ and $\rho = \sum\limits_{i=1}^{n} \rho_i$. So, estimate (2.3) holds on the set $(-\infty, t]$. Similarly, we show estimate (2.3) on the set $[t, +\infty)$. In this case we have $\zeta_i(\tau) = -\xi_i(\tau)$ for $\tau < t_i$, $\zeta_i(t_i) = 0$ and

$$\overline{T}_{ij} = \{\tau \in [t, t_i] \ : \ j \leqslant \zeta_i(\tau) < (j+1)\} \ \ (j = 0, \dots, k_i(t)+1).$$

The function ζ_i is non-increasing on $(-\infty, t]$.

Similarly, as above, we conclude that the estimate

$$\overline{\tau}_{ij+1} \geqslant v_i(\overline{\tau}_{ij}) \ \text{ for } \ (j = 0, \dots, k_i(t))$$

holds, where

$$\overline{\tau}_{i0} = t_i, \ \tau_{ij} = \begin{cases} \overline{\tau}_{ij-1} & \text{for } \ \overline{T}_{ij-1} = \varnothing, \\ \\ \inf \overline{T}_{ij-1} & \text{if } \ \overline{T}_{ij-1} \neq \varnothing \ (j = 1, \dots, k_i(t)+1). \end{cases}$$

Similarly, we verify that (3.14) holds. So, the corollary follows from Theorem 2.2. □

Proof of Corollary 2.3. First, as in the proof of Corollary 2.2, due to condition (2.17), we show that the estimates

$$\sup\Bigg\{\bigg|\int\limits_{t_i}^{t} |\gamma_{(\eta_i\alpha_i,\beta_i)}(t,\tau)||q_i(\tau)|\,d\tau + \sum_{t_i\leqslant\tau_l<t} |\gamma_{(\eta_i\alpha_i,\beta_i)}(t,\tau_l)||(1+\eta_i\beta_i(\tau_l))^{-1}u_i(\tau_l)|\bigg| \ : t \in \mathbb{R}\Bigg\} < +\infty$$

$$(i = 1, \dots, n). \tag{3.15}$$

Let $\xi_i(t) \equiv \xi_{(p_{ii},g_{ii})}(t)$ and $\gamma_i(t,\tau) = \xi_i(t)\xi_i^{-1}(\tau)$ $(i = 1, \dots, n)$.

Due condition (2.19), we have $\gamma_i(t, t_i) > 0$ $(i = 1, \dots, n)$.

In view of (2.18) and (2.19), it is not difficult to verify that

$$\int\limits_{s}^{t} p_{ii}(\tau)\,d\tau + \sum_{s\leqslant\tau_l<t} \ln|1+g_{ii}(\tau_l)|$$

$$\leqslant \operatorname{sgn}(t-s)\Bigg[\int\limits_{s}^{t} \eta_{ii}\alpha_i(\tau)\,d\tau + \sum_{s\leqslant\tau_l<t} \ln|1+\eta_{ii}\beta_i(\tau_l)|\Bigg] \ \ (i = 1, \dots, n). \tag{3.16}$$

Let us show that the estimates

$$\Big| \int_s^t |p_{ik}(\tau)|\,d\tau + \sum_{s\leqslant\tau_l<t} |(1+g_{ii}(\tau_l))^{-1} g_{ik}(\tau_l)|\Big|$$

$$\leqslant \frac{\eta_{ik}}{\eta_{ii}}\operatorname{sgn}(t-t_i)\Big[\int_s^t \eta_{ii}\alpha_i(\tau)\,d\tau + \sum_{s\leqslant\tau_l\leqslant t}\eta_{ii}(1+\eta_{ii}\beta_i(\tau_l))^{-1}\beta_i(\tau_l)\Big]\ (i\neq k;\ i,k=1,\dots,n) \quad (3.17)$$

hold on $\mathbb{R}$.

From (2.19), (2.20), (2.21), we find

$$\Big| \int_s^t p_{ik}(\tau)\,d\tau + \sum_{s\leqslant\tau_l<t} (1+g_{ii}(\tau))^{-1} g_{ik}(\tau_l)\Big|$$

$$\leqslant \eta_{ik}\int_s^t \alpha_i(\tau)\,d\tau + \sum_{s\leqslant\tau_l<t}(1+\eta_{ii}\beta_i(\tau_l))^{-1}\eta_{ik}\beta_i(\tau_l)$$

$$= \frac{\eta_{ik}}{\eta_{ii}}\Big[\int_s^t \eta_{ii}\alpha_i(\tau)\,d\tau + \sum_{s\leqslant\tau_l<t}(1+\eta_{ii}\beta_i(\tau_l))^{-1}\eta_{ii}\beta_i(\tau_l)\Big] \text{ for } s<t\ (i\neq k;\ i,k=1,\dots,n).$$

So, estimate (3.17) holds for $t_i<t$. Similarly, we show (3.17) for $t\leqslant t_i$, as well.

Moreover, using this way, we conclude that for $t\in\mathbb{R}$, we have

$$\Big| \int_s^t |q_i(\tau)|\,d\tau + \sum_{s\leqslant\tau_l<t} |(1+g_{ii}(\tau_l))^{-1} u_i(\tau_l)|\Big|$$

$$\leqslant \operatorname{sgn}(t-t_i)\Big[\int_s^t q_i(\tau)\,d\tau + \sum_{s\leqslant\tau_l<t}(1+\eta_{ii}\beta_i(\tau_l))^{-1}u_i(\tau_l)\Big]$$

$$(i\neq k;\ i,k=1,\dots,n). \quad (3.18)$$

Hence, due to (3.15), (3.16) and (3.18), we can conclude that conditions (2.3) and (2.4) hold.

On the other hand, by (1.5), (1.6), (3.16) and (3.17), we get

$$\Big| \int_{t_i}^t |\gamma_i(t,\tau)||p_{ik}(\tau)|\,d\tau + \sum_{t_i\leqslant\tau_l<t} |\gamma_i(t,\tau_l)||(1+g_{ii}(\tau_l))^{-1} g_{ik}(\tau_l)|\Big|$$

$$\leqslant \frac{\eta_{ik}}{\eta_{ii}}\exp\Big(\int_0^t \eta_{ii}\alpha_i(\tau)\,d\tau\Big)\prod_{0\leqslant\tau_l<t}|1+\eta_{ii}\beta_i(\tau_l)|$$

$$\times\Big[\int_{t_i}^t \exp\Big(-\int_0^\tau \eta_{ii}\alpha_i(s)\,ds\Big)\prod_{0\leqslant\tau_l<\tau}|1+\eta_{ii}\beta_i(\tau_l)|^{-1}\eta_{ii}\alpha_{ii}(\tau)\,d\tau$$

$$+\sum_{t_i\leqslant\tau_l<t}\exp\Big(-\int_0^{\tau_l}\eta_{ii}\alpha_i(s)\,ds\Big)\prod_{0\leqslant\tau_j<\tau_l}|1+\eta_{ii}\beta_i(\tau_j)|^{-1}(1+\eta_{ii}\beta_i(\tau_j))\eta_{ii}\beta_i(\tau_j)\Big]$$

$$=\frac{\eta_{ik}}{|\eta_{ii}|}\exp\Big(\int_0^t \eta_{ii}\alpha_i(\tau)\,d\tau\Big)\prod_{0\leqslant\tau_l<t}|1+\eta_{ii}\beta_i(\tau_l)|\Big[\exp\Big(-\int_0^t\eta_{ii}\alpha_i(\tau)\,d\tau\Big)\prod_{0\leqslant\tau_l<t}|1+\eta_{ii}\beta_i(\tau_l)|^{-1}$$

$$-\exp\Big(-\int_0^{t_i}\eta_{ii}\alpha_i(\tau)\,d\tau\Big)\prod_{0\leqslant\tau_l<t_i}|1+\eta_{ii}\beta_i(\tau_l)|^{-1}\Big]\leqslant\frac{\eta_{ik}}{\eta_{ii}} \text{ for } t>t_i\ (i\neq k;\ i,k=1,\dots,n).$$

Similarly, we show the estimate for $t \leqslant t_i$. So, we have $s_{ik} \leqslant \eta_{ik}|\eta_{ii}|^{-1}$ $(i,k=1,\dots,n)$, where s_{ik} is the left-hand side of estimate (2.2).

Hence, inequality (2.5) follow from (2.16). By (2.21) condition (2.1) holds. So conditions of Theorem 2.1 hold. The corollary follow from the theorem. □

Theorem 2.2′, Corollaries 2.2′ and 2.3′ are, respectively, the particular cases of Theorem 2.2, Corollaries 2.2 and 2.3 if we assume $N_0(t_1,\dots,t_n)=\varnothing$ therein.

Corollaries 2.4 and 2.4′ follow immediately from Theorems 2.2 and 2.2′ and Corollaries 2.2, 2.3, and 2.2′, 2.3′.

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