Transactions of A. Razmadze
Mathematical Institute
Vol. **179** (2025), issue 1, 121–132

A REPRESENTATION THEOREM OF A RIEMANN–LEBESGUE INTEGRABLE FUNCTION ASSOCIATED WITH A VECTOR MEASURE

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Abstract. We study several properties of the Banach lattice $\mathscr{RL}^1(m, \mathscr{X})$ of Riemann–Lebesgue integrable function space associated with a vector measure m. We also introduce weakly $\mathscr{RL}$-integrable function spaces endowed with a vector measure. A representation of the weakly Riemann–Lebesgue integral in terms of unconditionally convergent series is given. Finally, we discuss a weakly Riemann–Lebesgue integral that must coincide with the Bochner integral only if the series is absolutely convergent. In application, the conditional expectation of a weakly $\mathscr{RL}$-integrable function is shown.

1. Introduction

Riemann and Lebesgue integrals admit different extensions, some of them to the case of finitely additive, non-additive, or multi-valued measures (see [2]). The definition of one of these extensions by G. Birkhoff [5] makes use of countable sums for vector functions with respect to complete finite measures. This integral has thereafter undergone considerable study and been extended in [3, 6, 7, 9, 16, 23]. An important resource for the study of vector-valued functions in the integration theory are the vector measures (see [15]). D. R. Lewis [20] developed a measure-based theory of integration into a locally convex Hausdorff linear topological space by using a linear functional method. O. Delgado [14] developed this theory and examined subtle distinctions between vector measures defined on a σ-ring and σ-algebras in their L^1-spaces. She even demonstrated how the space $L^1(m)$ of a vector measure defined on a σ-ring is an order continuous Banach lattice that might not have a weak order unit. She considered the impact of strong additivity on $L^1(m)$, since a countably additive vector measure defined on a σ-ring may not be strongly additive. See [4, 11, 17, 24] for other related literature on the vector measure. V. M. Kadets et al. [19] introduced two integrals, called absolute Riemann–Lebesgue ($|\mathscr{RL}|$) and unconditional Riemann–Lebesgue ($\mathscr{RL}$), for functions with values in a Banach space, relative to a countably additive measure. The space of a Bochner-integrable $\mathscr{X}$-valued function is a closed subspace of $\mathscr{RL}^1(\Omega, \mathscr{X})$ (see [18]). In this article, V. M. Kadets et al. discussed various conditions on $\mathscr{X}$, the space of $\mathscr{X}$-valued Bochner functions is complemented in $\mathscr{RL}^1(\Omega, \mathscr{X})$. A. Croitoru et al. [12] offered a few limit theorems for collections of Riemann–Lebesgue integrable functions. The Fatou theorem and Lebesgue-type convergence were proved in more detail. The applications of these findings are then applied to the situation of Riemann–Lebesgue integrable interval-valued multifunctions. Although (countable) additivity is one of the most important notions in measure theory, it may be useless in many problems, for example, modeling different real aspects in data mining, computer science, economy, psychology, game theory, fuzzy logic, decision making, subjective evaluation. An important area of study is multifunction theory. Interval-valued multi-functions have also been applied to several novel signal and image processing approaches in a few recent works. In the true sense, digital images are the product of discretization of reality, or a sampling of a continuous stream (see [8, 10]). Some inequalities, such as the reverse Minkowski inequality and the reverse Hölder inequality, have been reproved for Riemann–Lebsegue integrable functions, where the integration under consideration is obtained by using a non-additive measure by Croitoru et al. in [13]. They extend these inequalities to the context of a multivalued case, specifically for Riemann–Lebesgue integrable interval-valued

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2020 *Mathematics Subject Classification.* 26A39, 46B20, 46B25.
Key words and phrases. Vector measure; Riemann–Lebesgue integrals; Weakly Riemann–Lebesgue integrals; Dunford–Pettis integrals.

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multifunctions. This leads to the emergence of several inequalities, including those of the Minkowski and Beckenbach types as well as several extensions of the Hölder inequalities.

The work of A. Fernarndez et al. [17] motivates us to develop weakly $\mathscr{RL}$-integrable functions with vector measure. The technique of W.J. Ricker [24] helps us to develop our setting in a broad sense.

The article is divided into four section. In Section 2, we recall several definitions and results that we use in our main section. In Section 3, we give several properties of $\mathscr{RL}$-integrable spaces. In the same section, we introduce weakly $\mathscr{RL}$-integrable functions associate with a vector measure. Section 4 discusses how to represent the weakly Riemann–Lebesgue integral using unconditionally convergent series.

2. Preliminaries

Throughout the article, $\mathscr{X}$ is a real Banach spaces with topological duals $\mathscr{X}^*$. $\mathscr{B}_{\mathscr{X}} and\ \mathscr{B}_{\mathscr{X}^*}$ denote the closed unit ball of $\mathscr{X}\, and\ \mathscr{X}^*$, respectively. We assume Σ is ring of subsets of a non-empty set Ω and $C(\Sigma)$ is the σ-algebra of sets locally in Σ. That is, $A \in C(\Sigma)$ if and only if $A \cap B \in \Sigma\ \forall\ B \in \Sigma$ (see [11]). Let (Ω, Σ, μ) be a measure space with finite measure μ. In our article, μ stands for the Lebesgue measure. We consider $m : \Sigma \to \mathscr{X}$ is a countably additive vector measure or simply a vector measure. Let $\mathscr{M}(\mathscr{X})$ denote the space of all $\mathscr{X}$-valued measurable functions on $(\Omega, C(\Sigma))$. If $\mathscr{X} = \mathbb{R}$, we simply denote $\mathscr{M}(\mathbb{R})$ by $\mathscr{M}_{\mathbb{R}}$. We denote the set of all extended real-valued measurable functions on $C(\Omega)$ by $\mathscr{M}^*_{\mathbb{R}}$. The space of all $\mathscr{X}$-valued Σ-simple functions is denoted by $\mathscr{S}(\Sigma, \mathscr{X})$. If $\mathscr{X} = \mathbb{R}$, we denote $\mathscr{S}(\Sigma, \mathbb{R})$ by $\mathscr{S}(\Sigma)$. Recall that a pseudometric space is a generalization of a metric space in which the distance between two distinct points can be zero. Pseudometric spaces were introduced by Duro Kurepa.

Definition 2.1 ([1])**.** Let H be a non-empty set. A pseudometric space is a pair (H, d), where $d : H \times H \to [0, \infty)$, that satisfies the following properties for all x, y, z in H :

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $\exists\ \alpha \geq 1$ such that $d(x, y) \leq \alpha d(x, z) + d(z, y)$.

It is easy to see metric space is pseudometric space but the following counter example shows that pseudometric space is not necessarily a metric space.

Example. Let $H = \{1, 2, 3\}$. Let us define $d(1, 1) = d(2, 2) = d(3, 3) = 0;\ d(1, 2) = d(2, 1) = d(2, 3) = d(3, 2) = 1;\ d(1, 3) = d(3, 1) = 4$. Clearly, d satisfies reflexivity and symmetry. Let $\alpha = 3$, then we can find for arbitrary $x, y, z \in H,\ d(x, y) \leq \alpha d(x, z) + d(z, x)$. So, (H, d) is a pseudometric space, but $d(1, 3) = 4 > d(1, 2) + d(2, 3) = 2$ shows that (H, d) is not a metric space.

Recalling a given measurable space (Ω, Σ) and a measure

$$m : \Sigma \to \mathscr{X},$$

we have the following associated pseudometric space (Σ, d_m), where for $A, B \in \Sigma$. We define $d_m(A, B) = ||m||(A \Delta B)$. Let λ be a control measure for m. The space (Σ, d_m) is homeomorphic to the pseudometric space (Σ, d_λ), associated to the measure space $(\Omega, \Sigma, \lambda)$. A vector measure m defined over Σ is said to be separable if so is the space (Σ, d_m) (see [24]).

Recalling Riemann sum as follows: For a given function $f : \Omega \to \mathscr{X}$, let $\mathscr{P} = \{E_i\}_{i=1}^{\infty}$ be a partition of Ω into a countable number of measurable subsets with $\bigcup_{i=1}^{\infty} E_i = E,\ E_i \cap E_j = \emptyset$ for $i \neq j$. The partition $\mathscr{P}$ follows partition $\mathscr{Q}$ or $\mathscr{P}$ is inscribed into $\mathscr{Q}$ denoted by $\mathscr{P} \succ \mathscr{Q}$ if $\mathscr{P}$ is a finer partition. Let $\Delta_i = \{t_i\}_{i=1}^{\infty}$ be the set of sampling points of $\mathscr{P}$, that is $t_i \in E_i$, we can define the formal series $\mathfrak{S}(f, \mathscr{P}) = \sum_{i=1}^{\infty} f(t_i)\mu(E_i)$. This series is called the absolute $\mathscr{RL}$ integral sum of f with respect to $\mathscr{P}$ and Δ_i, provided it is absolutely convergent. Recalling the upper Lebesgue integral of a real valued function $f : \Omega \to \mathbb{R}^+$ as follows:

Definition 2.2 ([18, Definition 4])**.** Let $f : \Omega \to \mathbb{R}^+$ be a function. The upper Lebesgue integral of f, $\overline{\int_\Omega} f d\mu$ is defined as

$$\overline{\int_\Omega} f d\mu = \inf \left\{ (L) \int_\Omega g d\mu : \ g(t) \geq \ f(t) \ \forall t, \ g \ \text{ is Lebesgue integrable} \right\}.$$

Also, $(L) \int_\Omega g d\mu$ is the Lebesgue integral of g. It is also known that $\overline{\int_\Omega} f d\mu = +\infty$, if f has no Lebesgue-integrable majorant.

Now, consider the definition of the Riemann–Lebesgue integral as follows:

Definition 2.3 ([19, 23])**.** A function $f : \Omega \to \mathscr{X}$ is called absolutely (unconditional, respectively) Riemann–Lebesgue $|\mathscr{RL}|$ ($\mathscr{RL}$, respectively) integrable over a measurable set $E \subset [a, b]$ if there exists a point $x \in \mathscr{X}$ such that for any $\epsilon > 0$, there exists a partition $\mathscr{P}$ of E and for any finer partition $\mathscr{Q} \succ \mathscr{P}$ and any set of sampling points Δ_i,

$$\left\| \mathfrak{S}(f, \mathscr{Q}) - x \right\|_{\mathscr{X}} < \epsilon$$

and the sum $\mathfrak{S}(f, \mathscr{Q})$ converges absolutely (unconditionally) over E. The vector x (necessarily unique) is called the Riemann–Lebesgue μ-integral of f on Ω and it is denoted by $|\mathscr{RL}| \int_\Omega f d\mu$ ($(\mathscr{RL}) \int_\Omega f d\mu$, respectively).

Remark 2.1. (i) If f is $|\mathscr{RL}|$-integrals, then it is also $\mathscr{RL}$-integrals.
(ii) If $\mathscr{X}$ is finite-dimensional, then $|\mathscr{RL}|$-integrability coincides with $\mathscr{RL}$-integrability.

Suppose that $\overline{\mathscr{L}}_1(\Omega, \Sigma, \mu, \mathscr{X})$ or $\overline{\mathscr{L}}_1(\Omega, \mathscr{X})$ is the space of all functions $f : \Omega \to \mathscr{X}$ such that $\overline{\int} ||f|| d\mu < +\infty$. It is well known that if $f : \Omega \to \mathscr{X}$ is an $\mathscr{RL}$-integrable function, then $f \in \overline{\mathscr{L}}_1(\Omega, \mathscr{X})$ (see [18, 23]). The norm of the space $\overline{\mathscr{L}}_1(\Omega, \Sigma, \mu)$ is known as $\mathscr{RL}$-norm, defined as follows:

$$||f||_{\overline{\mathscr{L}}_1} = \overline{\int} ||f||_{\mathscr{X}} d\mu.$$

We denote $\mathscr{RL}^1(\Omega, \mathscr{X})$, a space consisting of all those functions that are $\mathscr{RL}$-integrable. $\mathscr{RL}^1(\Omega, \mathscr{X})$ is a closed subspace of $\overline{\mathscr{L}}_1(\Omega, \mathscr{X})$ and also a Banach space (see [23]). In addition, a Bochner-integrable $\mathscr{X}$-valued function space is a closed subspace of $\mathscr{RL}^1(\Omega, \mathscr{X})$. The following theorem is known to us:

Theorem 2.1 ([2, Lemma 2])**.** *Let $f : [a, b] \to \mathscr{X}$ be Riemann–Lebesgue integrable, then f is:*

(1) *Birkhoff–integrable.*
(2) *Strongly–Pettis integrable.*
(3) *Pettis-integrable.*

Theorem 2.2 ([19, Theorem 1.9])**.** *Let $\mathscr{X}$ be a separable Banach space. Then for $\mathscr{X}$-valued functions, an unconditional $\mathscr{RL}$-integrability coincides with the Pettis integrability.*

We recall some properties of Banach spaces (see also [21, 22]).

A partially ordered Banach space $\mathscr{X}$, which is also a vector lattice, is a Banach lattice if $\|x\| \leq \|y\|$ for every x, $y \in \mathscr{X}$ whenever $|x| \leq |y|$.

A weak order unit of $\mathscr{X}$ is a positive element $e \in \mathscr{X}$ such that if $x \in \mathscr{X}$ and $x \wedge e = 0$, then $x = 0$.

Let $\mathscr{X}$ be a Banach lattice and $\emptyset \neq A \subset B \subset \mathscr{X}$. We say that A is solid in B if for each x, y with $x \in B$, $y \in A$ and $|x| \leq |y|$, it is $x \in A$.

Let μ be an extended real-valued measure on Σ. A Banach space $\mathscr{X}$ consisting of (classes of equivalence of) μ-measurable functions is called a Köthe function space with respect to μ if, for every $g \in \mathscr{X}$ and for each measurable function f with $|f| \leq |g|$ μ-almost everywhere, it is $f \in \mathscr{X}$ and $\|f\| \leq \|g\|$, and $\chi_A \in \mathscr{X}$ for every $A \in \Sigma$ with $\mu(A) < +\infty$.

3. $\mathscr{RL}^1(m,\mathscr{X})$: m is Countable Additive Measure

In this section, we discuss about $\mathscr{RL}^1(m,\mathscr{X})$; m is a countable additive measure. We assume $m:\Sigma\to\mathscr{X}$ a countable additive vector measure. The semi-variation of m is the set function

$$||m||(A)=\sup\{|x^*m|(A):\ x^*\in B_{\mathscr{X}^*}\}$$

where $|x^*m|$ is the variation of the scalar measure $x^*m,\ A\in\Sigma$. A Rybakov control measure for m is $\lambda=|x^*m|$ such that $\lambda(A)=0$ if and only if $||m||(A)=0$.

Definition 3.1 ([20, Definition 2.1])**.** A measurable function $f:\Omega\to\mathbb{R}$ is integrable with respect to m if

(1) f is x^*m integrable for every $x^*\in\mathscr{X}^*$ and
(2) for each $A\in\Sigma$, there exists an element of $\mathscr{X}$, denoted by $\int_A fdm$, such that $x^*\int_A fdm=\int_A fdx^*m$ for every $x^*\in\mathscr{X}^*$.

In terms of a vector measure m, we define the $\mathscr{RL}$-integrable functions space as follows.

Definition 3.2. Let $\mathscr{RL}^1(\Omega,\Sigma,m,\mathscr{X})$ or $\mathscr{RL}^1(\Omega,\mathscr{X})$ be the space of all $\mathscr{RL}$-integrable functions $f:\Omega\to\mathscr{X}$ such that $(\mathscr{RL})\overline{\int_\Omega}||f||_{\mathscr{X}}dm<+\infty$.

Identifying two functions. If the set, where they differ,, has null semi-variation, we obtain a linear space of classes of functions which, when endowed with the

$$||f||_m=\sup\left\{(\mathscr{RL})\int\limits_\Omega ||f||_{\mathscr{X}}d|x^*m|:\ x^*\in B_{\mathscr{X}^*}\right\},$$

becomes a Banach space. We denote it by $\mathscr{RL}^1(m,\mathscr{X})$.

Theorem 3.1. *The Σ-valued simple functions $\mathscr{S}(\Sigma,\mathscr{X})$ are dense in $\mathscr{RL}^1(m,\mathscr{X})$.*

Proof. Let $A\in\Sigma$ with $\sup(\mathscr{RL})\int_A ||g||_{\mathscr{X}}d(|x^*m|)<\frac{\epsilon}{2}$. Since $g\chi_A$ is Σ-measurable and concentrated on an element of Σ, there is a Σ-simple function vanishing off A and satisfying

$$\sup_{x\in A}||f-g||_{\mathscr{X}}<\frac{\epsilon}{2}\Big[||m||(A)+1\Big]^{-1}.$$

Hence the proof is complete. □

Since simple functions are dense in $\mathscr{RL}^1(m,\mathscr{X})$ and the identity is a one-to-one continuous mapping of the space of $||m||$-essentially bounded functions onto $\mathscr{RL}^1(m,\mathscr{X})$, we can define an equivalent norm on $\mathscr{RL}^1(m,\mathscr{X})$ as follows:

$$\big|||f|||_m=\sup\left\{\left|\left|(\mathscr{RL})\int\limits_A fdm\right|\right|_{\mathscr{X}}:\ A\in\Sigma\right\}$$

for which we have

$$\big|||f|||_m\le||f||_m\le 2\big|||f|||_m.$$

Next, we find an important convergent result as follows.

Lemma 3.1. *Let $\{f_n\}$ be a sequence in $\mathscr{RL}^1(m,\mathscr{X})$ that converges almost everywhere with respect to m to a function f and let $g\in\mathscr{RL}^1(m,\mathscr{X})$ such that $||f_n||\le g$ for every n. Then $f\in\mathscr{RL}^1(m,\mathscr{X})$ and $\{f_n\}$ converges to f in $\mathscr{RL}^1(m,\mathscr{X})$.*

Proof. The proof is similar to [4, Theorem 2.8]. □

We define the integral operator $m:\mathscr{RL}^1(m,\mathscr{X})\to\mathscr{X}$ as

$$m(f)=\int\limits_\Omega fdm, \tag{3.1}$$

for $f \in \mathscr{RL}^1(m, \mathscr{X})$. It is a continuous linear operator with the norm less than or equal to one, which is absolutely continuous with respect to $||m||$ or any control measure λ. If m is a vector measure and $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ is a Σ-simple function, where $\{A_i\} \subset \Sigma$ are pairwise disjoint sets, we can define another integral operator by

$$(\mathscr{RL}) \int_{\Omega} f dm = \sum_{i=1}^{n} \alpha_i m(A_i).$$

It is an easy exercise to see that this definition can be expanded to all elements of the space $\mathscr{RL}^1(m, \mathscr{X})$. Next, we prove that $\mathscr{RL}^1(m, \mathscr{X})$ is a continuous Banach lattice.

Theorem 3.2. *The Riemann–Lebesgue integrable function space $\mathscr{RL}^1(m, \mathscr{X})$ is an order continuous Banach lattice.*

Proof. By the Rybakov theorem (see also [15, Theorem IX.2.2]), there is $x^* \in B_{\mathscr{X}^*}$ with $||x^*|| \leq 1$ such that $\lambda = x^*m$ is a control measure of m. If $f,\ g \in \mathscr{RL}^1(m, \mathscr{X}),$ such that $|f| \leq |g|$ λ-a.e., and $x^* \in B_{\mathscr{X}^*}$ with $||x^*|| \leq 1$, then

$$(\mathscr{RL}) \int_{\Omega} ||f||_{\mathscr{X}} d|x^*m| \leq (\mathscr{RL}) \int_{\Omega} ||g||_{\mathscr{X}} d|x^*m|.$$

Hence

$$\sup \left\{ (\mathscr{RL}) \int_{\Omega} ||f||_{\mathscr{X}} d|x^*m| :\ x^* \in B_{\mathscr{X}^*} \right\} \leq \sup \left\{ (\mathscr{RL}) \int_{\Omega} ||g||_{\mathscr{X}} d|x^*m| :\ x^* \in B_{\mathscr{X}^*} \right\}.$$

Consequently, $||f||_m \leq ||g||_m$.
For order continuous: Let $\{f_n\}$ and $\{g_n\}$ are sequences in $\mathscr{RL}^1(m, \mathscr{X})$ and $f_n \to f;\ g_n \to g$ as $n \to \infty$. Then

$$\begin{aligned} \left\|f_n \wedge g_n - f \wedge g\right\|_m &\leq \left\|f_n \wedge g_n - f_n \wedge g\right\|_m + \left\|f_n \wedge g - f \wedge g\right\|_m \\ &\leq \left\|g_n - g\right\|_m + \left\|f_n - f\right\|_m. \end{aligned}$$

Since $||.||_m$ is a lattice norm, the conclusion follows immediately. (See also [22, Proposition 1.2.3 i]). □

Using (3.1), we will prove $\mathscr{RL}^1(m)$ is an order continuous Banach function space with weak unit over $(\Omega, \Sigma, \lambda)$ as follows.

Theorem 3.3. *$\mathscr{RL}^1(m)$ is an order continuous Banach function space with weak unit over $(\Omega, \Sigma, \lambda)$, where λ is a control measure for m.*

Proof. $\mathscr{RL}^1(m, \mathscr{X})$ is a Banach function space over the measure space $(\Omega, \Sigma, \lambda)$. Let $\{f_n\}$ be an order bounded increasing sequence in $\mathscr{RL}^1(m, \mathscr{X})$. Let $0 \leq f_n \leq f_{n+1} \leq g$, where $g \in \mathscr{RL}^1(m, \mathscr{X})$. If $f = \sup_n f_n$, then for every $x^* \in \mathscr{X}^*,\ \{f_n\}$ is order bounded and increasing in $\mathscr{RL}^1(|x^*m|, \mathscr{X})$ with $\|f_n\| \leq g$. Since $g \in \mathscr{RL}^1(m, \mathscr{X})$, so,

$$\begin{aligned} \left\| (\mathscr{RL}) \int_A f dm \right\|_m &= \sup \left\{ \left| (\mathscr{RL}) \int_A ||f_n||_{\mathscr{X}} d|x^*m| \right| :\ x^* \in B_{\mathscr{X}^*} \right\} \\ &\leq \sup \left\{ (\mathscr{RL}) \int_A ||f_n||_{\mathscr{X}} d|x^*m| :\ x^* \in B_{\mathscr{X}^*} \right\} \\ &\leq \sup \left\{ (\mathscr{RL}) \int_A ||g||_{\mathscr{X}} d|x^*m| :\ x^* \in B_{\mathscr{X}^*} \right\} \text{ (using Lemma 3.1).} \end{aligned}$$

Therefore, $m(f) \leq m(g)$.

Again, let $\epsilon > 0$. As $m(f-f_1)$ is absolutely continuous with respect to λ, there is a $\delta > 0$ such that $\lambda(A) < \delta, \implies ||m||(A) < \epsilon$. Since the sequence $\{f_n\}$ is increasing, we have

$$\begin{aligned}
||f_n - f||_m &= \sup\left\{(\mathscr{RL})\int\limits_{\Omega} ||f_n - f||_{\mathscr{X}} d|x^*m| : \; x^* \in B_{\mathscr{X}^*}\right\} \\
&\leq \sup\left\{(\mathscr{RL})\int\limits_{\Omega\setminus A} ||f_n - f||_{\mathscr{X}} d|x^*m| : \; x^* \in B_{\mathscr{X}^*}\right\} \\
&\qquad + \sup\left\{(\mathscr{RL})\int\limits_{A} ||f_1 - f||_{\mathscr{X}} d|x^*m| : \; x^* \in B_{\mathscr{X}^*}\right\} \\
&\leq \epsilon||m||(\Omega \setminus A) + ||m||(A) \\
&\leq \Big(\epsilon(1 + ||m||(\Omega)\Big) \text{ for } n \leq n_0(\epsilon).
\end{aligned}$$

Hence $f_n \to f$ in $\mathscr{RL}^1(m, \mathscr{X})$.

Again, $\inf\big\{f, \chi_\Omega\big\} = 0$. Implying $f = 0$ a.e., so, $\mathscr{RL}^1(m, \mathscr{X})$ is weak unit of the Banach lattice. □

Thus $\mathscr{RL}^1(\lambda, \mathscr{X})$, and $\mathscr{RL}^1(m, \mathscr{X})^*$ can be identified with the space of function g in $\mathscr{RL}^1(\lambda, \mathscr{X})$ such that $fg \in \mathscr{RL}^1(\lambda, \mathscr{X})$, for all f in $\mathscr{RL}^1(m, \mathscr{X})$, where the action of g over $\mathscr{RL}^1(m, \mathscr{X})$ is given by the integration with respect to λ.

We will now look into the spaces that can be produced as the $\mathscr{RL}^1$ of a vector measure in the space $\mathscr{RL}^1(m, \mathscr{X})$. In a natural way, there also arises the question whether the space $\mathscr{RL}^1(m, \mathscr{X})$ may be a Hilbert space or a reflexive space. The following theorem gives a complete answer to these problems, showing that the class of spaces obtained as $\mathscr{RL}^1$ of a vector measure coincides with the class of order continuous Banach lattices with weak unit.

Theorem 3.4. *Let $\mathscr{X}$ be an order continuous Banach lattice with weak unit. There exists a vector measure m, with values in $\mathscr{X}$, such that the space $\mathscr{RL}^1(m, \mathscr{X})$ is order isomorphic and isometric to $\mathscr{X}$.*

Proof. Consider $\mathscr{X}$ is an order continuous Banach lattice with weak unit. Then $\mathscr{X}$ is an order isomorphic and isometric to a Banach function space with respect to a probability space $(\Omega, \Sigma, \lambda)$. Let us consider

$$\Sigma \ni E \longmapsto m(E) = \chi_E \in \mathscr{X}. \tag{3.2}$$

Since $\mathscr{X}$ is a Banach function space, so expression (3.2) is well defined. Clearly, this is finitely additive. Let $\{E_n\}$ be a sequence of disjoint measurable sets. Assume $F_n = \bigcup_{i=1}^{n} E_i$ for every n and $F = \bigcup_{i=1}^{\infty} E_i$ are measurable. Since the sequence of sets $\{F_n\}$ are increasing, therefore $m(F_n) = \chi_{F_n}$ is increasing in $\mathscr{X}$. Also, for every n, $F_n \subset F$, and $m(F) < \infty$ in $\mathscr{X}$ for $m(F_n)$. Again, $\Big(m(F_n)\Big)$ is convergent in $\mathscr{X}$ to its supremum $m(F)$. Thus m is countable additive. Since $\mathscr{X}$ is order continuous, $\mathscr{X}^*$ coincides with Köthe dual (see [21, Theorem II.b.14]).

Consider $\mathscr{X}^* = \Big\{g : \mathscr{S}(\Sigma, \mathscr{X}) \to \mathbb{R} : \; g \text{ is measurable and } gf \in \mathscr{RL}^1(\lambda, \mathscr{X}) \text{ for every } f \in \mathscr{X}\Big\}$, where the action of these elements is given through integration with λ.

When $g \in \mathscr{X}^*$, $\Sigma \ni E \longmapsto gm(E) = (\mathscr{RL})\int_E g d\lambda \in \mathbb{R}$ is scalar integrable with respect to m. It is easy to see that $f \in \mathscr{RL}^1(m, \mathscr{X})$ and $(\mathscr{RL})\int_E f dm = f.\chi_E$ for every $E \in \Sigma$. So, $f \in \mathscr{X}$. Now,

$$\begin{aligned}
||f||_m &= \sup\left\{(\mathscr{RL})\int\limits_{E} ||f||_{\mathscr{X}} d|gm| : \; g \in B_{\mathscr{X}^*}\right\} \\
&= \sup\left\{(\mathscr{RL})\int\limits_{E} ||f||_{\mathscr{X}} ||g||_{\mathscr{X}} d\lambda : \; g \in B_{\mathscr{X}^*}\right\}
\end{aligned}$$

$$= \sup\left\{(\mathscr{RL})\int_{\Omega} ||fg||_{\mathscr{X}}\, d\lambda :\ g \in B_{\mathscr{X}^*}\right\}$$
$$= ||f||_{\mathscr{X}}. \qquad \square$$

So, there is an isometry between $\mathscr{X}$ and $\mathscr{RL}^1(m, \mathscr{X})$.

Corollary. *Let $\mathscr{X}$ be an order continuous Banach lattice. There exists a countably additive vector measure m defined over a δ-ring and with values in $\mathscr{X}$, such that the space $\mathscr{RL}^1(m, \mathscr{X})$ is order isomorphic and isometric to $\mathscr{X}$.*

We now study the possibility of obtaining spaces $\mathscr{RL}^1(m, \mathscr{X})$ from measures taking the values in a certain fixed Banach space. First we will study atoms and separability in $\mathscr{RL}^1(m, \mathscr{X})$.

Proposition 3.1. *Let $f \in \mathscr{RL}^1(m, \mathscr{X})$. Then f is an atom in $\mathscr{RL}^1(m, \mathscr{X})$ if and only if f is a multiple of χ_A, where $A \in \Sigma$ is an atom of m.*

Proof. Let f be an atom in $\mathscr{RL}^1(m, \mathscr{X})$. Assume f is non-null, non-constant, then there exists $a > 0$ such that the sets

$$E = \left\{x : f(x) \geq a\right\} \text{ and } F = \left\{x :\ 0 < f(x) < a\right\}$$

have non-empty semi-variation. Clearly, $0 < f.\chi_E < f$. Since $f.\chi_E$ is not a multiple of f, this gives a contradiction of the atom f in $\mathscr{RL}^1(m, \mathscr{X})$. Hence f is a multiple of χ_E, where $E = \left\{x :\ f(x) > 0\right\}$. Suppose E is not an atom of m, then there exists $F \subset E$ such that F and $E \setminus F$ have non-null semi-variation. This is also a contradiction by $f.\chi_F,\ f$ being an atom of $\mathscr{RL}^1(m, \mathscr{X})$. Hence, if E is an atom of m, then χ_E is an atom in $\mathscr{RL}^1(m, \mathscr{X})$. $\square$

Corollary. *$\mathscr{RL}^1(m, \mathscr{X})$ is atomic if and only if m is purely atomic.*

Lemma 3.2. *The injection $\mathscr{RL}^1(m, \mathscr{X}) \to \mathscr{RL}^1(\lambda, \mathscr{X})$ is continuous and has dense image.*

Proof. Let $f \in \mathscr{RL}^1(m, \mathscr{X})$. Then

$$\begin{aligned} ||f||_{\lambda} &= \sup\left\{(\mathscr{RL})\int_{\Omega} ||f||_{\mathscr{X}}\, d\lambda\right\} \\ &\leq \sup\left\{(\mathscr{RL})\int_{\Omega} ||f||_{\mathscr{X}}\, d|x^*m| :\ x^* \in B_{\mathscr{X}^*}\right\} \\ &\leq ||f||_m. \end{aligned}$$

Hence the proof is complete. $\square$

Theorem 3.5. *$\mathscr{RL}^1(m, \mathscr{X})$ is separable if and only if the pseudometric space (Σ, d_m) is separable.*

Proof. The separability of (Σ, d_m) is equivalent to that of the space (Σ, d_λ), where λ is a Rybakov control measure for m. Hence the separability of (Σ, d_λ) is equivalent to the separability of the space $\mathscr{RL}^1(\lambda, \mathscr{X})$. Let us assume $\mathscr{RL}^1(m, \mathscr{X})$ is separable. By Lemma 3.2, the injection $\mathscr{RL}^1(m, \mathscr{X}) \to \mathscr{RL}^1(\lambda, \mathscr{X})$ is continuous and has dense image. It follows that $\mathscr{RL}^1(\lambda, \mathscr{X})$ is separable. So, (Σ, d_m) is separable.

Conversely, since Σ-valued simple functions with rational coefficients over sets of sequence $\{E_n\}$ are dense in $\mathscr{RL}^1(m, \mathscr{X})$, so the result follows. $\square$

3.1. **Weakly $\mathscr{RL}$-integrable function spaces.** In this section, we discuss few important behaviours of weakly Riemann–Lebesgue integrable functions with respect to a vector measure m.

Definition 3.3. A measurable function $f : \Omega \to \mathbb{R}$ is called weakly $\mathscr{RL}$-integrable with respect to m if f is $\mathscr{RL}$-integrable with respect to $\left|\left\langle m, x^*\right\rangle\right|$ $\forall\ x^* \in \mathscr{X}^*$.

A weakly $\mathscr{RL}$ integrable function f is said to be $\mathscr{RL}$-integrable with respect to m if for each $E \in \Sigma$, there exists an element (necessarily unique) $(\mathscr{RL}) \int_E f dm \in \mathscr{X}$, satisfying

$$\Big\langle (\mathscr{RL}) \int\limits_E f dm, x^* \Big\rangle = (\mathscr{RL}) \int\limits_E f d\langle m, x^* \rangle, \quad x^* \in \mathscr{X}^*.$$

We denote a weakly $\mathscr{RL}$-integrable function space by $\mathscr{RL}^1_w(m, \mathscr{X})$. The space $\mathscr{RL}^1_w(m, \mathscr{X})$ of all (equivalence classes of) weakly $\mathscr{RL}$-integrable functions becomes a Banach space with respect to any Rybakov control measure for m, with the Fatou property associated with m-a.e., and the norm

$$||f||_{w,m} = \sup \Big\{ (\mathscr{RL}) \int\limits_\Omega |f| d|\langle m, x^* \rangle| : \ x^* \in B_{\mathscr{X}^*} \Big\}.$$

For simplicity, now and onwards we will denote $|\langle m, x^* \rangle| = |x^* m|$.

Theorem 3.6. $\big(\mathscr{RL}^1_w(m, \mathscr{X}), ||.||_{w,m}\big)$ *is a Banach space containing* $\mathscr{RL}^1(m, \mathscr{X})$ *as a closed linear subspace.*

Proof. Let $\{f_n\}$ be a $||.||_{w,m}$-Cauchy sequence in $\mathscr{RL}^1_w(m, \mathscr{X})$. Then $\{f_n\}$ is Cauchy sequence of the space $RL^1\big(|x^*m|, X\big)$, $x^* \in X^*$. Let $\lambda = |x_0^* m|$ be a control measure for m. Assume $f_0 = \lim_n f_n$ in $\mathscr{RL}^1(\lambda, \mathscr{X})$. Then for a subsequence $\{f_{n_j}\}$ in E_1 with $\lambda(E_1) = 0$ such that $f_{n_j}(w) \to f(w) \ \forall \ w \notin E_1$, where $E_1 \in \Sigma$. Let $x^* \in B_{\mathscr{X}^*}$. If $f_{x^*} = \lim_n f_n$ in $\mathscr{RL}^1\big(|x^*m|, \mathscr{X}\big)$, then $f_{x^*} = \lim_i f_{n_i}$. We can now have a subsequence $\{f_{n_j}\}$ of $\{f_{n_i}\}$ and a set E_{x^*} with $|x^*m|(E_{x^*}) = 0$ such that $f_{n_{ij}}(w) \to f_{x^*}(w) \ \forall \ w \notin E_{x^*}$. Since $|x^*m|(E_1 \cup E_{x^*}) = 0$, so, $f_{n_{ij}}(w) \to f_{x^*}(w)$ and $f_{n_{ij}}(w) \to f_0(w)$. Thus $\lim_n f_n = f_{x^*} = f_0$ in $\mathscr{RL}^1\big(|x^*m|, \mathscr{X}\big)$. So, $f_0 \in \mathscr{RL}^1_w(|x^*m|, \mathscr{X}) \ \forall \ x^* \in B_{\mathscr{X}^*}$. Hence $f_0 \in \mathscr{RL}^1_w(m, \mathscr{X})$ and $\lim_n ||f_0 - f_n||_{\mathscr{X}} = 0$.

Next, to show that $\mathscr{RL}^1(m, \mathscr{X})$ is a closed subspace of $\mathscr{RL}^1_w(m, \mathscr{X})$, let $f_n \in \mathscr{RL}^1(m, \mathscr{X})$. If U_n is the indefinite integral of f_n and U_0 is the indefinite integral of f_0, then $\{U_n\}$ is a sequence of $\mathscr{X}$-valued measures. Also, since

$$||U_n(E) - U_0(E)||_{\mathscr{X}} \leq ||f_n - f_0||_{\mathscr{X}} \to 0$$

holds for all $E \in \Sigma$, consequently, U_0 is $\mathscr{X}$-valued. Hence f_0 is m-valued. □

Theorem 3.7. *The following claims are equivalent:*

a. $\mathscr{RL}^1_w(m, \mathscr{X})$ *is order continuous.*
b. $\mathscr{RL}^1_w(m, \mathscr{X}) = \mathscr{RL}^1(m, \mathscr{X})$.
c. $\mathscr{RL}^1(m, \mathscr{X})$ *is weakly sequentially complete.*
d. $\mathscr{RL}^1_w(m, \mathscr{X})$ *is weakly sequentially complete.*

Proof. For $a. \implies b.$ Let $\mathscr{RL}^1_w(m, \mathscr{X})$ be order continuous and let $0 \leq f \in \mathscr{RL}^1_w(m, \mathscr{X})$. Let us assume an increasing sequence $\{f_n\}$ of simple functions such that $0 \leq f_n \leq f_{n+1} \leq \cdots \leq f$ and $f_n \to f$ a.e., since f_n is order bounded. The continuity gives $f_n \to f$ converges in norm. Again, $f_n \in \mathscr{RL}^1(m, \mathscr{X})$, and the closeness of $\mathscr{RL}^1(m, \mathscr{X})$ gives $f_n \to f \in \mathscr{RL}^1(m, \mathscr{X})$. So, $\mathscr{RL}^1_w(m, \mathscr{X}) \subseteq \mathscr{RL}^1(m, \mathscr{X})$. Consequently, $\mathscr{RL}^1_w(m, \mathscr{X}) = \mathscr{RL}^1(m, \mathscr{X})$.

For $b. \implies c.$ Let us assume f_n are all non-negative. For any $x^* \in B_{\mathscr{X}^*}$, f_n is norm bounded, non-negative and increasing sequence in $\mathscr{RL}^1\big(|x^*m|, \mathscr{X}\big)$. So, $f_n \to f$ in $\mathscr{RL}^1\big(|x^*m|, \mathscr{X}\big)$ for some $f_{x^*} \in \mathscr{RL}^1\big(|x^*m|, \mathscr{X}\big)$. Let $\lambda = |x_0^* m|$ be control measure for m, and $f_n \to f_0$ in $\mathscr{RL}^1(m, \mathscr{X})$. By Theorem 3.6, $f_0 \in \mathscr{RL}^1(|x^*m|, \mathscr{X}) \ \forall \ x^* \in B_{\mathscr{X}^*}$ and $f_0 = f_{x^*} \ |x^*m|$ a.e.. Hence $f_n \to f$ in each of the space $\mathscr{RL}^1(|x^*m|, \mathscr{X})$. So, $f_0 \in \mathscr{RL}^1_w(m, \mathscr{X})$. Also, we have $\mathscr{RL}^1_w(m, \mathscr{X}) = \mathscr{RL}^1(m, \mathscr{X})$ so, $f_0 \in \mathscr{RL}^1(m, \mathscr{X})$ which is order continuous. Hence $f_n \to f_0$ as a norm convergent.

For $c. \implies b.$ Let $\mathscr{RL}^1(m, \mathscr{X})$ be weakly sequentially complete. Assume $f \in \mathscr{RL}^1_w(m, \mathscr{X})$ and $f \geq 0$. Also, assume $0 \leq f_n \leq f_{n+1} \leq \cdots \leq f_n$ and $f_n \to f$ a.e.. Again, $\{f_n\}$ is norm bounded ($||f_n||_{\mathscr{X}} \leq ||f||_{\mathscr{X}}$ for all n) and increasing in $\mathscr{RL}^1(m, \mathscr{X})$. Now, by our assumption of the weak sequentially completeness of $\mathscr{RL}^1(m,\mathscr{X})$, $f_n \to f$ in norm. Hence f is $\mathscr{RL}$-integrable. Since f is an arbitrary element of $\mathscr{RL}^1(m,\mathscr{X})$, so, $\mathscr{RL}^1_w(m,\mathscr{X}) = \mathscr{RL}^1(m,\mathscr{X})$.

b. $\Longrightarrow$ *a.*; *a.* $\Longrightarrow$ *d.* are very obvious.

Finally, *d.* $\Longrightarrow$ *c.* follows due to the completeness of $\mathscr{RL}^1(m,\mathscr{X})$. □

4. Representation Theory of the Weakly Riemann–Lebesgue Integral

We will now discuss about a representation of the weakly Riemann–Lebesgue integral using unconditionally convergent series. We start with the following

Theorem 4.1. *Let $f:\Omega\to\mathscr{X}$ be a measurable weakly $\mathscr{RL}$-integrable function. Then $f=\phi+\psi$ a.e., where ϕ is a bounded Bochner integrable function and ψ is considered to be at most countably multi-valued in $\mathscr{X}$. In particular, if $\psi=\sum\limits_{i=1}^{\infty} x_i.\chi_{E_i}$ then*

$$(\mathscr{RL})\int\limits_E f dm = (B)\int\limits_E \phi dm + \sum_{i=1}^{\infty} x_i m(E_i\cap E),$$

provided

$$\sum_{i=1}^{\infty} x_i m(E_i\cap E) \tag{4.1}$$

converges unconditionally for each $E\in\Sigma$. The series (4.1) *converges absolutely if and only if f is Bochner integrable.*

Proof. Assume that $\{a_n\}$ is a decreasing sequence of positive numbers such that $\sum\limits_{i=1}^{\infty} a_i<\infty$. We construct an open ball $B(n,f(x))$ with radius a_n and centre at $f(x)$. Then $f(x)\subset \bigcup\limits_{x\in\Omega} B(n,f(x))$. By the Lindelöf theorem, $\{x_i^n\}_{i=1}^{\infty}\subset\Omega$ such that $f(\Omega)\subseteq \bigcup\limits_{i=1}^{\infty} B(n,f(x_i^n))$. As $||f-f(x_i^n)||_m$ is a real-valued function, so, $||f-f(x_i^n)||_m^{-1}[0,a_n)=A_i^n\in\Sigma$. If $E_i^n=A_i^n\setminus\big(\bigcup\limits_{j=1}^{i-1}A_j^n\big)$, $f_n=\sum\limits_{i=1}^{\infty} f(x_i^n)\chi_{E_i}$. Then $||f(x)-f_n(x)||_m<a_n$ on Ω. Hence $f_n\to f$ uniformly on Ω. Consider $\phi(x)=\sum\limits_{n=2}^{\infty}\big(f_n(x)-f_{n-1}(x)\big)$. Then

$$\begin{aligned}||\phi(x)||_m &\le \sum_{n=2}^{\infty}||f_n(x)-f_{n-1}(x)||_m\\ &\le 2\sum_{n=1}^{\infty} a_n \text{ on } \Omega.\end{aligned}$$

Also, $(B)\int_\Omega ||\phi)||_m dm \le (2\sum\limits_{n=1}^{\infty} a_n)m(\Omega)$. So, ϕ is a bounded Bochner integrable.

If $x_i=f(x_i^l)$, $l\in\mathbb{N}$ and $E_i=E_i^l$. Consider $\psi=\sum\limits_{i=1}^{\infty} x_i.\chi_{E_i}$. Since $f=\phi+\psi$, f, ϕ are weakly $\mathscr{RL}$-integrable, so, ψ is also weakly $\mathscr{RL}$-integrable. Indeed, for every $E\in\Sigma$, there is an element $x_E\in\mathscr{X}$ such that $x^*(x_E)=(B)\int_E x^*\psi dm$, $x^*\in B_{\mathscr{X}^*}$. Thus

$$\sum_{i=1}^{\infty}|x^*(x_i)||m|(E_i\cap E)=(B)\int\limits_E |x^*\psi|d|m|<\infty. \tag{4.2}$$

Next, to prove that

$$\sum_{i=1}^{\infty} x_i m(E_i\cap E) \tag{4.3}$$

converges unconditionally in $\mathscr{X}$ for every E (by [15, Corollary 4] we have to show that (4.3) converges to an element in $\mathscr{X}$. If $A = \bigcup_{i\in\mathbb{N}} E_i$, then

$$\begin{aligned} x^*(x_{E\cap A}) &= (B)\int\limits_{E\cap A} x^*\psi dm \\ &= \sum_{i=1}^{\infty} x^*(x_i)m(E\cap A\cap E_i) \\ &= \sum_{i\in\mathbb{N}} x^*(x_i)m(E\cap E_i) < \infty \text{ by (4.2).} \end{aligned}$$

So, (4.3) is unconditionally convergent. Hence (4.1) is established.

Further, if f is Bochner integrable, then

$$\begin{aligned} \sum_i ||x_i||_m|m(E\cap E_i)| &\le \sum_i ||x_i||_m|m|(E_i) \\ &= (B)\int\limits_{\Omega} ||f-\phi||_m d|m| < \infty. \end{aligned}$$

Let $F_1 > 0$ and $F_2 < 0$ be two measurable sets such that $F_1,\ F_2$ form Hahn decomposition of Ω. Then

$$\int\limits_{\Omega} ||\psi||d|m| = \sum_i ||x_i||m(E_i\cap F_1) - \sum_i ||x_i||m(E_i\cap F_2) < \infty.$$

So, ψ is Bochner integrable. □

Corollary. *Assume $f:\Omega\to\mathscr{X}$ is measurable. Then the following hold:*

a. *f is weakly $\mathscr{RL}$-integrable if and only if f can be expressed as $f = \sum_{i=1}^{\infty} x_i\chi_{E_i}$ for $x_i\in\mathscr{X}$, $E_i\in\Sigma$ and the series converges absolutely m-a.e.*

b. *$\sum_{i=1}^{\infty} x_i m(E\cap A_i)$ converges unconditionally for $E,\ A_i\in\Sigma$.*

In an example of Theorem 4.1, we consider the conditional expectation of a weakly $\mathscr{RL}$-integrable function as follows: Suppose λ is a probability measure on Σ and $\mathscr{H}$ is a sub σ-field of Σ. In any sample space S, consider $k:S\to\mathscr{X}$ is a weakly measurable and almost separably valued $\mathscr{RL}$-integrable function. Then it is easy to see that there exists a sequence k_n of finitely valued measurable functions $k_n(x)\to k(x)$ with probability 1 as $n\to\infty$;

$$(\mathscr{RL})\int\limits_H \big\|k_n(x)-k_m(x)\big\|d\lambda\to 0$$

as $n,m\to\infty$. So, $(\mathscr{RL})\int_H k_n(x)d\lambda = (\mathscr{RL})\int_H k(x)d\lambda$, λ-a.e.. Let us consider $k(.):H\to\mathscr{X}$ be finitely valued; consider $k(x)=\xi_i\in H_i$, $i=1,2,\ldots$ and $H_i\in\mathscr{H}$. We define $\mathscr{RL}$-integrable valued conditional expectation as

$$E^{\mathscr{H}}(k|\mathscr{H})(x) = \sum_{i=1}^{k}\xi_i.E(\chi_{H_i}|\mathscr{H})(x),$$

where $E(\chi_{H_i}|\mathscr{H})(x)$ is an ordinary conditional expectation of χ_{H_i} relative to $\mathscr{H}$. Thus $E^{\mathscr{H}}(k):S\to\mathscr{X}$ is $\mathscr{RL}$-integrable, $\mathscr{H}$-measurable, satisfies

$$(\mathscr{RL})\int\limits_H E^{\mathscr{H}}(k)d\lambda = (\mathscr{RL})\int\limits_H kd\lambda,\ H\in\mathscr{H} \tag{4.4}$$

and is unique λ-a.e.. We can prove (4.4) and its uniqueness in a similar technique of [25, Theorem 2.1]. With the help of Theorem 4.1, we can construct a representation of the conditional expectation when k is a weakly $\mathscr{RL}$-integrable function as below.

Theorem 4.2. *Let f be a measurable and weakly $\mathscr{RL}$-integrable. If $\mathscr{H}$ is a sub σ-field of Σ, then*

$$E^{\mathscr{H}}(f) = E^{\mathscr{H}}(g) + \sum_{i=1}^{\infty} x_i E^{\mathscr{H}}(\chi_{E_i}) \tag{4.5}$$

in the sense that $x^* E^{\mathscr{H}}(g) + \sum_{i=1}^{\infty} x^* x_i E^{\mathscr{H}}(\chi_{H_i})$ is $\mathscr{RL}$-integrable and $\mathscr{H}$-measurable with

$$(\mathscr{RL}) \int_H f d\lambda = (B) \int_H E^{\mathscr{H}}(g) d\lambda + \sum_{i=1}^{\infty} \int_H E^{\mathscr{H}}(x_i.\chi_{H_i}) d\lambda, \;\; H_i \in \mathscr{H}. \tag{4.6}$$

Proof. Suppose that (4.5) converges unconditionally $\lambda_{\mathscr{H}}$-a.e.. If

$$f = g_i + \sum_{i=1}^{\infty} x'_i E^{\mathscr{H}}(\chi_{E'_i})$$

is another representation of f in sense of Theorem 4.1, then $f = g_i + \sum_{i=1}^{\infty} x'_i E^{\mathscr{H}}(\chi_{E'_i})$ converges unconditionally $\lambda_{\mathscr{H}}$-a.e..

Equation (4.6) follows from the definition of conditional expectation and Theorem 4.1. □

Conclusion

The $\mathscr{RL}$-integrable function spaces with countable additive measure is discussed in Section 3. An order continuous Banach lattice of $\mathscr{RL}^1(m, \mathscr{X})$ is discussed along with the separability of $\mathscr{RL}^1(m, \mathscr{X})$. In the sequel, weakly $\mathscr{RL}$-integrable functions are introduced in the same section. In Theorem 3.7, we have shown that the weakly $\mathscr{RL}$-integrable function space is an order continuous and weakly sequentially complete. In the last section, a representation theorem of the weakly $\mathscr{RL}$-integrable function is discussed in Theorem 4.1.

Acknowledgement

The first author like to thanks Turkish Academy of Sciences for support this research work. The authors would like to thank Prof. Anca Croitoru, Faculty of Mathematics, University “Alexandru Ioan Cuza”, Romania for her constructive comments and suggestions.

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(Received 13.07.2023)

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