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A CHARACTERIZATION OF MODULE CATEGORIES OVER QUANTALES

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Abstract. It is shown that module categories over quantales are precisely those quantaloids (i.e., categories enriched in the symmetric monoidal closed category of sup-lattices) that are monadic over the category of sets.

As a background to the subject, we refer to S. MacLane [5] for generalities on the (ordinary) category theory and to M. Kelly [4] for enriched category theory.

1. Preliminaries

1.1. For a monad $\mathscr{T} = (T, \mu, \eta)$ on a category $\mathscr{X}$, we write

- $\mathscr{X}^{\mathscr{T}}$ for the Eilenberg–Moore category of $\mathscr{T}$–algebras;
- $U^{\mathscr{T}} : \mathscr{X}^{\mathscr{T}} \to \mathscr{X},\ (X, h) \to X$, for the underlying (forgetful) functor;
- $F^{\mathscr{T}} : \mathscr{X} \to \mathscr{X}^{\mathscr{T}},\ X \to (T(X), \mu_X)$, for the free $\mathscr{T}$–algebra functor, and
- $\eta^{\mathscr{T}}, \varepsilon^{\mathscr{T}} : F^{\mathscr{T}} \dashv U^{\mathscr{T}} : \mathscr{X}^{\mathscr{T}} \to \mathscr{X}$ for the forgetful–free adjunction. (Recall that $\eta^{\mathscr{T}} = \eta$ and $(\varepsilon^{\mathscr{T}})_{(X,h)} = h$ for all $(X, h) \in \mathscr{X}^{\mathscr{T}}$).

Let $\eta, \varepsilon \colon F \dashv U \colon \mathscr{A} \to \mathscr{X}$ be an adjunction, $\mathscr{S} = (UF, U\varepsilon F, \eta)$ be the monad on $\mathscr{X}$ generated by the adjunction and $K^{\mathscr{S}} : \mathscr{A} \to \mathscr{X}^{\mathscr{S}}$ be the comparison functor. Recall that $K^{\mathscr{S}}$ assigns to each object $A \in \mathscr{A}$ the $\mathscr{T}$-algebra $(U(A), U(\varepsilon_A))$, and to each morphism $f : A \to A'$ the morphism $U(f) : U(A) \to U(A')$. Moreover, $U^{\mathscr{T}} K^{\mathscr{T}} = U$ and $K^{\mathscr{T}} F = F^{\mathscr{T}}$. One says that the functor U is *monadic* if $K^{\mathscr{T}}$ is an equivalence of categories.

1.2. Let $\mathcal{S}l$ denote the category of sup-lattices, i.e., complete lattices and sup-preserving morphisms. It is well known (see, [3]) that $\mathcal{S}l$ is a symmetric monoidal closed category, the tensor unit being $2 = \{0 \leq 1\}$, and $\mathcal{S}l$ is a monadic category over the category of sets, Set. The forgetful functor $\mathcal{S}l(2, -) : \mathcal{S}l \to \mathsf{Set}$ admits as a left adjoint the functor $2^{(-)}$ sending a set X to 2^X, and the Eilenebrg–Moore category of algebras w.r.t. the monad generated by this adjunction is (isomorphic to) $\mathcal{S}l$. An important property of the category $\mathcal{S}l$ is that (small) coproducts are biproducts in $\mathcal{S}l$; that is, if $\{X_i\}$ is a (small) family of objects of $\mathcal{S}l$, then the comparison morphism $\coprod X_i \to \prod X_i$ is an isomorphism.

1.3. A *quantale* is a monoid in the symmetric monoidal category $\mathcal{S}l$. Therefore, a quantale is a sup-lattice A equipped with a monoid structure $(A, *, 1)$ such that $*$ distributes over suprema. As in any symmetric monoidal category, we can talk about (right) actions of a quantale A on objects of $\mathcal{S}l$. We call them *(right) A-modules* and write $\mathsf{Mod}_A(\mathcal{S}l)$ for the corresponding category. It is well known that the evident forgetful functor $U^A : \mathsf{Mod}_A(\mathcal{S}l) \to \mathsf{Set}$ is monadic. Its left adjoint takes a set X to the coproduct $\coprod_X A$ of X copies of A.

1.4. A *quantaloid* is a locally small category enriched in the symmetric monoidal closed category of sup-lattices. If $\mathscr{Q}$ is a quantaliod, then one has the following commutative diagram of categories and

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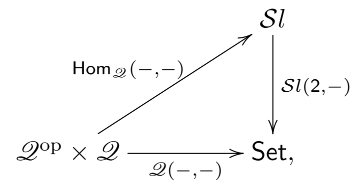
functors

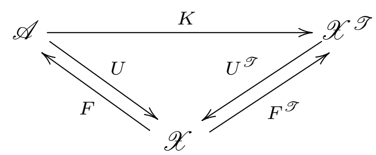


in which $\mathsf{Hom}_{\mathscr{Q}}(-,-)$ is the so-called *lifted hom-functor*. Typical examples of quantaloids are module categories over quantales. The property that (small) coproducts are biproducts in $\mathcal{S}l$, transfers to any quantaloid $\mathscr{Q}$ (see, e.g., [2]). We write $\oplus Q_i$ for the biproduct of the family $(Q_i, i \in I)$.

2. The Characterization Theorem

We now give the conditions under which a quantaloid can be recognized as a module category over a quantale.

2.1. Let $\eta, \varepsilon : F \dashv U : \mathscr{A} \to \mathscr{X}$ be an adjunction, $\mathscr{T}$ be a monad on $\mathscr{X}$ and $K : \mathscr{A} \to \mathscr{X}^{\mathscr{T}}$ be a functor such that $U^{\mathscr{T}} K = U$. The situation may be pictured as



(2.1)

Write $\mathscr{S}$ for the monad on $\mathscr{X}$ generated by the adjunction $F \dashv U$. It is shown in [1] that the composite

$$s_K : U^{\mathscr{T}} F^{\mathscr{T}} \xrightarrow{U^{\mathscr{T}} F^{\mathscr{T}} \eta} U^{\mathscr{T}} F^{\mathscr{T}} U F = U^{\mathscr{T}} F^{\mathscr{T}} U^{\mathscr{T}} K F \xrightarrow{U^{\mathscr{T}} \varepsilon^{\mathscr{T}} K F} U^{\mathscr{T}} K F = U F$$

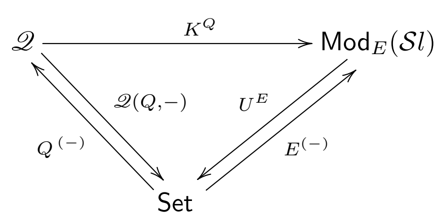
is a monad morphism $\mathscr{T} \to \mathscr{S}$. When s_K is a natural isomorphism, one says that the functor U is $\mathscr{T}$*-Galois* (see [7, Definition 1.3]). Some criteria for the functor U to be $\mathscr{T}$-Galois can be found in [8, 9].

Applying the dual of [6, Theorem 4.4] (or [7, Proposition 2.1]) to the data given in the diagram (2.1), we get the following *comparison theorem*.

Theorem 2.1. *In the situation described above, K is an equivalence of categories if and only if*

(i) *U is monadic and*
(ii) *the induced monad morphism $s_K : \mathscr{T} \to \mathscr{S}$ is an isomorphism (U is $\mathscr{T}$-Galois).*

2.2. Let $\mathscr{Q}$ be a quantaloid and Q an arbitrary object of $\mathscr{Q}$. To Q one associates the functor $\mathsf{Hom}_{\mathscr{Q}}(Q,-) : \mathscr{Q} \to \mathcal{S}l$ and the quantale $E = \mathsf{Hom}_{\mathscr{Q}}(Q,Q)$ of endomorphisms of Q. Then for any object $A \in \mathscr{Q}$, the sup-lattice $\mathsf{Hom}_{\mathscr{Q}}(Q,A)$ naturally has the structure of a right E-module given by the composition and thus the assignment $A \longmapsto \mathsf{Hom}_{\mathscr{Q}}(Q,A)$ yields a functor $K^Q : \mathscr{Q} \to \mathsf{Mod}_E(\mathcal{S}l)$ such that $U_E K^Q = \mathscr{Q}(Q,-)$. If $\mathscr{A}$ admits small copowers of Q (i.e. biproducts $Q^{(X)} = \oplus_X Q$ of X copies of Q for all set X), then $\mathscr{Q}(Q,-)$ admits a left adjoint $Q^{(-)}$ which associates to a set X the copower $Q^{(X)}$ and we have the following particular instance of diagram (2.1):



(2.2)

We write $\mathscr{T}^E$ for the monad on Set generated by the adjunction $E^{(-)} \dashv U^E$.

Proposition 2.1. *In the situation of diagram* (2.2), *the functor $\mathscr{Q}(Q,-)$ is $\mathscr{T}^E$-Galois.*

Combining Theorem 2.1 and Proposition 2.1, we obtain our characterization theorem.

Theorem 2.2. *A quantaloid $\mathscr{Q}$ is (equivalent to) a module category over a quantale if and only if there exists an object $Q \in \mathscr{Q}$ such that the functor $\mathscr{Q}(Q,-) : \mathscr{Q} \to \mathsf{Set}$ is monadic. When this is the case, $\mathscr{Q}$ is equivalent to $\mathsf{Mod}_E(\mathcal{S}l)$, where $E = \mathscr{Q}(Q,Q)$.*

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