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A NOTE ON THE BERIKASHVILI FUNDAMENTAL GROUP

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Dedicated to the memory of Nodar Berikashvili

Abstract. We make a comparison between Berikashvili's group $\pi(M, \overline{m})$ defined with respect to a family M of path connected subsets of a path connected topological space (and fixed $\overline{m} \in M$) and the Galois groupoid of an appropriate local homeomorphism associated with M. In fact, this compares Berikashvili's approach to the notion of fundamental group with the categorical-Galois-theoretic approach.

1. Introduction

I should have seen Nodar Berikashvili's paper [1] forty years ago, when I started to think about a new categorical approach to Galois theory, but I first saw that paper only a couple of months ago...

Given a set M of path connected subsets of a path connected topological space X and a fixed element $\overline{m}$ in M, Berikashvili constructs a certain group $\pi(M, \overline{m})$, and proposes conditions under which it is nicely related to the fundamental group of X. Berikashvli's theory here is seemingly similar to the known ones, but in fact it is very different and interesting. Most importantly, I immediately thought that there are easily visible conditions under which the group $\pi(M, \overline{m})$ is equivalent to the groupoid constructed in the following steps:

- Take $p : E \to B$ (I changed the letter X by B) to be the canonical map from the coproduct of all elements of M to B, but assume B to be locally connected, as well.
- Take the kernel pair $Eq(p)$ of p as an internal category in the category LCTop of locally connected topological spaces, and apply the connected-component-functor $I : \mathsf{LCTop} \to \mathsf{Sets}$ to get a *precategory* $I(Eq(p))$ (recall that a precategory is a kind of a truncated simplicial set).
- Take the category $\mathcal{L}(I(Eq(p)))$ associated to $I(Eq(p))$ (which is the same as the fundamental groupoid of $I(Eq(p))$ considered as a simplicial set).

But, searching for the "easily visible conditions", I could not find anything better than two independent conditions found by Berikashvili for a different purpose, each of which turned out to be sufficient. The aim of this paper is to show how (each of) these conditions make $\pi(M, \overline{m})$ equivalent to $\mathcal{L}(I(Eq(p)))$.

Apart from this Introduction, the paper has three sections; the first two of them recall necessary material from Berikashvili's paper [1] and from the author's papers, respectively; the last section is devoted to the equivalence $\pi(M, \overline{m}) \sim \mathcal{L}(I(Eq(p)))$.

2. The Berikashvili Fundamental Group

In this section, we only recall Berikashvili's construction [1], using mostly his notation. There are places where, I think, Berikashvili means "path connected" but says just "connected"; in those cases I'll write "(path) connected". Let us also agree that "path connected" means "non-empty path connected".

Let X be a fixed path connected topological space, M a fixed set of path connected subsets of X, and $\overline{m}$ a fixed element of M. Then:

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- A *chain* is a sequence $(m_1, m_2, m_3, \dots, m_n)$ in M such that $m_i \cap m_{i+1}$ is path connected for each $i = 1, \dots, n-1$, and $m_1 = m_n = \overline{m}$.
- A chain $(m_1, m_2, m_3, \dots, m_n)$ is said to be *elementary equivalent* to the sequence obtained by removing m_i from it if that sequence is also a chain and $m_{i-1} \cap m_i \cap m_{i+1}$ is (path) connected.
- The *equivalence* of chains is defined as the smallest equivalence relation (on their set) containing the elementary equivalence. The set of equivalence classes of chains is denoted by $\pi(M, \overline{m})$. It is a group whose multiplication is induced by composing chains via juxtaposition.
- For a fixed $y \in \overline{m}$, let $H(X, M)$ be the subgroup of the fundamental group $\pi(X, y)$ generated by (the equivalent classes of) paths of the form CDC^{-1}, where C begins at y and ends where D begins, while D is a loop.

Further, Berikashvili observes:

- It is easy to see that the group $\pi(M, \overline{m})$ coincides with the fundamental group $\pi(N_\omega(M), \overline{m})$, where $N_\omega(M)$ is the simplicial complex, in which: the set of vertexes is M; $\{m, m'\}$ is a 1-simplex if $m \cap m'$ is (path) connected; $\{m, m', m''\}$ is a 2-simplex if $\{m, m'\}$, $\{m, m''\}$ and $\{m', m''\}$ are 1-simplexes, and $m \cap m' \cap m''$ is (path) connected; and there are no higher-dimensional simplexes.
- It is also easy to see that $H(X, M)$ is a normal subgroup of $\pi(X, y)$, and that it is trivial whenever so are all the canonical homomorphisms $\pi(m) \to \pi(X)$ $(m \in X)$.
- Given a chain $(m_1, m_2, m_3, \dots, m_k)$, choose: a sequence $(y, x_1, x_2, \dots, x_k, y)$ of points with $x_i \in m_i \cap m_{i+1}$; and then a sequence $(Q, C_{12}, \dots, C_{k-1,k}, P)$ of paths, in which $C_{i,i+1}$ connects x_i and x_{i+1} inside m_{i+1}, Q connects y with x_1 inside $m_1 = \overline{m}$, and P connects x_k with y inside $\overline{m} = m_k$. This defines a group homomorphism

$$\pi(M, \overline{m}) \to \pi(X, y)/H(X, M).$$

And then Berikashvili proves:

Theorem 2.1 (Theorems 1 and 2 of [1])**.** *Suppose all elements of M are open and $\bigcup M = X$. Then the homomorphism* (1) *is an isomorphism whenever any of the following two conditions hold:*

(a) *each non-empty intersection of two or three elements of M is path connected;*
(b) *each* (*path*) *component of each intersection of two elements of M belongs to M.*

Let us omit his Theorem 3, but recall his several observations, made at various places, as:

Remark 2.2. Assuming again that $\bigcup M = X$ and each element of M is open:

(a) Under the assumptions of Theorem 2.1(a), $N_\omega(M)$ coincides with the 2-skeleton of the nerve of M and so, Theorem 2.1(a) says that the group $\pi(X, y)/H(X, M)$ is isomorphic to the fundamental group of the nerve of M (and Berikashvili suggests: "cf. [8, 9]").
(b) (Corollary of Theorem 2 in [1]) Let X be not just path connected but also locally path connected and locally simply connected, and let L be the set of components of finite intersections of elements of M. Then L (replacing M) satisfies the conditions of Theorem 2.1(b). Further, since X is locally simply connected, we can choose the elements of M to be 'small enough' to make the group $H(X, L)$ trivial. For such L, Theorem 2.1(b) gives $\pi(M, \overline{m}) \approx \pi(X, y)$.
(c) Everything done above has a simplicial counterpart, where X is an abstract simplicial complex, the fundamental group is defined combinatorially, M is a set of closed subcomplexes, and the role of path connectedness is played by combinatorial connectedness.

Note that Remark 2.2(c) is just a copy (well, almost) of the last sentence of [1]; doing that in detail would be a material for another article... .

3. Galois Theory of Locally Connected Spaces

In this section, we recall a special case of the categorical form of *Fundamental Theorem of Galois Theory* (FTGT, for short) from [3] and [4], which will be done in two steps: first at an abstract-categorical level, and then for locally connected topological spaces.

A *Galois structure* consists of an adjunction $(I,H,\eta,\varepsilon):\mathcal{C}\to\mathcal{X}$ and classes $\mathcal{E}\subseteq\mathrm{Mor}(\mathcal{C})$ and $\mathcal{Z}\subseteq\mathrm{Mor}(\mathcal{X})$ such that $\mathcal{C}$ and $\mathcal{X}$ admit pullbacks along morphisms from $\mathcal{E}$ and $\mathcal{Z}$ (respectively), $\mathcal{E}$ and $\mathcal{Z}$ are pullback stable, $I(\mathcal{E})\subseteq\mathcal{Z}$, and $H(\mathcal{Z})\subseteq\mathcal{E}$.

Assuming such a structure fixed, consider, for any given object B in $\mathcal{C}$, the induced adjunction $(I^B,H^B,\eta^B,\epsilon^B):\mathcal{E}(B)\to\mathcal{Z}(I(B))$, in which:

- $\mathcal{E}(B)$ is the full subcategory of $(\mathcal{C}\downarrow B)$ with objects all (A,α) with $\alpha\in\mathcal{E}$, and, similarly, $\mathcal{Z}(I(B))$ is the full subcategory of $(\mathcal{X}\downarrow I(B))$ with objects all (X,ζ) with $\zeta\in\mathcal{Z}$;
- $I^B:\mathcal{E}(B)\to\mathcal{Z}(I(B))$ is defined by $I^B(A,\alpha)=(I(A),I(\alpha))$;
- $H^B:\mathcal{Z}(I(B))\to\mathcal{E}(B)$ is defined by $I^B(X,\zeta)=(B\times_{HI(B)}H(X),\pi_1)$, using the pullback

$$\begin{array}{ccc} B\times_{HI(B)}H(X) & \xrightarrow{\pi_2} & H(X)\\ {\scriptstyle\pi_1}\big\downarrow & & \big\downarrow{\scriptstyle H(\zeta)}\\ B & \xrightarrow[\eta_B]{} & HI(B)\end{array}$$

  (here and below, all pullback projections are denoted by π's with indices);
- $\eta^B_{(A,\alpha)}=\langle\alpha,\eta_A\rangle:(A,\alpha)\to(B\times_{HI(B)}HI(A),\pi_1)$;
- $\varepsilon^B_{(X,\zeta)}=\varepsilon_X I(\pi_2):(I(B\times_{HI(B)}H(X)),I(\pi_1))\to(X,\zeta)$.

When all counits of such induced adjunctions are isomorphisms, the Galois structure is called *admissible*. If it is the case, given a morphism $p:E\to B$ in $\mathcal{C}$, the category $\mathrm{Spl}_I(E,p)$ of coverings of B split over p is defined as the full subcategory of $\mathcal{E}(B)$ with objects all $(A,\alpha)\in\mathcal{E}(B)$ for which the diagram

$$\begin{array}{ccc} E\times_B A & \xrightarrow{\eta_{E\times_B A}} & H(E\times_B A)\\ {\scriptstyle\pi_1}\big\downarrow & & \big\downarrow{\scriptstyle H(\pi_1)}\\ E & \xrightarrow[\eta_E]{} & HI(E)\end{array}$$

is a pullback.

An *internal precategory* in $\mathcal{X}$ is a diagram in $\mathcal{X}$ of the form $P=$

$$P_2 \;\begin{array}{c}\xrightarrow{p_1}\\ \xrightarrow{m}\\ \xrightarrow[p_2]{}\end{array}\; P_1 \;\begin{array}{c}\xrightarrow{d}\\ \xleftarrow{e}\\ \xrightarrow[c]{}\end{array}\; P_0$$

in which $de=1_{P_0}=ce$, $dp_1=cp_2$, $dm=dp_2$, and $cm=cp_1$. An *internal action* of P is a triple $F=(F_0,\gamma,\xi)$ as in the diagram

$$\begin{array}{ccccccc} P_2\times_{(dp_2,\gamma)}F_0 & \xrightarrow{\langle p_1,p_2\rangle\times 1} & P_1\times_{(d,c)}P_1\times_{(d,\gamma)}F_0 & \xrightarrow{1\times\xi} & P_1\times_{(d,\gamma)}F_0 & \xleftarrow{\langle e\gamma,1\rangle} & F_0\\ {\scriptstyle m\times 1}\big\downarrow & & & & \big\downarrow{\scriptstyle\xi} & \swarrow\!\!\!= & \\ P_1\times_{(d,\gamma)}F_0 & & \xrightarrow{\qquad\qquad\xi\qquad\qquad} & & F_0 & & \\ {\scriptstyle\pi_1}\big\downarrow & & & & \big\downarrow{\scriptstyle\gamma} & & \\ P_1 & & \xrightarrow[\qquad\qquad c\qquad\qquad]{} & & P_0 & & \end{array}$$

which is required to commute; we assume here that all the pullbacks involved do exist. The category of all internal actions (F_0,γ,ξ) of P with $\gamma\in\mathcal{Z}$ is denoted by $\mathcal{X}^P\cap\mathcal{Z}$.

A *monadic extension* (=an *effective descent morphisms* in the terminology of [6] and related papers) in $\mathcal{C}$ is a morphism $p:E\to B$ in $\mathcal{C}$ for which the pullback functor $p^*:(\mathcal{C}\downarrow B)\to(\mathcal{C}\downarrow E)$ is monadic.

Theorem 3.1 (FTGT, the admissible case)**.** *For an admissible Galois structure above and a monadic extension $p:E\to B$ in $\mathcal{C}$, there is a canonical category equivalence $\mathrm{Spl}_I(E,p)\sim\mathcal{X}^{I(Eq(p))}\cap\mathcal{Z}$, where $Eq(p)$ is the kernel pair of p considered as an internal category in $\mathcal{C}$.*

Note that, since the functor I is not required to preserve pullbacks, $I(Eq(p))$ is not an equivalence relation, but only a precategory in general (internal to $\mathcal{X}$); in fact, it is what is called in [3] an internal pregroupoid. But, when $\mathcal{X} = \mathsf{Sets}$, and assuming for simplicity that $\mathcal{Z}$ is the class of all morphisms in $\mathcal{X}$, for an arbitrary precategory P and then for $P = I(Eq(p))$, we have:

- $\mathcal{X}^P$ is nothing but the category of precategory morphisms $P \to \mathsf{Sets}$;
- the forgetful functor from the category of categories to the category of precategories has a left adjoint, which we denote by $\mathcal{L}$;
- it easily follows that $\mathcal{X}^P$ is canonically isomorphic to $\mathcal{X}^{\mathcal{L}(P)}$;
- it is also easy to see that $\mathcal{L}(I(Eq(p)))$ is a groupoid, and we call it the Galois groupoid of (E, p) and denote it by $\mathrm{G}(E, p)$.

Thus, from Theorem 3.1, we obtain

Corollary 3.2. *Under the assumptions of Theorem 3.1, if $\mathcal{Z}$ is the class of all morphisms in $\mathcal{X}$, then the category $\mathrm{Spl}_I(E, p)$ is equivalent to the category $\mathsf{Sets}^{\mathrm{G}(E,p)}$ of actions of the Galois groupoid $\mathrm{G}(E, p)$.*

Consider the special case examined in detail in Chapter 6 of [2], where we take:

- $\mathcal{C}$ to be the category of locally connected topological spaces;
- $\mathcal{X} = \mathsf{Sets}$;
- I to be defined by $I(A)$ = the set of connected components of A;
- accordingly, H carries the sets to themselves equipped with the discrete topology;
- $\mathcal{E}$ to be the class of local homeomorphisms of locally connected topological spaces;
- $\mathcal{Z}$ to be the class of all morphisms in $\mathcal{X}$, that is, all maps of sets.
- $p : E \to B$ to be a surjective local homeomorphism (of locally connected spaces).

In this case, Corollary 3.2 can be reformulated as follows:

Corollary 3.3. *The category $\mathrm{Spl}_I(E, p)$ of pairs (A, α), where $\alpha : A \to B$ is a continuous map such that for every connected component C of E the pullback projection $C \times_B A \to C$ is a trivial covering map (in the classical sense), is equivalent to the category $\mathsf{Sets}^{\mathrm{G}(E,p)}$ of actions of the Galois groupoid $\mathrm{G}(E, p)$.*

4. Galois-theoretic Interpretation of Berikashvili's Construction

Putting together the contexts of Sections 2 and 3, let us write B instead of X (for X from Section 2), and assume this space to be not only path connected, but also locally path connected. Then, in particular, for the subsets of B, path connectedness becomes the same as connectedness. Let us also assume that all elements of M are not only connected, but also open, and that $\bigcup M = B$.

Let E be the coproduct of elements of M and $p : E \to B$ be the canonical map, which is obviously a surjective local homeomorphism. Consider the precategory $I(Eq(p)) =$

$$I(E \times_B E \times_B E) \overset{\mathsf{p}_1,\ \mathsf{m},\ \mathsf{p}_2}{\Rrightarrow} I(E \times_B E) \underset{\mathsf{c}}{\overset{\mathsf{d}}{\rightrightarrows}} I(E), \qquad \mathsf{e} : I(E) \to I(E \times_B E)$$

and observe:

- $I(E) = M$, since each element of M is connected.
- A morphism $n : m_1 \to m_2$ in $I(Eq(p))$, that is, an element n of $I(E \times_B E)$ with $\mathsf{d}(n) = m_1$ and $\mathsf{c}(n) = m_2$, is a connected component of $m_1 \cap m_2$.
- A triangle t in $I(Eq(p))$ with the boundary

$$\begin{array}{ccc} m_1 & \xrightarrow{\ n\ } & m_3 \\ & {\scriptstyle n_1}\searrow \qquad \nearrow{\scriptstyle n_2} & \\ & m_2 & \end{array}$$

that is, an element t of $I(E \times_B E \times_B E)$ with $\mathsf{p}_1(t) = n_2$, $\mathsf{m}(t) = n$, $\mathsf{p}_2(t) = n_1$, $\mathsf{dp}_1(t) = m_2 = \mathsf{cp}_2(t)$, $\mathsf{dm}(t) = m_1 = \mathsf{dp}_2(t)$, and $\mathsf{cm}(t) = m_3 = \mathsf{dp}_1(t)$, is a connected component of $m_1 \cap m_2 \cap m_3$.

Remark 4.1. Conditions 2.1(a) and 2.1(b) are characterized in terms of $I(Eq(p))$ as follows:

(a) 2.1(a) is satisfied if and only if $I(Eq(p))$ is a relation, that is, parallel morphisms in $I(Eq(p))$ are always equal;

(b) the following conditions are equivalent:
- (b_1) condition 2.1(b);
- (b_2) all morphisms in $I(Eq(p))$ belong to M;
- (b_3) all triangles in $I(Eq(p))$ belong to M.

Theorem 4.2. *Under the assumptions above and any of the conditions* 2.1(a) *and* 2.1(b)*, the Galois groupoid* $\mathrm{G}(E,p)$ *is a connected groupoid, equivalent to the Berikashvili group* $\pi(M,\overline{m})$*, for any choice of* $\overline{m} \in M$*.*

Proof. We have $\mathrm{G}(E,p) = \mathcal{L}(I(Eq(p)))$, and we use freely the construction of $\mathcal{L}$ described in [4], since it is just a simplified version of the well-known construction of the fundamental groupoid of a simplicial set.

We are going to prove that $\mathrm{G}(E,p)$ is isomorphic to what we are going to call *Berikashvili groupoid* of M and denote it by $\pi(M)$. Its objects are the elements of M, and its morphisms are equivalent classes of the chains

$$(m_1, m_2, m_3, \dots, m_n) : m_1 \to m_n,$$

exactly as in Berikashvili's construction, except that $m_1 = m_n$ is not required. Of course, $\pi(M)$ is equivalent to $\pi(M,\overline{m})$ as a category. Our proof requires seven preliminary steps:

Step 1: Construction of a functor

$$\pi(M) \to \mathrm{G}(E,p),$$

which we denote by P*.*

A chain $(m_1, m_2, m_3, \dots, m_n) : m_1 \to m_n$ determines a morphism in $\mathrm{G}(E,p)$ represented by

$$m_1 \xrightarrow{m_1 \cap m_2} m_2 \xrightarrow{m_2 \cap m_3} \dots \xrightarrow{m_{n-1} \cap m_n} m_n$$

and, having in mind how we compose the morphisms in $\pi(M)$ and $\mathrm{G}(E,p)$, all we need to show is:

If a triple (m_1, m_2, m_3) has connected $m_1 \cap m_2$, $m_2 \cap m_3$, $m_1 \cap m_3$, and $m_1 \cap m_2 \cap m_3$, then

$$(m_1 \xrightarrow{m_1 \cap m_2} m_2 \xrightarrow{m_2 \cap m_3} m_3) = (m_1 \xrightarrow{m_1 \cap m_3} m_3)$$

in $\mathrm{G}(E,p)$.

However, this follows from the fact that the assumptions on (m_1, m_2, m_3) make $m_1 \cap m_2 \cap m_3$ a triangle in $I(Eq(p))$ with the boundary

$$\begin{array}{ccc} m_1 & \xrightarrow{m_1 \cap m_3} & m_3\ , \\ & {\scriptstyle m_1 \cap m_2} \searrow \quad \nearrow {\scriptstyle m_2 \cap m_3} & \\ & m_2 & \end{array}$$

where $m_i \cap m_j$ ($i < j$ in $\{1, 2, 3\}$), being connected, can be considered as connected components of $E \times_B E$, and $m_1 \cap m_2 \cap m_3$, being connected, can be considered as a connected component of $E \times_B E \times_B E$.

That is, P is defined by

$$P(\mathrm{cls}(m_1, m_2, m_3, \dots, m_n) : m_1 \to m_n)$$
$$= \ \mathrm{cls}(m_1 \xrightarrow{m_1 \cap m_2} m_2 \xrightarrow{m_2 \cap m_3} \dots \xrightarrow{m_{n-1} \cap m_n} m_n).$$

Step 2. Construction of a precategory morphism

$$I(Eq(p)) \to \pi(M)$$

under condition 2.1(a)*; we denote this morphism by* Q_1*.*

Under condition 2.1(a), a morphism $n : m_1 \to m_2$ in $I(Eq(p))$ must have $n = m_1 \cap m_2$ connected and so it makes (m_1, m_2) a chain. It remains to prove that, for triangle t with the boundary (2), the chains (m_1, m_2, m_3) and (m_1, m_3) are equivalent. However, this is the case, since $n_1 = m_1 \cap m_2$, $n_2 = m_2 \cap m_3$, $n = m_1 \cap m_3$, and $m_1 \cap m_2 \cap m_3$ are connected by 2.1(a).

That is, Q_1 is defined by

$$Q_1(n : m_1 \to m_2) = \mathrm{cls}(m_1, m_2).$$

Step 3. Construction of a precategory morphism

$$I(Eq(p)) \to \pi(M)$$

under condition 2.1(b)*; we will denote this morphism by* Q_2*.*

Under condition 2.1(b), a morphism $n : m_1 \to m_2$ in $I(Eq(p))$ must have n in M, which implies that (m_1, n, m_2) is a chain. It remains to prove that, for a triangle t in $I(Eq(p))$ with the boundary (2), the chains $(m_1, n_1, m_2, n_2, m_3)$ and (m_1, n, m_3) are equivalent. As follows from 2.1(b), t belongs to M, and since (2) is the boundary of t, $t \subseteq n_1 \cap n \cap n_2$. This gives us the following equivalences $\sim$ of chains:

$$\begin{aligned}(m_1, n_1, m_2, n_2, m_3) &\sim (m_1, n_1, t, m_2, n_2, m_3) \sim (m_1, n_1, t, m_2, t, n_2, m_3)\\ &\sim (m_1, n_1, t, t, n_2, m_3) \sim (m_1, n_1, t, n_2, m_3) \sim (m_1, n_1, t, m_3) \sim (m_1, t, m_3)\\ &\sim (m_1, n, t, m_3) \sim (m_1, n, t, n, m_3) \sim (m_1, n, n, m_3) \sim (m_1, n, m_3).\end{aligned}$$

That is, Q_2 is defined by

$$Q_2(n : m_1 \to m_2) = \mathrm{cls}(m_1, n, m_2).$$

Step 4. Under condition 2.1(a)*, we have* $PQ_1 = F$*, where* F *is the canonical morphism*

$$I(Eq(p)) \to \mathrm{G}(E, p).$$

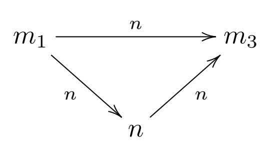
We need to show that a morphism in $\mathrm{G}(E, p)$ represented by $n : m_1 \to m_2$ is the same as the one represented by $m_1 \cap m_2 : m_1 \to m_2$, but $n = m_1 \cap m_2$ by condition 2.1(a), since n must be a connected component in $m_1 \cap m_2$.

Step 5. Under condition 2.1(b)*, we have* $PQ_2 = F$*.*

We need to show that a morphism in $\mathrm{G}(E, p)$ represented by $n : m_1 \to m_2$ is the same as the one represented by

$$m_1 \xrightarrow{\;n\;} n \xrightarrow{\;n\;} m_2,$$

but this follows from the fact that n can be considered as a triangle in $I(Eq(p))$ with the boundary



Step 6. Under condition 2.1(a)*, the unique functor*

$$\overline{Q_1} : \mathrm{G}(E, p) \to \pi(M)$$

with $\overline{Q_1}F = Q_1$ *is surjective.*

This is the case, since $\pi(M)$ is generated by the set of equivalence classes of two-component chains (m_1, m_2) and since for such a chain, we have

$$\mathrm{cls}(m_1, m_2) = Q_1(m_1 \cap m_2 : m_1 \to m_2).$$

Step 7. Under condition 2.1(b)*, the unique functor*

$$\overline{Q_2} : \mathrm{G}(E, p) \to \pi(M)$$

with $\overline{Q_2}F = Q_2$ *is surjective.*

This is the case since $\pi(M)$ is generated by the set of equivalence classes of two-component chains (m_1, m_2) and since for such a chain, we have

$$\mathrm{cls}(m_1, m_2) = \mathrm{cls}(m_1, m_1 \cap m_2, m_2) = Q_2(m_1 \cap m_2 : m_1 \to m_2).$$

Now, we are ready to complete our proof. By the results of Steps 6 and 7, it suffices to prove that $P\overline{Q_1} = 1_{\mathrm{G}(E,p)}$ under condition 2.1(a), and $P\overline{Q_2} = 1_{\mathrm{G}(E,p)}$ under condition 2.1(b). But this follows from the universal property of F and the results of previous steps that give $P\overline{Q_i}F = PQ_i = F$ $(i = 1, 2)$. □

Remark 4.3. In addition to Theorem 4.2 let us briefly mention:

(a) It would be natural to modify Berikashvili's construction of $\pi(M, \overline{m})$ replacing "path connected" with "connected" everywhere. Then Remark 4.1 and Theorem 4.2 could be copied assuming that B is connected and locally connected instead of assuming it to be path connected and locally path connected.

(b) When, in addition to all our assumptions, B $(=X)$ is locally simply connected, we could choose M consisting of simply connected (open) subsets of B. In that case, $\mathrm{G}(E, p)$ coincides with the fundamental groupoid of B defined via the Galois theory up to a category equivalence. According to Theorem 4.2, this agrees with the isomorphism $\pi(M, \overline{m}) \approx \pi(X, y)$ mentioned in Remark 2.2.

(c) Suppose: B is path connected, locally path connected and locally simply connected, as above; either 2.1(a) or 2.1(b) is satisfied; $p : E \to B$ is chosen using M as before; and $q : \tilde{B} \to B$ is a universal covering map. Then:

(c_1) $\mathrm{Spl}_I(\tilde{B}, q)$ is the category of all covering maps with codomain B, and, as follows from the results of [5], $\mathrm{Spl}_I(E, p)$ is a full reflective subcategory of $\mathrm{Spl}_I(\tilde{B}, q)$.

(c_2) Let (E', p') be the image of $(\tilde{B}, q)$ under the reflection

$$\mathrm{Spl}_I(\tilde{B}, q) \to \mathrm{Spl}_I(E, p).$$

Suppose $p' : E' \to B$ happened to be a regular covering map. Then, suitably choosing base points, we obtain a surjective homomorphism from the fundamental group of B to the Galois group of (E', p') which can be identified with Berikashvili's group $\pi(M, \overline{m})$ by Theorem 4.2. Then Theorem 2.1 tells us that Berikashvili's $H(B, M)$ (written in Section 2 as $H(X, M)$) should coincide with the fundamental group of E'. This, together with some obvious further questions needs to be worked out in detail.

(d) Although Berikashvili mentions the relationship of his construction with simplicial ones (see the 5th bullet point in Section 2 and Remark 2.2(a)), further comparisons with old and recent topological, localic, and topos-theoretic Čech-type constructions of fundamental groups would be interesting: see, e.g., Definition 4 in Section 1 of [7].

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