Transactions of A. Razmadze
Mathematical Institute
Vol. **179** (2025), issue 1, 169–171

ON EXTENSIONS OF G-VOLUMES AND THEIR UNIQUENESS PROPERTY

ALEKS KIRTADZE[1,2] AND LIKA BERAIA[2]

Abstract. The method for extending G-volumes on the Euclidean space $\mathbf{R}^d$ with its applications is considered. In particular, the uniqueness property of G-volumes is discussed.

The concept of G-volumes for geometric figures is one of the most important in the geometry of Euclidean space and, more generally, in the theory of G-measures. One of the important topics in the modern theory of G-volumes is the problem of the existence of a volume in a sufficiently large class of geometric figures of $\mathbf{R}^d$ ($d \geq 1$). Note that it is impossible to define a volume in the family of all subsets of $\mathbf{R}^d$ (see, [3, 4, 6, 7]).

Two possible approaches to the theory of G-volumes should be mentioning:

(a) the approach from the geometrical point of view;

(b) the approach from the measure-theoretical view-point.

The present paper is devoted to certain extensions of G-volumes on the Euclidean space $\mathbf{R}^d$ using some methods motivated by the techniques of measure theory. In particular, certain profound connections of the theory of G-volumes with general methods of the theory of G-invariant measures are discussed.

Let D_d denote the group of all isometric transformations of $\mathbf{R}^d$ and let S_d be the ring generated by the collection of all coordinate parallelepipeds of $\mathbf{R}^d$, i.e., the sets of the forms

$$[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_b].$$

Let G be a subgroup D_d. A functional V_d defined on S_d is called an elementary G-volume on $\mathbf{R}^d$ if it is normalized, non-negative, finitely additive and G-invariant.

The above-mentioned conditions are usually treated as Axioms of Invariant Finitely Additive Measure. The ring S_d is called the class of elementary figures in the Euclidean space. Any element $X \in S_d$ is called an elementary figure. If the condition of finite additivity is replaced by the countable additivity condition, then we obtain the definition of a G-measure (see, e. g., [5]). From a standard course of mathematical analysis it is well known that the classical Jordan measure on $\mathbf{R}^d$ is a natural example of G-volume in $\mathbf{R}^d$. Respectively, a certain extension of Jordan measure to a sufficiently large class of subsets of $\mathbf{R}^d$ is the standard Lebesgue measure. Note that the latter class of sets is maximal, since it is impossible to extend this class without using an uncountable form of the Axiom of Choice (see [11]).

From the geometrical point of view, there are many interesting and important facts concerning G-volumes (see, for example, [3, 5–7]). The most famous among them is due to Banach.

Theorem 1. *In the case $d = 1$ and $d = 2$, there exists a non-negative additive functional defined in the family of all bounded subsets of the Euclidean space $\mathbf{R}^d$, invariant under the group of all isometries of $\mathbf{R}^d$ and extending the Lebesgue measure λ_d.*

The proof of Theorem 1 can be found in [5, 12].

One of the principal problems arising here is to extend a G-volume on the Euclidean space to a G-volume on the same space defined in a maximally large class of figures. This problem is successfully solved within the framework of the modern theory of invariant measures and its solution largely depends on the purely algebraic properties of a basic group of transformations of the Euclidean space. Note that by using the Kuratowski–Zorn lemma, any G-volume on $\mathbf{R}^d$ can be extended to a maximal (by the inclusion relation) G-volume on $\mathbf{R}^d$, but the descriptive structure of the domain of such a

2020 *Mathematics Subject Classification.* 51M05, 51M25, 28C10.

Key words and phrases. G-volume; Extension of volume; Uniqueness property.

maximal G-volume is not known and this problem seems to be of interest. A sufficiently general method of extending measures was suggested by E. Marczewski. This method is purely set-theoretical because no specific properties of the given measurable space are used. According to a result of Marczewski, we can always extend Lebesgue measure to an isometrically-invariant countably additive measure.

The following statement is a Marczewski type theorem for the G-volumes.

Theorem 2. *Let V_d be a G-volume on $\mathbf{R}^d$ and let $\mathbf{I}_d$ be a family of subsets of $\mathbf{R}^d$, which is closed under finite unions, invariant under transformations from G and every set $X \in \mathbf{I}_d$ has V_d-inner volume zero. Then V_d admits an extension V_d' such that $\mathbf{I}_d \subset \operatorname{dom}(V_d')$ and $V_d'(X) = 0$ for every $X \in \mathbf{I}_d$.*

Indeed, consider the algebra $\mathbf{S}$ of subsets of $\mathbf{R}^d$, generated by $\mathbf{I}_d^* \cup \operatorname{dom}(V_d)$, where $\mathbf{I}_d^*$ is the ideal generated by $\mathbf{I}_d$. Obviously, any set $Z \in \mathbf{S}$ can be represented in the form

$$Z = (X \cup Y_1) \setminus Y_2,$$

where $X \in V_d$ and Y_1 and Y_2 are the members of I_d.

Let us put

$$V_d'(Z) = V_d(X).$$

It can be checked that the functional V_d' is well defined on $\mathbf{S}$ and V_d' turns out to be a G-volume on $\mathbf{R}^d$ extending V_d. Also, it follows from the definition of V_d' that $V_d'(X) = 0$ for every $X \in \mathbf{I}_d$.

In our context, let us consider as an example an application of Theorem 2.

Example 1. Let $\mathbf{R}^d$ be the d-dimensional Euclidean space, j_d be the Jordan measure on $\mathbf{R}^d$ and let $\{X_i : i \in I\}$ be an arbitrary finite family of bounded and nowhere dense subsets of $\mathbf{R}^d$. Then there exists a measure j_d' on $\mathbf{R}^d$ extending j_d and satisfying the following relations:

$$\{X_i : i \in I\} \subset \operatorname{dom}(j_d'), \quad (\forall i)\,(i \in I \Rightarrow j_d'(X_i) = 0).$$

As a typical example of a bounded and nowhere dense subset of $\mathbf{R}^d$, we can consider the well-known Jordan curve L in $\mathbf{R}^d$ possessing a positive d-dimensional Lebesgue measure (see [1, 8, 9]).

For our further purposes, we will need several auxiliary notions.

Let $\mathbf{R}^d$ be again the Euclidean space, G be a group of isometric transformations of $\mathbf{R}^d$ and let M be a class of all G-volumes on $\mathbf{R}^d$.

We say that a G-volume $V_1 \in M$ has the uniqueness property, if for every G-volume $V_2 \in M$ such that $\operatorname{dom}(V_1) = \operatorname{dom}(V_2)$, we have $V_1 = V_2$.

Remark. If the group G is sufficiently rich, then all elementary volumes on $\mathbf{R}^d$ coincide with each other. In particular, if G contains everywhere a dense set of translations of $\mathbf{R}^d$, then all elementary volumes coincide with the restriction of the Jordan measure to S_d.

We say that a figure $X \subset \mathbf{R}^d$ has the uniqueness property in M, if for every two G-volumes $V_1 \in M$ and $V_2 \in M$ such that $X \in \operatorname{dom}(V_1) \cap \operatorname{dom}(V_2)$, we have the equality $V_1(X) = V_2(X)$.

We say that a G-volume $V \in M$ has the strong uniqueness property in M if each figure $X \in \operatorname{dom}(V)$ has the uniqueness property on M.

In other words, a G-volume $V \in M$ possesses the strong uniqueness property with respect to M, if for every $X \in \operatorname{dom}(V)$ and for every two G-volumes $V_1 \in M$ and $V_2 \in M$ such that $X \in \operatorname{dom}(V_1)$ and $X \in \operatorname{dom}(V_2)$, we have the following equality $V_1(X) = V_2(X)$.

The above-mentioned definitions can be found, e. g., in [2, 5, 10].

It is clear that if a G-volume has the strong uniqueness property, then the same volume has the uniqueness property, too.

From the definition of G-volumes it trivially follows that the unit cube on $\mathbf{R}^d$ has the uniqueness property on M.

Example 2. It is well known that every subset $X \subset \mathbf{R}^d$, measurable with respect to the classical Jordan measure, has the strong uniqueness property in the class of π_d-volumes, where π_d denotes the group of all translations of $\mathbf{R}^d$.

Example 3. It is well known that a Jordan curve is a homeomorphic image of the unit circle. According to the Denjoy–Riesz theorem, each compact zero-dimensional set C in $\mathbf{R^d}$ is contained in a Jordan curve. Taking as C a Cantor-type set in $\mathbf{R^d}$ with $\lambda_d(C) > 0$, we get the desired curve L (see, [1]). According to the Banach theorem, the Jordan curve L does not have the uniqueness property with respect to the class of all D_d-volumes of $\mathbf{R^d}$.

Theorem 3. *There exists a π_d-volume defined in the family of all bounded subsets of the Euclidean space on $\mathbf{R}^d$ which extends the Jordan measure j_d of $\mathbf{R}^d$ and which does not possess the strong uniqueness property in the class of π_d-volumes.*

References

1. M. Balcerzak, A. Kharazishvili, On uncountable unions and intersections of measurable sets. *Georgian Math. J.* **6** (1999), no. 3, 201–212.
2. Sh. Beriashvili, T. Kasrashvili, A. Kirtadze, On the strong uniqueness of elementary volumes on $\mathbf{R}^2$. *Rep. Enlarged Sess. Semin. I. Vekua Appl. Math.* **32** (2018), 11–14.
3. H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie.* (German) Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957.
4. P. R. Halmos, *Measure Theory.* D. Van Nostrand Co., Inc., New York, 1950.
5. A. Kharazishvili, *Invariant Extensions of Lebesgue Measure.* (Russian). Tbilis. Gos. Univ., Tbilisi, 1983.
6. A. Kharazishvili, On uniform sets in Euclidean space $\mathbf{R}^m$ and G-volumes. *Trans. A. Razmadze Math. Inst.* **178** (2024), no. 1, 161–163.
7. A. Kirtadze, On volume type functional in Euclidean geometry. *God. Sofiĭ. Univ. "Sv. Kliment Okhridski." Fac. Mat. Inform.* **103** (2016), 45–53.
8. N. N. Luzin, *The Theory of Real Functions.* (Russian) Moscow, 1948.
9. William F. Osgood, A Jordan curve of positive area. *Trans. Amer. Math. Soc.* **4** (1903), no. 1, 107–112.
10. S. S. Pkhakadze, The theory of Lebesgue measure. (Russian). *Akad. Nauk Gruzin. SSR Trudy Tbiliss. Mat. Inst. Razmadze* **25** (1958), 3–271.
11. R. M. Solovay, A model of set-theory in which every set of reals is Lebesgue measurable. *Ann. of Math. (2)* **92** (1970), 1–56.
12. S. Wagon, *The Banach-Tarski Paradox.* With a foreword by Jan Mycielski. Encyclopedia of Mathematics and its Applications, 24. Cambridge University Press, Cambridge, 1985.

(Received 29.01.2025)

[1]A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 2 Merab Aleksidze II Lane, Tbilisi 0193, Georgia

[2]Georgian Technical University, 77 Kostava Str., Tbilisi, 0160, Georgia
Email address: kirtadze2@yahoo.com
Email address: likaberaia444@gmail.com