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ON ADDITIVE FUNCTIONS WITH THE SUPER-STRONG DARBOUX PROPERTY

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Abstract. Some versions of the well-known Darboux property for real-valued additive functions are considered and their connections with the concepts of relative measurability and absolute non-measurability of functions are examined.

Let $\mathbf{Q}$ denote, as usual, the field of all rational numbers, let $\mathbf{R}$ denote the field of all real numbers, and let $\mathbf{c}$ stand for the cardinality of the continuum.

Recall that a function $g : \mathbf{R} \to \mathbf{R}$ has the Darboux property, if for any two distinct real numbers a and b, the line segment with the end-points $g(a)$ and $g(b)$ is entirely contained in $g([a, b])$.

As widely known, many authors devoted their research works to this property and investigated it from various points of view. We mention only few of those works (see [1–5, 8, 13]).

We say that a function $g : \mathbf{R} \to \mathbf{R}$ has the strong Darboux property if $g(\triangle) = \mathbf{R}$ for every non-degenerate subinterval $\triangle$ of $\mathbf{R}$.

We say that a function $g : \mathbf{R} \to \mathbf{R}$ has the super-strong Darboux property if $g(P) = \mathbf{R}$ for every nonempty perfect subset P of $\mathbf{R}$.

Remark 1. It is not difficult to verify that:

(a) there are additive functions acting from $\mathbf{R}$ into $\mathbf{Q}$, which trivially do not have the Darboux property (such functions are easily obtained by using a Hamel basis of $\mathbf{R}$);

(b) there are additive functions acting from $\mathbf{R}$ into itself, which have the Darboux property, but do not have the strong Darboux property (for example, any nonzero linear function from $\mathbf{R}$ into itself is such a function).

Lemma 1. *Let $h : \mathbf{R} \to \mathbf{R}$ be an additive surjective function satisfying the equality $h(1) = 0$.*

Then h has the strong Darboux property.

Proof. Take arbitrarily $y \in \mathbf{R}$. Since h is a surjection, there is $x \in \mathbf{R}$ for which $h(x) = y$. Let $\triangle$ be any non-degenerate subinterval of $\mathbf{R}$. Obviously, there exists a rational number q such that $x + q$ is in $\triangle$. So, in view of $h(q) = 0$, one has

$$y = h(x) = h(x + q), \quad y \in h(\triangle),$$

whence the strong Darboux property of h follows.

Denote by $\mathcal{M}$ the class of the completions of all those nonzero σ-finite Borel measures on $\mathbf{R}$, which vanish at the singletons of $\mathbf{R}$.

We say that a function $f : \mathbf{R} \to \mathbf{R}$ is relatively measurable with respect to $\mathcal{M}$ if there exists at least one measure $\mu \in \mathcal{M}$ such that f is μ-measurable.

Accordingly, we shall say that a function $f : \mathbf{R} \to \mathbf{R}$ is absolutely nonmeasurable with respect to $\mathcal{M}$ if there exists no measure $\mu \in \mathcal{M}$ such that f is μ-measurable. □

Lemma 2. *There exists an additive function $g : \mathbf{R} \to \mathbf{R}$ satisfying the following two conditions:*

(1) *g has the strong Darboux property;*

(2) *g is relatively measurable with respect to the class $\mathcal{M}$.*

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Proof. Our argument will be based on one interesting fact of classical point set theory, namely, there exists a nonempty perfect set P in $\mathbf{R}$ which is linearly independent over the field $\mathbf{Q}$ (see, for instance, [10]). Clearly, P can be expanded to some Hamel basis B of $\mathbf{R}$. Without loss of generality, we may assume that

$$\mathrm{card}(B \setminus P) = \mathbf{c}, \quad 1 \in B \setminus P.$$

So, we can write $B = \{e_i : i \in I\} \cup \{e_j : j \in J\} \cup \{1\}$, where

$$\{e_i : i \in I\} = P, \quad \{e_j : j \in J\} = B \setminus (P \cup \{1\}).$$

Let ϕ be a bijection of J onto $\mathbf{R}$. We define the values of g on B as follows:

$$(\forall i \in I)(g(e_i) = e_i), \quad (\forall j \in J)(g(e_j) = \phi(j)), \quad g(1) = 0.$$

Then we extend this partial function g to an additive function on $\mathbf{R}$ (denoted by the same symbol g). We now assert that the extended g satisfies both conditions (1) and (2). Indeed, (1) is fulfilled in view of Lemma 1. In order to show the validity of (2), observe that the restriction $g|P$ of g to P is the identical mapping of P onto itself, hence trivially is continuous. There exists a measure $\mu \in \mathcal{M}$ which is concentrated on P, i.e., $\mu(\mathbf{R} \setminus P) = 0$. Obviously, g turns out to be a μ-measurable function. Thus, (2) takes place, which ends the proof. □

To show the existence of additive functions which possess the super-strong Darboux property, a more delicate argument is necessary.

Let $\{P_t : t \in T\}$ be some bijective enumeration of all nonempty perfect subsets of $\mathbf{R}$ (so, $\mathrm{card}(T) = \mathbf{c}$).

Lemma 3. *There is a disjoint family* $\{E_t : t \in T\}$ *of subsets of* $\mathbf{R}$ *such that:*

(i) $(\forall t \in T)(E_t \subset P_t)$*;*

(ii) $(\forall t \in T)(\mathrm{card}(E_t) = \mathbf{c})$*;*

(iii) *the set* $\cup\{E_t : t \in T\}$ *is linearly independent over the field* $\mathbf{Q}$.

Remark 2. In connection with Lemma 3, see also [6, 7]. A more general result can be established. Let V be an uncountable vector space over $\mathbf{Q}$ and let $\{A_k : k \in K\}$ be a family of subsets of V such that $\mathrm{card}(K) \le \mathrm{card}(V)$ and the cardinalities of all sets A_k are equal to $\mathrm{card}(V)$. Then there exists a disjoint family $\{C_k : k \in K\}$ satisfying the following relations:

$$(\forall k \in K)(C_k \subset A_k \ \&\ \mathrm{card}(C_k) = \mathrm{card}(V)),$$

$$\cup\{C_k : k \in K\} \text{ is linearly independent over } \mathbf{Q}.$$

In fact, the formulated above result may be treated as an algebraic version of the well-known Sierpiński's theorem from general set theory (see [14]).

Theorem 1. *There exist real-valued additive functions on* $\mathbf{R}$ *possessing the super-strong Darboux property.*

Any function $g : \mathbf{R} \to \mathbf{R}$ *with the super-strong Darboux property satisfies the following conditions:*

(1) *g is absolutely nonmeasurable with respect to the class $\mathcal{M}$;*

(2) *if $\mu \in \mathcal{M}$ and X is a μ-nonmeasurable subset of $\mathbf{R}$, then the set $g^{-1}(X)$ is also μ-nonmeasurable.*

Proof. Consider again the family $\{P_t : t \in T\}$ of all nonempty perfect subsets of $\mathbf{R}$ and the family $\{E_t : t \in T\}$ described in Lemma 3. It is not difficult to define an additive function $h : \mathbf{R} \to \mathbf{R}$ such that $h(E_t) = \mathbf{R}$ for each index $t \in T$. Consequently, we get $h(P) = \mathbf{R}$ for every nonempty perfect set P in $\mathbf{R}$. This circumstance implies that h has the super-strong Darboux property.

Now, let $g : \mathbf{R} \to \mathbf{R}$ be an arbitrary function possessing the super-strong Darboux property and let μ be any measure from the class $\mathcal{M}$. It is clear that g cannot be μ-measurable (because g is unbounded on each uncountable compact subset of $\mathbf{R}$). Take any μ-nonmeasurable set $X \subset \mathbf{R}$ and suppose to the contrary that the set $g^{-1}(X)$ is μ-measurable. Only two cases are possible here.

1. $\mu(g^{-1}(X)) > 0$.

In this case, the set $g^{-1}(X)$ contains a nonempty perfect subset. So, we get the relations

$$\mathbf{R} = g(g^{-1}(X)) \subset X, \quad X = \mathbf{R},$$

which is impossible, because of the μ-nonmeasurability of X.

2. $\mu(g^{-1}(X)) = 0$.

In this case, $\mu(\mathbf{R} \setminus g^{-1}(X)) > 0$ and the set $\mathbf{R} \setminus g^{-1}(X)$ contains a nonempty perfect subset. So, we come to the equality $g(\mathbf{R} \setminus g^{-1}(X)) = \mathbf{R}$, which contradicts the relation $X \cap g(\mathbf{R} \setminus g^{-1}(X)) = \emptyset$.

The contradiction obtained in both above cases completes the proof. □

Remark 3. Note that the analogue of Theorem 1 in terms of the Baire property is also valid (see, e.g., [8–10] or [12] for extensive information about the Baire property). Actually, every function $g : \mathbf{R} \to \mathbf{R}$ possessing the super-strong Darboux property does not have the Baire property, and if X is an arbitrary subset of $\mathbf{R}$ lacking the Baire property, then the set $g^{-1}(X)$ does not have the Baire property either. These facts can be established by using an argument similar to the previous one.

Remark 4. Lemma 2 and Theorem 1 imply at once that there are additive functions acting from $\mathbf{R}$ into itself, which possess the strong Darboux property, but do not possess the super-strong Darboux property.

Let λ denote the standard Lebesgue measure on $\mathbf{R}$ and let $\lambda_2 = \lambda \otimes \lambda$.

Lemma 4. *The graph of any function $f : \mathbf{R} \to \mathbf{R}$ with the super-strong Darboux property is λ_2-thick in $\mathbf{R}^2$, i.e., the graph of f intersects every λ_2-measurable set in $\mathbf{R}^2$ having strictly positive measure.*

Notice that the assertion of Lemma 4 can be deduced from Mycielski's theorem on inscribed products of perfect sets (see [11]).

Theorem 2. *If $f : \mathbf{R} \to \mathbf{R}$ is an additive function with the super-strong Darboux property, then there exists a translation quasi-invariant measure ν on $\mathbf{R}$ such that:*

(1) *ν extends λ;*

(2) *f is ν-measurable.*

In contrast to Theorem 1, the above theorem shows that any additive function having the super-strong Darboux property turns out to be relatively measurable with respect to the class of all those measures on $\mathbf{R}$ which extend λ and are translation quasi-invariant.

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