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ON THE STABILITY OF SOLUTIONS OF NONLINEAR DIFFERENTIAL SYSTEMS WITH RETARDED ARGUMENTS

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Abstract. The systems of unilateral functional differential inequalities are considered. The effective conditions guaranteeing the boundedness, stability and asymptotic stability of solutions of nonlinear differential systems with retarded arguments are established.

In the present paper we consider the systems of unilateral functional differential inequalities of the types that arise in the theory of boundary value problems and in the theory of stability (see [1–5]). The effective conditions that guarantee the boundedness, uniform stability and uniform asymptotic stability of solutions of nonlinear differential systems with retarded arguments, are established.

Throughout the paper, the following designations are used:

– $\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$;
– δ_{ik} is the Kronecker symbol, that is,

$$\delta_{ik} = \begin{cases} 1 & \text{for } \ i = k, \\ 0 & \text{for } \ i \neq k; \end{cases}$$

– $x = (x_i)_{i=1}^n$ and $X = (x_{ik})_{i,k=1}^n$ are n-dimensional column vector and $n\times n$-matrix respectively, with the elements x_i and $x_{ik} \in \mathbb{R}$ $(i, k = 1, \dots, n)$ and the norms

$$\|x\| = \sum_{i=1}^n |x_i|, \quad \|X\| = \sum_{i,k=1}^n |x_{ik}|;$$

– X^{-1} is an inverse matrix to X;
– $r(X)$ is the spectral radius of X;
– E is the unit matrix;
– I is a compact or noncompact interval;
– $C(I)$ is the space of bounded continuous functions $x : I \to \mathbb{R}$ with the norm $\|x\|_{C(I)} = \sup\{|x(t)| : \ t \in I\}$;
– $\widetilde{C}_{\rm loc}(I)$ is the space of functions $x : I \to \mathbb{R}$, absolutely continuous on every compact interval contained in I;
– $L(I)$ is the space of the Lebesgue integrable functions $x : I \to \mathbb{R}_+$;
– $L_{\rm loc}(I)$ is the space of functions $x : I \to \mathbb{R}$, Lebesgue integrable on every compact interval contained in I.

On a finite or infinite interval, we consider a system of functional differential inequalities

$$\sigma_1 u_i'(t) \leq p_i(t)u_i(t) + \sum_{k=1}^n p_{ik}(t)\|u_k\|_{C(I)} + q_i(t) \ \ (i = 1, \dots, n), \tag{1}$$

where $\sigma_i \in \{-1; 1\}$, $p_i \in L_{\rm loc}(I)$, $p_{ik} \in L_{\rm loc}(I)$, $q_i \in L_{\rm loc}(I)$ $(i = 1, \dots, n)$, $p_i(t) \leq 0$, $p_{ik}(t) \geq 0$, $q_i(t) \geq 0$ a.e. on I $(i, k = 1, \dots, n)$.

Definition 1. The vector function $(u_i)_{i=1}^n : I \to \mathbb{R}^n$ is said to be a nonnegative solution of system (1) if $u_i \in \widetilde{C}_{\rm loc}(I)$, $u_i(t) \geq 0$ for $t \in I$ $(i = 1, \dots, n)$ and almost everywhere on I, inequality (1) is satisfied.

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Theorem 1. *Let* $-\infty < a < b < +\infty$, $I = [a, b]$, $t_i = 0$ *for* $\sigma_i = 1$, $t_i = b$ *for* $\sigma_i = -1$,

$$h_{ik}(t) = \left| \int_{t_i}^{t} \exp\left(\sigma_i \int_s^t p_i(x)\,dx\right) p_{ik}(s)\,ds \right|, \quad h_i(t) = \left| \int_{t_i}^{t} \exp\left(\sigma_i \int_s^t p_i(x)\,dx\right) q_i(s)\,ds \right| \quad (i = 1, \dots, n)$$

and

$$r(H) < 1, \quad \textit{where} \ H = \left(\|h_{ik}\|_{C(I)}\right)_{i,k=1}^n. \tag{2}$$

Then an arbitrary nonnegative solution $(u_i)_{i=1}^n$ *of system* (1) *admits the estimate*

$$\sum_{i=1}^n \|u_i\|_{C(I)} \le \rho \sum_{i=1}^n \left(u_i(t_i) + \|h_i\|_{C(I)}\right),$$

where

$$\rho = \left\|(E - H)^{-1}\right\|. \tag{3}$$

If $I = \mathbb{R}_+$, *then we assume that*

$$h_{ik} \in C(\mathbb{R}_+), \quad h_i \in C(\mathbb{R}_+) \ (i, k = 1, \dots, m), \tag{4}$$

where

$$h_{ik}(t) = \int_0^t \exp\left(\int_s^t p_i(x)\,dx\right) p_{ik}(s)\,ds, \quad h_i(t) = \int_0^t \exp\left(\int_s^t p_i(x)\,dx\right) q_i(s)\,ds \ \textit{ for } \ \sigma_i = 1,$$

$$h_{ik}(t) = \int_t^{+\infty} \exp\left(\int_t^s p_i(x)\,dx\right) p_{ik}(s)\,ds, \quad h_i(t) = \int_t^{+\infty} \exp\left(\int_t^s p_i(x)\,dx\right) q_i(s)\,ds \ \textit{ for } \ \sigma_i = -1.$$

Of our interest is the case in which, along with (4), *one of the following three conditions:*

$$\sigma_i = 1,$$

$$m \in \{1, \dots, n-1\}, \ \sigma_1 = 1 \ (i = 1, \dots, m), \ \sigma_i = -1, \ \int_0^{+\infty} p_i(s)\,ds = -\infty \ (i = m+1, \dots, n) \tag{5}$$

$$\sigma_i = -1, \quad \int_0^{+\infty} p_i(s)\,ds = -\infty \ (i = 1, \dots, n)$$

is fulfilled. In these cases, we establish the following a priori estimates:

$$\begin{aligned} &\sum_{i=1}^n \|u_i\|_{C(\mathbb{R}_+)} \le \rho \sum_{i=1}^n \left(u_i(0) + \|h_i\|_{C(\mathbb{R}_+)}\right), \\ &\sum_{i=1}^n \|u_i\|_{C(\mathbb{R}_+)} \le \rho \Big(\sum_{i=1}^n u_i(0) + \sum_{i=1}^n \|h_i\|_{C(\mathbb{R}_+)}\Big), \\ &\sum_{i=1}^n \|u_i\|_{C(\mathbb{R}_+)} \le \rho \sum_{i=1}^n \|h_i\|_{C(\mathbb{R}_+)}. \end{aligned} \tag{6}$$

Theorem 2. *Let* $I = \mathbb{R}_+$ *and along with* (2) *and* (4) *condition* (5) *be fulfilled. Then an arbitrary nonnegative solution of system* (1) *admits the estimate* (6), *where* ρ *is a number prescribed by equality* (3).

Consider the differential systems

$$x_i'(t) + g_i\big(t, x_1(\tau_{i1}(t)), \dots, x_n(\tau_{in}(t))\big) x_i(t) = f_i\big(t, x_1(\tau_{i1}(t)), \dots, x_n(\tau_{in}(t))\big) \ (i = 1, \dots, n) \tag{7}$$

and

$$x_i'(t) + g_{0i} x_i(\tau_i(t)) = f_i\big(t, x_1(\tau_{i1}(t)), \dots, x_n(\tau_{in}(t))\big) \ (i = 1, \dots, n). \tag{8}$$

Here, $g_i : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, $f_i : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ $(i = 1, \dots, n)$ are the functions satisfying the local Carathéeodory conditions

$$g_{0i} \in L_{\mathrm{loc}}(\mathbb{R}_+), \quad g_{0i}(t) \geq 0 \ \text{ for } \ t \in \mathbb{R}_+ \ \ (i = 1, \dots, n),$$

and $\tau_i : \mathbb{R} \to \mathbb{R}$, $\tau_{ik} : \mathbb{R}_+ \to \mathbb{R}$ $(i, k = 1, \dots, n)$ are the measurable on every interval functions such that $\tau_i(t) \leq t$, $\tau_{ik}(t) \leq t$ for $t \in \mathbb{R}_+$ $(i, k = 1, \dots, n)$.

Let $a \in \mathbb{R}_+$, $c_i \in C(]-\infty, +\infty[)$, $c_{0i} \in \mathbb{R}$ $(i = 1, \dots, n)$. For systems (7) and (8), consider the Cauchy problem

$$x_i(t) = c_i(t) \ \text{ for } \ t < a, \ \ x_i(a) = c_{0i} \ \ (i = 1, \dots, n). \tag{9}$$

Suppose

$$\chi_a(t) = \begin{cases} 1 & \text{for } \ t \geq a, \\ 0 & \text{for } \ t < a, \end{cases} \qquad \tau_{aik}(t) = \begin{cases} \tau_{ik} & \text{for } \ t \geq a, \\ 0 & \text{for } \ t < a \end{cases} \ \ (i, k = 1, \dots, n),$$

$$\tau_{ai}(t) = \begin{cases} \tau_i(t) & \text{for } \ t \geq a, \\ a & \text{for } \ t < a \end{cases} \ \ (i = 1, \dots, n)$$

and introduce the following

Definition 2. Let $a < b \leq +\infty$ and $I = [a, b[$, or $a < b < +\infty$ and $I = [a, b]$. A vector function $(x_i)_{i=1}^n : I \to \mathbb{R}^n$ is said to be a solution of problem (7), (9) (of problem (8), (9)) defined on I if $x_i \in \widetilde{C}_{\mathrm{loc}}(I)$, $x_i(a) = c_i$ $(i = 1, \dots, n)$, and almost everywhere on I, equality (7) (equality (8)) is satisfied, where

$$x_i(\tau_{ik}(t)) = \big(1 - \chi_a(\tau_{ik}(t))\big) c_i(\tau_{ik}(t)) + \chi_a(\tau_{ik}(t)) x_i(\tau_{aik}(t)) \ \ (i, k = 1, \dots, n),$$
$$x_i(\tau_i(t)) = \big(1 - \chi_a(\tau_i(t))\big) c_i(\tau_i(t)) + \chi_a(\tau_i(t)) x_i(\tau_{ai}(t)) \ \ (i = 1, \dots, n).$$

Definition 3. Let $a < b < +\infty$ and $I = [a, b[$ $(i = [a, b])$. A solution $(x_i)_{i=1}^n$ of problem (7), (9) (of problem (8), (9)) **is said to be continuable** if there exists $\overline{b} \in [b, +\infty[$ $(\overline{b} \in]b, +\infty[)$ and the solution $(\overline{x}_i)_{i=1}^n$ of this problem is defined on $[a, b]$ such that $\overline{x}_i(t) = x_i(t)$ for $t \in I$ $(i = 1, \dots, n)$. Otherwise, the solution $(x_i)_{i=1}^n$ is called **non-continuable**.

Definition 4. A trivial solution of system (7) (of system (8)) is called uniformly stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for arbitrary numbers and functions $a \in \mathbb{R}_+$, $c_{0i} \in \mathbb{R}$ and $c_i \in C(]-\infty, a[)$ satisfying the condition

$$\sum_{i=1}^n \big(|c_{0i}| + \|c_i\|_{C(]-\infty, a[)} \big) < \delta, \tag{10}$$

every non-continuable solution of problem (7), (9) (of problem (8), (9)) is defined on $[a, b]$ and admits the estimate

$$\sum_{i=1}^n \|x_i\|_{C([a, +\infty[)} < \varepsilon. \tag{11}$$

Definition 5. A trivial solution of system (7) (of system (8)) is called uniformly asymptotically stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for arbitrary numbers and functions $a \in \mathbb{R}_+$, $c_{0i} \in \mathbb{R}$ and $c_i \in C(]-\infty, a[)$ satisfying condition (10), every non-continuable solution of problem (7), (9) (of problem (8), (9)) is defined on $[a, +\infty[$, admits the estimate (11) and is vanishing at infinity, that is,

$$\lim_{i \to \infty} x_i(t) = 0 \ \ (i = 1, \dots, n).$$

Theorem 3. *Let there exist the numbers $b_{ik} \in \mathbb{R}_+$, $\ell_i \in \mathbb{R}_+$ $(i, k = 1, \dots, n)$ and non-negative functions f_{0i} and $g_{0i} \in L_{\mathrm{loc}}([a, +\infty[)$ $(i = 1, \dots, n)$ such that*

$$r(\varphi) < 1, \ \ \textit{where} \ \ \varphi = (\ell_{ik})_{i,k=1}^n, \tag{12}$$

$$\ell_{0i} = \sup \bigg\{ \int_a^t \exp\bigg(-\int_s^t g_{0i}(x)\, dx \bigg) f_{0i}(s)\, ds : \ \ t \geq a \bigg\} < +\infty \ \ (i = 1, \dots, n),$$

and on $[a,+\infty[\,\times\mathbb{R}^n$ *the inequality*

$$g_i(t,x_1,\dots,x_n)\geq g_{0i}(t),\quad |f_i(t,x_1,\dots,x_n)|\leq g_i(t,x_1,\dots,x_n)\Big(\sum_{k=1}^n \ell_{ik}|x_k|+\ell_i\Big)+f_{0i}(t)$$
$$(i=1,\dots,n)$$

is satisfied. Then every non-continuable solution of problem (7), (9) *defined on* $[a,+\infty[$ *is bounded and admits the estimate*

$$\sum_{i=1}^n \|x_i\|_{C([0,+\infty[)}\leq \rho\Big(\sum_{i,k=1}^n \ell_{ik}\|c_k\|_{C(]-\infty,a[)}+\sum_{i=1}^n \big(|c_{0i}|+\ell_{0i}+\ell_i\big)\Big),$$

where $\rho=|(E-\mathscr{L})^{-1}\|$.

Corollary 1. *Let for some* $\delta_0>0$, *on the set* $\{(t,x_1,\dots,x_n):\ t\in\mathbb{R}_+,\ |x_k|\leq\delta_0\ (k=1,\dots,n)\}$ *the inequality*

$$|f_i(t,x_1,\dots,x_n)|\leq g_i(t,x_1,\dots,x_n)\sum_{k=1}^n \ell_{ik}|x_k|\ \ (i=1,\dots,n)$$

be fulfilled, where ℓ_{ik} $(i,k=1,\dots,n)$ *are the nonnegative constants satisfying condition* (12). *Then the trivial solution of system* (7) *is uniformly stable.*

Corollary 2. *Let for some* $\delta_0>0$, *on the set* $\{(t,x_1,\dots,x_n):\ t\in\mathbb{R}_+,\ |x_k|\leq\delta_0\ (k=1,\dots,n)\}$, *the inequalities*

$$g_i(t,x_1,\dots,x_n)\geq g_0(t),\quad \exp\bigg(\int_0^t g_0(x)\,dx\bigg)|f_i(t,x_1,\dots,x_n)|$$
$$\leq \big(g_i(t,x_1,\dots,x_n)-g_0(t)\big)\sum_{k=1}^n \ell_{ik}\gamma_{ik}(t)|x_k|\ \ (i=1,\dots,n)$$

be fulfilled, where

$$\gamma_{ik}(t)=\exp\bigg(\int_0^{\tau_{0ik}(t)} g_0(x)\,dx\bigg)\ \ (i,k=1,\dots,n),$$

whereas ℓ_{ik} $(i,k=1,\dots,n)$ *and* $g_0\in L_{\rm loc}(\mathbb{R}_+)$ *are, respectively, the nonnegative constants and nonnegative function satisfying condition* (12) *and* $\int_0^{+\infty} g_0(s)\,ds=+\infty$. *Then the trivial solution of system* (7) *is uniformly asymptotically stable.*

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