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THE BREADTH OF BERIKASHVILI'S FUNCTOR $\mathcal{D}$

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To the memory of Nodar Berikashvili

Abstract. We discuss variants of Berikashvili's functor that arise in differential homological algebra, from simplicial bundles, from ordinary topological bundles and in more general categorical settings. We prove that under suitable circumstances, the value of Berikasvili's functor parametrizes isomorphism classes of bundles in various contexts.

1. Introduction

In [2], N. Berikashvili introduced the functor $\mathcal{D}$ in terms of "twisting elements" or "twisting cochains" in a differential graded algebra. At that time twisting cochains already had a history in topology and differential homological algebra, see, e.g., [45] (where the terminology is "twisting morphism") and the references there to [5] and [7,8]. With hindsight we see that twisting cochains make precise a certain piece of structure behind the notion of transgression [7,8]. The original version of Berikashvili's functor $\mathcal{D}$ assigns to a differential graded algebra $\mathcal{A}$ the set $\mathcal{D}(\mathcal{A}) = \mathcal{T}(\mathcal{A})/\mathcal{G}$ of orbits of twisting elements (homogeneous degree -1 members τ of $\mathcal{A}$ that satisfy the identity $d\tau = \tau\tau$ or master equation relative to the differential d in $\mathcal{A}$) in $\mathcal{A}$ with respect to the group $\mathcal{G}$ of inner automorphisms of $\mathcal{A}$. For a differential graded algebra $\mathcal{A}$, the set $\mathcal{D}(\mathcal{A})$ resembles a *moduli space of gauge equivalence classes of flat connections*. Such a moduli space is sometimes considered as a first non-abelian cohomology space. The results in the present paper suggest that, in the situations we discuss here, the value of Berikashvili's functor could also be viewed as a first non-abelian cohomology set. Remark 7.5 renders this observation explicit.

The paper [25] explains some of the significance of Berikashvili's functor within ordinary differential homological algebra and how it relates to deformation theory, and [30] develops a small aspect of that relationship further. Here we discuss variants of Berikashvili's functor in various contexts.

We begin by reviewing bundles in the category of chain complexes. Applying Berikashvili's functor $\mathcal{D}$ to the differential graded algebra $\operatorname{Hom}(C, A)$ associated to a differential graded coalgebra C and a differential graded algebra A (endowed with the cup or convolution product) yields the set $\mathcal{D}(C, A)$ of orbits of twisting cochains $\tau \colon C \to A$ relative to the group of inner automorphisms of the differential graded algebra $\operatorname{Hom}(C, A)$. We prove (Theorem 3.6) that, for a differential graded coalgebra C and a differential graded algebra A, the set $\mathcal{D}(C, A)$ parametrizes isomorphism classes of bundles (principal twisted tensor products) having C as base and A as fiber. Thereafter, guided by the idea that, for a group G, a principal G-bundle admits a characterization in terms of a certain G-valued functor defined on a certain category, cf. [49], we proceed in a more abstract manner within the appropriate categorical framework. Theorem 6.8 says that, for a simplicial set B and a simplicial group K, the value $\mathcal{D}(B, K)$ of Berikashvili's functor on the pair (B, K) parametrizes isomorphism classes of simplicial principal K-bundles on B. Theorem 6.9 establishes the fact that, for a simplicial set B and a simplicial group K, the assignment to a twisting function $\rho \colon B \to K$ of the twisting cochain which ρ determines via the perturbation lemma yields a map from isomorphism classes of simplicial principal bundles to the isomorphism classes of the associated twisted tensor products. Theorem 7.1 yields the same kind of parametrization result, but phrased over a category. This recovers principal bundles

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over a simplicial complex (Example 7.2) and over the simplicial category arising from an open cover of a manifold (Example 7.3) and hence ordinary principal bundles over a manifold endowed with a partition of unity subordinate to the open cover: A construction in [49] assigns to an open cover $\mathcal{U}$ of a manifold M a (topological or smooth) category $\mathcal{M}_{\mathcal{U}}$ in such a way that a functor F from $\mathcal{M}_{\mathcal{U}}$ to a group G (viewed as a category in the standard way) determines a simplicial principal G-bundle on the nerve of $\mathcal{M}_{\mathcal{U}}$. Theorem 7.4 says that evaluating Berikasvili's functor $\mathcal{D}$ at $\mathcal{M}_{\mathcal{U}}$ and G yields a set $\mathcal{D}(\mathcal{M}_{\mathcal{U}}, G)$ which recovers the non-abelian cohomology set $\mathrm{H}^1(B\mathcal{U}, G)$ of equivalence classes of G-transition functions relative to the open cover $B\mathcal{U}$ of M arising as the 'barycentric subdivision' $B\mathcal{U}$ of $\mathcal{U}$; when M admits a partition of unity subordinate to $B\mathcal{U}$, in particular, when M is paracompact, by a classical result, this cohomology set characterizes isomorphism classes of principal G-bundles on the manifold M. Section 8 explores Berikashvili's functor in a general categorical setting. This relates Berikashvili's functor to A_∞-functors and provides, perhaps, clarification and simplification for the theory of dg categories. For example, the paper [10] explores dg quotients of dg categories; in [10, p. 687, p. 689], the terminology is 'Maurer-Cartan functor' for the assignment to a coalgebra and algebra of the family of twisting cochains between the two. Thus there is a huge unexplored territory in this area.

2. Preliminaries

References for notation and terminology are [3, 4, 12, 13, 18–22, 25–29, 32, 33, 35, 45–48]. The ground ring R is a commutative ring with 1. We take *chain complex* to mean *differential graded R-module*. A chain complex is not necessarily concentrated in non-negative or non-positive degrees. The differential of a chain complex is always of degree -1. For a filtered chain complex X, a *perturbation* of the differential d of X is a (homogeneous) morphism ∂ of the same degree as d such that ∂ lowers filtration and $(d+\partial)^2 = 0$ or, equivalently,

$$[d, \partial] + \partial\partial = 0. \tag{2.1}$$

Thus, when ∂ is a perturbation on X, the sum $d+\partial$, referred to as the *perturbed differential*, endows X with a new differential. When X has a graded coalgebra structure such that (X, d) is a differential graded coalgebra, and when the *perturbed differential* $d+\partial$ is compatible with the graded coalgebra structure, we refer to ∂ as a *coalgebra perturbation*; the notion of *algebra perturbation* is defined similarly. Given a differential graded coalgebra C and a coalgebra perturbation ∂ of the differential d on C, occasionally we denote the new or *perturbed* differential graded coalgebra by C_∂. Given a differential graded algebra A and an algebra perturbation ∂ of the differential on A, occasionally we denote the new or *perturbed* differential graded algebra likewise by A_∂.

Given two chain complexes X and Y, recall that $\mathrm{Hom}(X, Y)$ inherits the structure of a chain complex by the operator D defined by

$$D\phi = d\phi - (-1)^{|\phi|}\phi d \tag{2.2}$$

where ϕ is a homogeneous homomorphism from X to Y and where $|\phi|$ refers to the degree of ϕ. The notation D for the Hom-differential and $\mathcal{D}$ for Berikashvili's functor might be slightly confusing but is, perhaps, unavoidable, for reasons of consistency with notation established in the literature.

A *contraction*

$$(M \underset{g}{\overset{\nabla}{\rightleftarrows}} N, h) \tag{2.3}$$

of chain complexes [12] consists of

– chain complexes N and M,

– chain maps $g\colon N \to M$ and $\nabla\colon M \to N$,

– a morphism $h\colon N \to N$ of the underlying graded modules of degree 1,

subject to

$$g\nabla = \mathrm{Id}, \tag{2.4}$$

$$Dh = \nabla g - N, \tag{2.5}$$

$$gh = 0, \quad h\nabla = 0, \quad hh = 0. \tag{2.6}$$

It is common to refer to the requirements (2.6) as *annihilation properties* or *side conditions*. We say a contraction (2.3) having M and N filtered chain complexes and ∇, g, h filtration preserving is *filtered*.

Remark 2.1. Given M, N, g, ∇, h, subject to (2.4) and (2.5) but not necessarily (2.6), substituting $g - gdh$ and hdh for g and h, we obtain a contraction in the strict sense, that is, the data satisfy (2.6) as well.

For later reference, we recall the ordinary perturbation lemma.

Lemma 2.2. *Let*

$$(M \underset{g}{\overset{\nabla}{\rightleftarrows}} N, h) \tag{2.7}$$

be a filtered contraction. Let ∂ be a perturbation of the differential on N, and let

$$\mathcal{D} = \sum_{n\geq 0} g\partial(h\partial)^n\nabla = \sum_{n\geq 0} g(\partial h)^n\partial\nabla \tag{2.8}$$

$$\nabla_\partial = \sum_{n\geq 0}(h\partial)^n\nabla \tag{2.9}$$

$$g_\partial = \sum_{n\geq 0} g(\partial h)^n \tag{2.10}$$

$$h_\partial = \sum_{n\geq 0}(h\partial)^n h = \sum_{n\geq 0} h(\partial h)^n. \tag{2.11}$$

When the filtrations on M and N are complete, these infinite series converge, the operator $\mathcal{D}$ is a perturbation of the differential on M and, with the notation N_∂ and $M_\mathcal{D}$ for the new chain complexes,

$$(M_\mathcal{D} \underset{g_\partial}{\overset{\nabla_\partial}{\rightleftarrows}} N_\partial, h_\partial) \tag{2.12}$$

constitute a new filtered contraction that is natural in terms of the given data.

Proof. See [6] or [19, 4.3 Lemma Section 4 p. 404]. □

Under the circumstances of Lemma 2.2, we refer to (2.12) as the *perturbed contraction*.

Remark 2.3. In (2.5), the sign of h is the same as in [19, Section 4 p. 403] and [32, Section 1 p. 247], opposite to that in [26, 2.2 p. 164] (in (2.2) of that paper, M and N should be exchanged). This explains the appearance of the minus sign in [26, (9.2)–(9.5), p. 183].

Let (C, η, Δ) be a *coaugmented* differential graded coalgebra and let $\mathrm{J}C = \operatorname{coker}(\eta)$ denote its *coaugmentation* coideal. Recall the counit $\varepsilon\colon C \to R$ and the coaugmentation map η determine a direct sum decomposition $C = R \oplus \mathrm{J}C$, and the diagonal Δ induces a diagonal $\mathrm{J}\Delta\colon \mathrm{J}C \to \mathrm{J}C \otimes \mathrm{J}C$. The ascending sequence

$$R \subseteq \cdots \subseteq \mathrm{F}_nC \subseteq \mathrm{F}_{n+1}C \subseteq \cdots \ (n \geq 1) \tag{2.13}$$

formed by the kernels

$$\mathrm{F}_nC = \ker(C \xrightarrow{\mathrm{pr}} \mathrm{J}C \xrightarrow{(\mathrm{J}\Delta)^{\otimes(n+1)}} (\mathrm{J}C)^{\otimes(n+1)}) \ (n \geq 0)$$

yields the *coaugmentation* filtration $\{\mathrm{F}_nC\}_{n\geq 0}$ of C, cf., e.g., [22, Section 1 p. 11], well known to turn C into a *filtered* coaugmented differential graded coalgebra; thus, in particular, $\mathrm{F}_0C = R$. We recall that C is said to be *cocomplete* when $C = \cup \mathrm{F}_nC$.

Write s for the *suspension* operator and accordingly s^{-1} for the *desuspension* operator. Thus, given the chain complex X, $(sX)_j = X_{j-1}$, etc., and the differential $d\colon sX \to sX$ on the suspended object sX is defined in the standard manner so that $ds + sd = 0$.

Consider a simplicial R-module W. We take its *Moore complex* $(\mathrm{Mo}(W), d)$ to be

$$\begin{aligned} \mathrm{Mo}(W)\colon \ \cdots \xrightarrow{d_{p+1}} \mathrm{Mo}_p(W) \xrightarrow{d_p} \cdots \xrightarrow{d_1} \mathrm{Mo}_0(W) = W_0,\\ \mathrm{Mo}_p(W) = \cap_{0\leq j<p} \ker(\partial_j\colon W_p \to W_{p-1}), \ \ p \geq 1,\\ d_p = \partial_p|_{\mathrm{Mo}_p(W)}, \ \ p \geq 1. \end{aligned} \tag{2.14}$$

The canonical injection $\iota\colon \mathrm{Mo}(W) \to W$ extends to a contraction

$$(\mathrm{Mo}(W) \underset{g}{\overset{\iota}{\rightleftarrows}} W, h) \tag{2.15}$$

that is natural in terms of the data, the kernel of g coincides with the degeneracy subcomplex $D(W)$ of W, and g induces an isomorphism $|W| = W/D(W) \to \mathrm{Mo}(W)$ from the normalized R-chain complex $|W|$ onto the Moore complex $\mathrm{Mo}(W)$ of W; see, e.g., [16, Ch. III Theorem 2.1 p. 146]. We identify $|W|$ with $\mathrm{Mo}(W)$.

3. Bundles in the Category of Chain Complexes

The classical notion of *twisting cochain* goes back to [5]; this notion arises by abstraction from properties of the transgression [7, 39, 40]. The defining equation of a twisting cochain also occurs in the literature as *master equation*; see [33] and the literature there.

Let C be a differential graded coalgebra and A a differential graded algebra. Write the unit as $\eta\colon R \to A$, the counit as $\varepsilon\colon C \to R$, the multiplication map of A as $\mu\colon A \otimes A \to A$, and the diagonal map of C as $\Delta\colon C \to C \otimes C$. The Hom differential D and *cup product*

$$\cup\colon \mathrm{Hom}(C, A) \otimes \mathrm{Hom}(C, A) \longrightarrow \mathrm{Hom}(C, A),$$
$$\alpha \cup \beta\colon C \xrightarrow{\Delta} C \otimes C \xrightarrow{\alpha\otimes\beta} A \otimes A \xrightarrow{\mu} A,\ \ \alpha, \beta\colon C \to A,$$

[35, II.1.1 Definition p. 135] turn $\mathrm{Hom}(C, A)$ into a differential graded algebra having unit the composite $\eta\varepsilon$. The cup product has also come to be known as *convolution product* [44, p. 6]. When C is coaugmented, with coaugmentation $\eta\colon R \to C$, and A augmented with augmentation $\varepsilon\colon A \to R$,

$$\mathrm{Hom}(C, A) \longrightarrow R,\ \ \varphi \mapsto \varepsilon\varphi\eta \tag{3.1}$$

is an augmentation map for $\mathrm{Hom}(C, A)$ as a differential graded algebra.

A (C, A)-*bundle* is a differential graded right A-module left C-comodule N together with an isomorphism $\lambda\colon N \to C \otimes A$ of graded right A-modules and left C-comodules, differentials being neglected, the right A-module and left C-comodule structures on $C \otimes A$ being the extended ones [35, II.1.6 p. 137]. For a morphism $\phi\colon A_1 \to A_2$ of differential graded algebras and a morphism $\psi\colon C_1 \to C_2$ of differential graded coalgebras, a (ϕ, ψ)-*morphism*

$$(C_1 \otimes A_1, d_1) \longrightarrow (C_2 \otimes A_2, d_2) \tag{3.2}$$

from the (C_1, A_1)-bundle $(C_1 \otimes A_1, d_1)$ to the (C_2, A_2)-bundle $(C_2 \otimes A_2, d_2)$ is a morphism of chain complexes which is, furthermore, a morphism of differential graded A_1-modules via $\phi\colon A_1 \to A_2$ and of differential graded C_2-comodules via $\psi\colon C_1 \to C_2$. With the obvious notions of composition of morphisms and identity, (C, A)-bundles and, more generally, bundles constitute a category. In particular, isomorphism classes of bundles are well defined.

For an augmented differential graded algebra (A, ε), we say a (C, A)-bundle $(C \otimes A, d)$ is *augmented* when $C \otimes \varepsilon\colon (C \otimes A, d) \to C$ is a morphism of differential graded C-comodules. For a coaugmented differential graded coalgebra (C, η), we say a (C, A)-bundle $(C \otimes A, d)$ is *coaugmented* when $\eta \otimes A\colon A \to (C \otimes A, d)$ is a morphism of differential graded A-modules, and we say a (C, A)-bundle $(C \otimes A, d)$ is *supplemented* when it is both augmented and coaugmented. Now, for a (ϕ, ψ)-morphism $(C_1 \otimes A_1, d_1) \to (C_2 \otimes A_2, d_2)$ of coaugmented bundles, the diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\ \ \phi\ \ } & A_2 \\ {\scriptstyle \eta\otimes\mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \eta\otimes\mathrm{Id}} \\ (C_1 \otimes A_1, d_1) & \longrightarrow & (C_2 \otimes A_2, d_2) \end{array} \tag{3.3}$$

is commutative and, for a (ϕ,ψ)-morphism $(C_1\otimes A_1,d_1)\to(C_2\otimes A_2,d_2)$ of augmented bundles, the diagram

$$\begin{array}{ccc} (C_1\otimes A_1,d_1) & \longrightarrow & (C_2\otimes A_2,d_2) \\ \Big\downarrow{\scriptstyle \mathrm{Id}\otimes\varepsilon} & & \Big\downarrow{\scriptstyle \mathrm{Id}\otimes\varepsilon} \\ C_1 & \xrightarrow[\psi]{} & C_2 \end{array} \tag{3.4}$$

is commutative. In practice, morphisms of bundles arising from simplicial fiber bundles are augmented; they are, furthermore, coaugmented after a choice of base point of the base.

We define a homogeneous degree -1 morphism $\tau\colon C\to A$ of the underlying graded R-modules to be a *twisting cochain* when

$$D\tau+\tau\cup\tau=0. \tag{3.5}$$

For a coaugmented differential graded coalgebra (C,η), we say a twisting cochain $\tau\colon C\to A$ is *coaugmented* when $\tau\eta=0$ and, for an augmented differential graded algebra (A,ε), we say a twisting cochain $\tau\colon C\to A$ is *augmented* when $\varepsilon\tau=0$. We refer to a twisting cochain that is both coaugmented and augmented as being *supplemented*.

Remark 3.1. This definition of a twisting cochain is that in [19, Section 2, p. 401]. The terminology in [35] is 'twisting morphism' for a supplemented twisting cochain, with opposite sign, and that in [24] and in the updated version [31] thereof is twisting cochain, still with the sign opposite to the present one. The present sign for a twisting cochain is most convenient for bundles of the kind $C\otimes A$ (as opposed to those of the kind $A\otimes C$).

Let $d^{\otimes}$ denote the tensor product differential. The cap product

$$\begin{gathered}\cap\colon \mathrm{Hom}(C,A)\otimes C\otimes A\longrightarrow C\otimes A,\\ \varphi\cap\cdot\colon C\otimes A\overset{\Delta\otimes A}{\longrightarrow}C\otimes C\otimes A\overset{C\otimes\varphi\otimes A}{\longrightarrow}C\otimes A\otimes A\overset{C\otimes\mu}{\longrightarrow}C\otimes A,\ \varphi\colon C\to A,\end{gathered} \tag{3.6}$$

[19, 2.3 Definitions p. 401] yields an action of $\mathrm{Hom}(C,A)$ on $C\otimes A$. For a twisting cochain $\tau\colon C\to A$, let $d^{\tau}=\cap\tau$. Then the sum

$$d^{\otimes}+d^{\tau}\colon C\otimes A\to C\otimes A \tag{3.7}$$

is a differential on $C\otimes A$ which, relative to the obvious structures, turns $C\otimes A$ into a (C,A)-bundle. It is common to denote this (C,A)-bundle by $C\otimes_\tau A$; we refer to it as an *A-principal twisted tensor product on C*. See, e.g., [35] for details. By construction, the A-principal twisted tensor product $C\otimes_\tau A$ on C is a (C,A)-bundle in a canonical manner, the right A-module and left C-comodule structures on $C\otimes A$ being the extended ones. The terminology in [19] is "principal twisted object". The following reproduces [19, 2.2 Proposition p. 400].

Proposition 3.2. *For a differential graded algebra A and a differential graded coalgebra C, the assignment to a bundle differential D on $C\otimes A$ of the degree -1 morphism*

$$\tau_D\colon C\overset{\mathrm{Id}\otimes\eta}{\longrightarrow}C\otimes A\overset{D}{\longrightarrow}C\otimes A\overset{\varepsilon\otimes\mathrm{Id}}{\longrightarrow}A \tag{3.8}$$

of the underlying graded objects establishes, in the general, augmented, coaugmented, and supplemented case, a bijection between twisting cochains from C to A and bundle differentials D on $C\otimes A$ in such a way that $(C\otimes A,D)=C\otimes_{\tau_D}A$. Thus any (C,A)-bundle structure on $C\otimes A$ is of the kind $C\otimes_\tau A$, for some uniquely determined twisting cochain $\tau\colon C\to A$. □

As before, let C be a differential graded coalgebra and A a differential graded algebra. Consider two twisting cochains $\tau_1,\tau_2\colon C\to A$. A homogeneous morphism $h\colon C\to A$ of the underlying graded modules having degree zero is a *homotopy of twisting cochains* from τ_1 to τ_2, written $h\colon\tau_1\simeq\tau_2$, when

$$\tau_2\cup h=h\cup\tau_1-Dh. \tag{3.9}$$

In the augmented case, a homotopy $h\colon\tau_1\simeq\tau_2$ of twisting cochains is *augmented* when $\varepsilon h=\varepsilon$, in the coaugmented case, a homotopy h of twisting cochains is *coaugmented* when $h\eta=\eta$ and, in the supplemented case, a homotopy h of twisting cochains is *supplemented* when it is both augmented and coaugmented. An augmented homotopy h necessarily satisfies the identity $\varepsilon(Dh) = 0$ and a coaugmented homotopy h satisfies the identity $(Dh)\eta = 0$.

For $h\colon \tau_1 \simeq \tau_2\colon C \to A$, the resulting identity

$$(Dh)\cap = (h \cup \tau_1) \cap -(\tau_2 \cup h)\cap = h \cap \tau_1 \cap - \tau_2 \cap h\cap$$

says that the diagram

$$\begin{array}{ccc} C \otimes A & \xrightarrow{h\cap} & C \otimes A \\ {\scriptstyle d^{\otimes}+\tau_1\cap}\downarrow & & \downarrow{\scriptstyle d^{\otimes}+\tau_2\cap} \\ C \otimes A & \xrightarrow[h\cap]{} & C \otimes A \end{array} \tag{3.10}$$

is commutative. Hence $h\cap\colon C \otimes_{\tau_1} A \to C \otimes_{\tau_2} A$ is an $(\mathrm{Id}, \mathrm{Id})$-morphism of bundles.

Remark 3.3. In [24] and in the updated version [31] thereof, the defining identity of a homotopy of twisting cochains $k\colon \tau_1 \cong \tau_2$ reads

$$\tau_1 \cup k = k \cup \tau_2 + Dk. \tag{3.11}$$

The following is again straightforward.

Proposition 3.4. *For a differential graded algebra A, a differential graded coalgebra C, and two twisting cochains $\tau_1, \tau_2\colon C \to A$, the assignment to an $(\mathrm{Id}, \mathrm{Id})$-bundle morphism*

$$\Psi\colon C \otimes_{\tau_1} A \to C \otimes_{\tau_2} A \tag{3.12}$$

of the degree 0 morphism

$$h_\Psi\colon C \stackrel{\mathrm{Id}\otimes\eta}{\longrightarrow} C \otimes A \stackrel{\Psi}{\longrightarrow} C \otimes A \stackrel{\varepsilon\otimes\mathrm{Id}}{\longrightarrow} A \tag{3.13}$$

of the underlying graded objects establishes, in the general, augmented, coaugmented, and supplemented case, a bijection between homotopies of twisting cochains from τ_1 to τ_2 and $(\mathrm{Id}, \mathrm{Id})$-bundle morphisms in such a way that

$$\Psi = h_\Psi\cap\colon C \otimes_{\tau_1} A \longrightarrow C \otimes_{\tau_2} A. \tag{3.14}$$

Thus any $(\mathrm{Id}, \mathrm{Id})$-bundle morphism from $C \otimes_{\tau_1} A$ to $C \otimes_{\tau_2} A$ is of the kind

$$h\cap\colon C \otimes_{\tau_1} A \longrightarrow C \otimes_{\tau_2} A,$$

for some uniquely determined homotopy $h\colon \tau_1 \simeq \tau_2$ of twisting cochains. □

Proposition 3.5. *Let A_1, A_2 be differential graded algebras, C_1, C_2 differential graded coalgebras, $\tau_1\colon C_1 \to A_1$ and $\tau_2\colon C_2 \to A_2$ twisting cochains, $\phi\colon A_1 \to A_2$ a morphism of augmented differential graded algebras, $\chi\colon C_1 \to C_2$ a morphism of coaugmented differential graded coalgebras, and $h\colon \phi\tau_1 \simeq \tau_2\chi : C_1 \to A_2$. Then the composite*

$$[\chi, h, \phi]\colon C_1 \otimes_{\tau_1} A_1 \stackrel{\mathrm{Id}\otimes\phi}{\longrightarrow} C_1 \otimes_{\phi\tau_1} A_2 \stackrel{h\cap}{\longrightarrow} C_1 \otimes_{\tau_2\chi} A_2 \stackrel{\chi\otimes\mathrm{Id}}{\longrightarrow} C_2 \otimes_{\tau_2} A_2 \tag{3.15}$$

is a (ϕ, χ)-morphism of bundles, and every (ϕ, χ)-morphism $C_1 \otimes_{\tau_1} A_1 \to C_2 \otimes_{\tau_2} A_2$ of bundles arises in this manner from a suitable homotopy $\phi\tau_1 \simeq \tau_2\chi : C_1 \to A_2$ of twisting cochains. This claim holds as well in the augmented, cooaugmented, and supplemented case.

Proof. Any (ϕ, χ)-morphism

$$C_1 \otimes_{\tau_1} A_1 \longrightarrow C_2 \otimes_{\tau_2} A_2$$

factors as

$$C_1 \otimes_{\tau_1} A_1 \stackrel{\mathrm{Id}\otimes\phi}{\longrightarrow} C_1 \otimes_{\phi\tau_1} A_2 \longrightarrow C_1 \otimes_{\tau_2\chi} A_2 \stackrel{\chi\otimes\mathrm{Id}}{\longrightarrow} C_2 \otimes_{\tau_2} A_2.$$

By Proposition 3.4, the middle bundle morphism is of the kind

$$C_1 \otimes_{\phi\tau_1} A_2 \stackrel{h\cap}{\longrightarrow} C_1 \otimes_{\tau_2\chi} A_2,$$

for a unique homotopy $\phi\tau_1 \simeq \tau_2\chi : C_1 \to A_2$ of twisting cochains. □

When a homotopy $h\colon \tau_1 \simeq \tau_2$, viewed as a homogeneous degree zero member of the algebra $(\mathrm{Hom}(C,A),\cup)$, is invertible, (3.9) is equivalent to

$$\tau_2 = h \cup \tau_1 \cup h^{-1} - (Dh) \cup h^{-1}. \tag{3.16}$$

As before, let C be a differential graded coalgebra and A a differential graded algebra. Let $\mathcal{T}(C,A)$ denote the set of twisting cochains from C to A. The invertible members φ of $\mathrm{Hom}(C,A)$ (invertible degree zero morphisms of the underlying graded R-modules) form a group $\mathcal{G}(C,A)$, and a straightforward verification shows that the association

$$\begin{gathered}\mathcal{G}(C,A) \times \mathrm{Hom}(C,A) \longrightarrow \mathrm{Hom}(C,A),\\ (\varphi,\rho) \mapsto \varphi * \rho = \varphi \cup \rho \cup \varphi^{-1} - (D\varphi)\varphi^{-1}\end{gathered} \tag{3.17}$$

yields an action of $\mathcal{G}(C,A)$ on $\mathcal{T}(C,A)$.

The assignment to (C,A) of the $\mathcal{G}(C,A)$-orbits $\mathcal{D}(C,A) = \mathcal{T}(C,A)/\mathcal{G}(C,A)$ is Berikashvili's functor under the present circumstances. In the same vein, in the augmented, coaugmented, and supplemented case, let $\mathcal{T}_{\mathrm{aug}}(C,A)$, $\mathcal{T}_{\mathrm{coaug}}(C,A)$, $\mathcal{T}_{\mathrm{supp}}(C,A)$, denote the set of, respectively, augmented, coaugmented, supplemented twisting cochains and let

$$\begin{aligned}\mathcal{G}_{\mathrm{aug}}(C,A) &= \{\varphi; \varepsilon\varphi = \varepsilon\} \subseteq \mathcal{G}(C,A), & \mathcal{D}_{\mathrm{aug}}(C,A) &= \mathcal{T}_{\mathrm{aug}}(C,A)/\mathcal{G}_{\mathrm{aug}}(C,A),\\ \mathcal{G}_{\mathrm{coaug}}(C,A) &= \{\varphi; \varphi\eta = \eta\} \subseteq \mathcal{G}(C,A), & \mathcal{D}_{\mathrm{coaug}}(C,A) &= \mathcal{T}_{\mathrm{coaug}}(C,A)/\mathcal{G}_{\mathrm{coaug}}(C,A),\\ \mathcal{G}_{\mathrm{supp}}(C,A) &= \{\varphi; \varepsilon\varphi = \varepsilon, \varphi\eta = \eta\} \subseteq \mathcal{G}(C,A), & \mathcal{D}_{\mathrm{supp}}(C,A) &= \mathcal{T}_{\mathrm{supp}}(C,A)/\mathcal{G}_{\mathrm{supp}}(C,A).\end{aligned}$$

Theorem 3.6. *For a differential graded coalgebra C and a differential graded algebra A, the assignment to a twisting cochain $t\colon C \to A$ of the A-principal twisted tensor product $C \otimes_t A$ on C induces a bijection between $\mathcal{D}(C,A)$, $\mathcal{D}_{\mathrm{aug}}(C,A)$, $\mathcal{D}_{\mathrm{coaug}}(C,A)$, $\mathcal{D}_{\mathrm{supp}}(C,A)$ and isomorphism classes of, respectively, general, augmented, coaugmented, and supplemented (C,A)-bundles. Thus the value $\mathcal{D}(C,A)$ of Berikashvili's functor $\mathcal{D}$ on (C,A) parametrizes isomorphism classes of (C,A)-bundles.*

Proof. Proposition 3.4 implies that homotopic twisting cochains determine isomorphic (C,A)-bundles, that is, the map from $\mathcal{D}(C,A)$ to the set of isomorphism classes of (C,A)-bundles is well defined and that, furthermore, this map is injective. Proposition 3.2 implies that this map is surjective. □

Proposition 3.7. *For a cocomplete coaugmented differential graded coalgebra (C,η) and a differential graded algebra A, a degree zero morphism $h\colon C \to A$ such that $h\eta = \eta\colon R \to A$ is invertible, i.e., belongs to $\mathcal{G}_{\mathrm{coaug}}(C,A)$.*

Proof. Write $h = \eta\varepsilon + \widetilde{h}\colon C \to A$ such that $\widetilde{h}\eta = 0$. Then $h^{-1} = \eta\varepsilon + \sum_{j\geq 1}(-\widetilde{h})^{\cup j}$.

Indeed, for $p \geq 1$, the term $\widetilde{h}^{\cup p}$ is zero on $\mathrm{F}_{p-1}C$ whence, restricted to $\mathrm{F}_{p-1}C$, the infinite sum $\eta\varepsilon + \sum_{j\geq 1}(-\widetilde{h})^{\cup j}$ has only finitely many non-zero terms. This implies the claim since C is cocomplete, i.e., $C = \cup \mathrm{F}_j C$. □

Thus, for a cocomplete coaugmented coalgebra C and a differential graded algebra A, there is no need to distinguish between the members of $\mathcal{G}_{\mathrm{coaug}}(C,A)$ and coaugmented homotopies of coaugmented twisting cochains from C to A, and this is, likewise, true in the supplemented case.

4. Enriched Categories, dg Categories and dg Cocategories

Henceforth we take every object, e.g., category, cocategory, etc. upon which we carry out an algebraic construction to be *small* without explicitly saying so. Thus a (directed) graph (precategory, or quiver) of the kind mentioned above is supposed to be small. However the universe, that is, the closed monoidal category of graphs (or quivers or precategories), enriched in the closed monoidal category of chain complexes over R, is not taken to be small. For more details, the reader may consult, e.g., [17, 38, 43].

A small category is an *interpretation* [50] of the category axioms within set theory, cf. [42]. Thus an (oriented) *graph* (O,A,s,t) consists of a set O of *objects*, a set A of *arrows*, and two maps $s,t\colon A \to O$, the map s being referred to as *source* and the map t as *target* map. The *product*

$$A \times_O A = \{(g,f); g,f \in A,\ t(g) = s(f)\}$$

of A with itself *over* O is the set of *composable* arrows. When the set O is fixed, it is common to refer to the graph (O, A, s, t) as an *O-graph* [42]. A *morphism* of graphs and, likewise, a *morphism* of O-graphs, is defined in the obvious way. With this notion of morphism, graphs constitute a category $\mathcal{G}$ and O-graphs form a subcategory $\mathcal{G}_O$ thereof. A graph is *discrete* when its only arrows are the identity maps between objects, so that the set of arrows coincides with its set of objects and so that the source and target maps are necessarily the identity. A set O determines a unique discrete graph $(O, O, \mathrm{Id}, \mathrm{Id})$ in an obvious way and, with a slight abuse of notation, we denote this graph by O as well and refer to it as the *discrete graph* O.

A (small) *category* is a graph (O, A, s, t) together with two maps

$$\mathrm{Id}\colon O \longrightarrow A, \quad c\colon A \times_O A \longrightarrow A,$$

referred to as *identity* and *composition*, subject to the familiar constraints. The set of arrows A is then commonly referred to as that of *morphisms*. Thus, relative to the product over objects, a category is a *monoid in the category of graphs*. A *cocategory* is, likewise, a comonoid in the category of graphs. The discrete graph $O = (O, O, \mathrm{Id}, \mathrm{Id})$ is a category in an obvious manner, the *discrete category*. A *topological* category is a small category having as objects and morphisms topological spaces with continuous structure maps. A *smooth* category is, likewise, a small category having as objects and morphisms smooth manifolds with smooth structure maps.

Let $R\mathcal{G}$ denote the category of graphs enriched in the closed monoidal category Mod_R of R-modules. An *R-graph* is an object of the category $R\mathcal{G}$. Given the two R-graphs $\mathcal{A}$ and $\mathcal{B}$, a *morphism* $f\colon \mathcal{A} \to \mathcal{B}$ of *R-graphs* is defined in the obvious way: It consists of a map

$$\mathrm{Ob}(f)\colon \mathrm{Ob}\mathcal{A} \longrightarrow \mathrm{Ob}\mathcal{B}$$

and, for each ordered pair (x, y) of objects of $\mathcal{A}$, of a morphism

$$f_{x,y}\colon \mathcal{A}(x, y) \longrightarrow \mathcal{B}(fx, fy)$$

of R-modules. We refer to an R-graph having object set O as an *RO-graph*. Plainly, RO-graphs constitute a subcategory of $R\mathcal{G}$.

A particular RO-graph $R[O]$ arises from the discrete O-graph determined by O, with source and target maps the identity map of O; we refer to this R-graph as the *discrete* RO-graph. With the obvious notions of composition and identity, $R[O]$ becomes a category, indeed, $R[O]$ acquires a *ringoid* structure in an obvious manner and, with the obvious interpretation of the term "module", a general RO-graph is then a *module* over $R[O]$; frequently we will use the terminology $R[O]$-*module* rather than *module* over $R[O]$. Thus $R[O]$-module and RO-graph are synonymous notions. We can realize the ringoid $R[O]$ as a functor from O to the category Mod_R of R-modules.

Likewise, let $\mathrm{Chain}_{R\mathcal{G}}$ be the closed monoidal category of *graphs* enriched in the closed monoidal category Chain_R of R-chain complexes. With these preparations out of the way, an *R-category* is a unital associative algebra in the closed monoidal category $R\mathcal{G}$ of R-graphs; a *differential graded R-category* (dg category) is a unital associative algebra in the category $\mathrm{Chain}_{R\mathcal{G}}$. We will also refer to an object of $\mathrm{Chain}_{R\mathcal{G}}$ having object set O as an $R[O]$-*chain complex*.

The discrete R-graph $R[O]$ acquires obvious R-category, R-cocategory and, more generally, dg category and dg cocategory structures. The dg category $\mathcal{A}$ with object set O being *unital* signifies that the unit $\eta\colon R[O] \to \mathcal{A}$ is a morphism of dg categories. This unit encodes of course the identities in $\mathcal{A}$. In the same vein, a (counital) *differential graded* R-cocategory (dg cocategory) is a counital coassociative coalgebra in the category $\mathrm{Chain}_{R\mathcal{G}}$. The dg cocategory $\mathcal{C}$ with object set O being *counital* signifies that the counit $\varepsilon\colon \mathcal{C} \to R[O]$ is a morphism of dg cocategories. A dg category $\mathcal{A}$ with object set O endowed with a morphism $\varepsilon\colon \mathcal{A} \to R[O]$ of dg categories such that $\varepsilon\eta$ is the identity of the dg category $R[O]$ is defined to be *augmented*; likewise a dg cocategory $\mathcal{C}$ with object set O endowed with a morphism $\eta\colon R[O] \to \mathcal{C}$ of dg cocategories such that $\varepsilon\eta$ is the identity of the dg cocategory $R[O]$ is defined to be *coaugmented*.

5. The Nerve of a Category

5.1. **Preliminaries.** Let Ord denote the category of finite ordered sets $[p] = (0, 1, \dots, p)$, $p \geq 0$, and monotone maps. Let $\mathcal{C}$ be a category. A *simplicial object* in $\mathcal{C}$ is a contravariant functor from Ord to $\mathcal{C}$; a *cosimplicial object* in $\mathcal{C}$ is a (covariant) functor from Ord to $\mathcal{C}$. The assignment to $[p]$ ($p \geq 0$) of the standard simplex $\nabla[p]$ yields a cosimplicial space ∇.

Let $\mathcal{C}$ be a category, not necessarily small. Let X be an object of $\mathcal{C}$. This object defines a "trivially" simplicial object in $\mathcal{C}$, and we denote this simplicial object by X again. It has $X_p = X$, for $p \geq 0$, and every arrow the identity.

5.2. **The nerve.** The *nerve* $N\mathcal{C}$ of a small category $\mathcal{C}$ is the simplicial set having the objects as vertices, the morphisms as edges, the triangular commutative diagrams as 2-simplicies, etc. More formally, the nerve arises in the following way [49]: Regard an ordered set S as a category with S as set of objects and with just one morphism from $x \in S$ to $y \in S$ whenever $x \leq y$. Given the category $\mathcal{C}$, let $N\mathcal{C}(S)$ be the set of functors from S to $\mathcal{C}$. As S ranges over the finite ordered sets, this construction yields the *nerve* of the category $\mathcal{C}$.

For convenience, here is the standard elementary description of the face and degeneracy operators on the degree p constituent $N_p(\mathcal{C}) = \mathrm{Mor}_{\mathcal{C}} \times_O \dots \times_O \mathrm{Mor}_{\mathcal{C}}$ ($p \geq 1$ factors) of the nerve $N(\mathcal{C})$ of $\mathcal{C}$, with $N_0(\mathcal{C}) = O$, and with the notation $[x_0|\dots|x_{p-1}] \in N_p(\mathcal{C})$ for $p \geq 1$ for the members of $N_p(\mathcal{C})$, in particular p composable morphisms for $p \geq 2$:

$$
\begin{aligned}
\partial_0[x] &= s(x) \in O, \\
\partial_1[x] &= t(x) \in O, \\
s_0(y) &= [\mathrm{Id}_y] \in \mathrm{Mor}_{\mathcal{C}}(y, y), \\
\partial_j[x_0|\dots|x_{p-1}] &= \begin{cases} [x_1|\dots|x_{p-1}], & j = 0, \\ [x_0|\dots|x_{j-1}x_j|\dots|x_{p-1}], & 1 \leq j \leq p-1, \\ [x_0|\dots|x_{p-2}], & j = p, \end{cases} \\
s_j[x_0|\dots|x_{p-1}] &= [x_0|\dots|x_{j-1}|\mathrm{Id}_{t(x_{j-1})}|x_j|\dots|x_{p-1}], \ 0 \leq j \leq p.
\end{aligned}
\tag{5.1}
$$

5.3. **Bar and cobar constructions.** Let $\mathcal{A}$ be an augmented small dg category having object set O. Its *nerve* $N\mathcal{A}$ carried out relative to the operation of taking the tensor product over $R[O]$ (the appropriate coend) is a simplicial differential graded $R[O]$-module, that is, a simplicial object in $\mathrm{Chain}_{R\mathcal{G}}$; cf. [42, IX.6 p. 226 ff.] for the notion of coend. Condensation, that is, totalization and normalization, yields the differential graded $R[O]$-module

$$\mathcal{B}\mathcal{A} = |N\mathcal{A}|, \tag{5.2}$$

by construction, a differential graded $R[O]$-graph having, in particular, O as its set of objects. The ordinary *Alexander-Whitney* diagonal Δ turns $\mathcal{B}\mathcal{A}$ into a dg cocategory having O as its set of objects. The resulting dg cocategory $\mathcal{B}\mathcal{A}$ is the *reduced normalized bar construction* for $\mathcal{A}$. When O consists of a single element so that $\mathcal{A}$ is an ordinary augmented dg algebra, the cocategory $\mathcal{B}\mathcal{A}$ has a single object and is the ordinary reduced normalized bar construction for $\mathcal{A}$.

An alternate construction of the reduced normalized bar construction relies on the observation that, given the set O and the dg category $\mathcal{A}$ with object set O, the functor which assigns to a dg $R[O]$-module the induced $\mathcal{A}$-module is left adjoint to the forgetful functor, and the adjunction determines a comonad. The bar construction then arises from the associated standard construction.

In the same vein, let $\mathcal{C}$ be a coaugmented small dg cocategory having object set O . The construction dual to the bar construction yields the dg category $\Omega\mathcal{C}$, the *reduced normalized cobar construction* for $\mathcal{C}$. When O consists of a single element so that $\mathcal{C}$ is an ordinary coaugmented dg coalgebra, $\Omega\mathcal{C}$ is the ordinary reduced normalized cobar construction for $\mathcal{C}$. Similarly as before, an alternate construction relies on the observation that the appropriate adjunction determines a monad. The cobar construction then arises from the associated dual standard construction.

5.4. **The path object, its nerve, and twisted objects.** Let $\mathcal{C}$ be a small category. Let X be an object of $\mathcal{C}$. Let $\mathcal{P}X$ be the category having $\mathrm{Ob}(\mathcal{P}X) = X$ and $\mathrm{Mor}(\mathcal{P}X) = X \times X$, with

$$\begin{gathered} s,t\colon X \times X \longrightarrow X,\ s(x_1,x_2) = x_1,\ \ t(x_1,x_2) = x_2,\ \ x_1,x_2 \in X, \\ \mathrm{Id}\colon X \longrightarrow X \times X,\ \mathrm{Id}(x) = (x,x),\ \ x \in X, \\ c\colon \mathrm{Mor}(\mathcal{P}X) \times_X \mathrm{Mor}(\mathcal{P}X) \longrightarrow \mathrm{Mor}(\mathcal{P}X),\ \ c((x_1,x_2),(x_2,x_3)) = (x_1,x_3),\ \ x_1,x_2,x_3 \in X. \end{gathered}$$

Thus $\mathcal{P}X$ has a unique morphism between each pair of members of X. To have a name, we refer to $\mathcal{P}X$ as the *path category* associated to X.

Let $PX = N\mathcal{P}X$, the nerve of $\mathcal{P}X$. This is the familiar simplical object in $\mathcal{C}$ having $(PX)_p = X^{\times(p+1)}$, for $p \geq 0$, with the standard face and degeneracy maps

$$\begin{aligned} \partial_j(y_0,\dots,y_p) &= \begin{cases} (y_1,\dots,y_p), & j=0, \\ (y_0,\dots,y_{j-1},y_{j+1},\dots,y_p), & 1 \leq j \leq p-1, \\ (y_0,\dots,y_{p-1}), & j=p, \end{cases} \\ s_j(y_0,\dots,y_p) &= (y_0,\dots,y_{j-1},y_j,y_j,y_{j+1},\dots,y_p),\ 0 \leq j \leq p, \end{aligned} \tag{5.3}$$

cf., e.g., [3, Section 2 p. 294], also known as the *path object* associated to X.

The path category $\mathcal{PC}$ of $\mathcal{C}$ has

$$\begin{aligned} \mathrm{Ob}(\mathcal{PC}) &= \mathrm{Mor}_{\mathcal{C}} \\ \mathrm{Mor}(\mathcal{PC}) &= \mathrm{Mor}_{\mathcal{C}} \times \mathrm{Mor}_{\mathcal{C}},\ \text{written as } \{(y_0,y_1),\ y_0,y_1 \in \mathrm{Mor}_{\mathcal{C}}\} \\ s&\colon \mathrm{Mor}_{\mathcal{C}} \times \mathrm{Mor}_{\mathcal{C}} \to \mathrm{Mor}_{\mathcal{C}},\ s(y_0,y_1) = y_0, \\ t&\colon \mathrm{Mor}_{\mathcal{C}} \times \mathrm{Mor}_{\mathcal{C}} \to \mathrm{Mor}_{\mathcal{C}},\ t(y_0,y_1) = y_1, \\ \mathrm{Id}&\colon \mathrm{Mor}_{\mathcal{C}} \to \mathrm{Mor}_{\mathcal{C}} \times \mathrm{Mor}_{\mathcal{C}},\ \mathrm{Id}(y) = (y,y) \\ c&\colon (\mathrm{Mor}_{\mathcal{C}} \times \mathrm{Mor}_{\mathcal{C}}) \times_{\mathrm{Mor}_{\mathcal{C}}} (\mathrm{Mor}_{\mathcal{C}} \times \mathrm{Mor}_{\mathcal{C}}) \to \mathrm{Mor}_{\mathcal{C}},\ c((y_0,y_1),(y_1,y_2)) = (y_0,y_2). \end{aligned}$$

Then $P\mathcal{C} = N\mathcal{PC}$ is the simplicial category having $\mathrm{Ob}_p(P\mathcal{C}) = O^{\times(p+1)}$ and $\mathrm{Mor}_p(P\mathcal{C}) = \mathrm{Mor}_{\mathcal{C}}^{(p+1)}$, for $p \geq 0$.

For $p \geq 1$, let $N_p(\mathcal{C}) \times_O \mathrm{Mor}_{\mathcal{C}}$ denote the set of $(p+1)$-tuples $[x_0|x_1|\dots|x_{p-1}]x$ of composable morphisms in $\mathcal{C}$ and interpret $N_0(\mathcal{C}) \times_O \mathrm{Mor}_{\mathcal{C}}$ as $\mathrm{Mor}_{\mathcal{C}}$ in the obvious way. The map

$$\begin{gathered} N_p(\mathcal{C}) \times_O \mathrm{Mor}_{\mathcal{C}} = \mathrm{Mor}_{\mathcal{C}}^{\times_O p} \times_O \mathrm{Mor}_{\mathcal{C}} \longrightarrow \mathrm{Mor}_{\mathcal{C}}^{\times(p+1)} = \mathrm{Mor}_p(P\mathcal{C}) \\ [x_0|x_1|\dots|x_{p-1}]x \mapsto (y_0,\dots,y_p) \\ (y_0,\dots,y_p) = (x_0x_1\dots x_{p-1}x, x_1\dots x_{p-1}x,\dots,x_{p-1}x,x) \end{gathered} \tag{5.4}$$

is well defined. Setting, for $p \geq 0$,

$$\begin{aligned} \partial_j[x_0|\dots|x_{p-1}]x &= \begin{cases} [x_1|\dots|x_{p-1}]x, & j=0, \\ [x_0|\dots|x_{j-1}x_j|\dots|x_{p-1}]x, & 1 \leq j \leq p-1, \\ [x_0|\dots|x_{p-2}]x_{p-1}x, & j=p, \end{cases} \\ s_j[x_0|\dots|x_{p-1}]x &= [x_0|\dots|x_{j-1}|\mathrm{Id}_{t(x_{j-1})}|x_j|\dots|x_{p-1}]x,\ 0 \leq j \leq p, \end{aligned} \tag{5.5}$$

defines a simplicial structure on $N(\mathcal{C}) \times_O \mathrm{Mor}_{\mathcal{C}} = (N_p(\mathcal{C}) \times_O \mathrm{Mor}_{\mathcal{C}})_{p\geq 0}$, and the canonical projection

$$N(\mathcal{C}) \times_O \mathrm{Mor}_{\mathcal{C}} \longrightarrow N(\mathcal{C}) \tag{5.6}$$

and the maps (5.4) are compatible with the simplicial structures. We say that $N(\mathcal{C}) \times_O \mathrm{Mor}_{\mathcal{C}}$ is the *principal simplicial twisted object associated to* $\mathcal{C}$.

The degeneracy operators of $N(\mathcal{C})$ plainly determine those of $N(\mathcal{C}) \times_O \mathrm{Mor}_{\mathcal{C}}$, and this is also true of the face operators apart from the last one. For $p \geq 1$, the last face operator $\partial_p\colon N_p(\mathcal{C}) \to N_{p-1}(\mathcal{C})$ together with the map

$$\begin{gathered} \rho_p\colon N_p(\mathcal{C}) \longrightarrow \mathrm{Mor}_{\mathcal{C}}, \\ \rho_p[x_0|\dots|x_{p-1}] = x_{p-1}, \end{gathered} \tag{5.7}$$

determines the last face operator of $N(\mathcal{C})\times_O \mathrm{Mor}_{\mathcal{C}}$ as

$$\begin{gathered}\partial^\rho_p\colon N_p(\mathcal{C})\times_O \mathrm{Mor}_{\mathcal{C}} \longrightarrow N_{p-1}(\mathcal{C})\times_O \mathrm{Mor}_{\mathcal{C}}\\ \partial^\rho_p[x_0|\ldots|x_{p-1}]x = \partial_p[x_0|\ldots|x_{p-1}]\rho_p[x_0|\ldots|x_{p-1}]x.\end{gathered} \tag{5.8}$$

The sequence $\rho_1,\rho_2,\ldots$ of maps $\rho_p\colon N_p(\mathcal{C})\to \mathrm{Mor}_{\mathcal{C}}$ $(p\geq 1)$ enjoys the properties

$$\rho(\partial_p b)\rho(b) = \rho(\partial_{p-1}(b)),\ b\in N_p(\mathcal{C}),\ p\geq 2, \tag{5.9}$$

$$\rho(s_p(b)) = \mathrm{Id}_{t(\rho(b))}\in \mathrm{Mor}_{\mathcal{C}}(t(\rho(b),t(\rho(b)),\ b\in N_p(\mathcal{C}),\ p\geq 1. \tag{5.10}$$

For a simplicial principal bundle, such a map has come to be known as a *twisting function*, cf. (6.1) and (6.2) in Section 6 below and [1,9], [19, p. 406], [36]. For $p\geq 2$, the commutative diagram

$$\begin{array}{ccccccc} N_p(\mathcal{C}) & \xrightarrow{\Delta} & N_p(\mathcal{C})\times N_p(\mathcal{C}) & \xrightarrow{(\partial^{\mathrm{last}},\rho_p)} & N_{p-1}(\mathcal{C})\times_O \mathrm{Mor}_{\mathcal{C}} & \xrightarrow{(\rho,\partial^{\mathrm{last}})} & \mathrm{Mor}_{\mathcal{C}}\times_O \mathrm{Mor}_{\mathcal{C}}\\ \big\downarrow{\scriptstyle \partial^{\text{next to last}}} & & & & & & \big\downarrow{\scriptstyle c}\\ N_{p-1}(\mathcal{C}) & & & \xrightarrow[\rho_{p-1}]{} & & & \mathrm{Mor}_{\mathcal{C}} \end{array} \tag{5.11}$$

depicts property (5.9). We refer to ρ as the *universal twisting function for* $\mathcal{C}$ and write the principal simplicial twisted object $N(\mathcal{C})\times_O \mathrm{Mor}_{\mathcal{C}}$ associated to $\mathcal{C}$ as $N(\mathcal{C})\times_\rho \mathrm{Mor}_{\mathcal{C}}$.

5.5. **Group case.** Let G be a group, discrete, topological, or a Lie group, and consider G as a category with a single object having each morphism an isomorphism. Below we simultaneously treat the discrete, topological and smooth cases without further mention. With G substituted for $\mathcal{C}$ and for $\mathrm{Mor}_{\mathcal{C}}$, consider the universal twisting function $\rho\colon N(G)\to G$ for G; it has constituents $\rho_p\colon N_p(G)\to G$ $(p\geq 1)$, and this twisting function determines the total space $N(G)\times_\rho G$ of the simplicial principal (right) G-bundle

$$N(G)\times_\rho G\to N(G) \tag{5.12}$$

(discrete, topological, smooth).

Consider the path category $\mathcal{P}G$ of G. Segal [49] writes this category as $\overline{G}$. The group G acts freely on $\mathcal{P}G$ from the right, by right translation on $\mathrm{Ob}(\mathcal{P}G)=G$ and diagonal right translation on $\mathrm{Mor}(\mathcal{P}G)=G\times G$. The association

$$(G\times G,G)\longrightarrow (G,\{e\}),\ (x_1,x_2,x)\mapsto (x_1x_2^{-1},e),\ x_1,x_2,x\in G, \tag{5.13}$$

determines a smooth functor $\Pi\colon \mathcal{P}G\to G$ inducing an isomorphism $(\mathcal{P}G)/G\to G$ of categories, a homeomorphism in the topological case and a diffeomorphism in the smooth case. Taking nerves, we obtain the simplicial principal right G-bundle

$$N\Pi\colon N(\mathcal{P}G)\longrightarrow NG. \tag{5.14}$$

The (lean) geometric realization of $N\Pi$ yields the universal principal G-bundle $EG\to BG$ over the classifying space BG [49], see also [11]. The geometric realization of $N(\mathcal{P}G)$ is contractible for completely formal reasons.

The constituent $N_0\Pi\colon N_0(\mathcal{P}G)\to N_0G$ is the trivial map $G\to\{e\}$, viewed as a principal G-bundle and, for $q\geq 1$, the constituent $N_q\Pi\colon N_q(\mathcal{P}G)\to N_qG$ is the principal right G-bundle

$$N_q\Pi\colon G^{q+1}\longrightarrow G^q,\ (y_0,y_1,\ldots,y_q)\mapsto (y_0y_1^{-1},y_1y_2^{-1},\ldots,y_{q-1}y_q^{-1}). \tag{5.15}$$

By construction, the group G acts by diagonalwise right translation, that is

$$G^{q+1}\times G\longrightarrow G^{q+1},\ (y_0,y_1,\ldots,y_q,y)\mapsto (y_0y,y_1y,\ldots,y_qy). \tag{5.16}$$

The simplicial morphism (5.4) now takes the form

$$N(G)\times_\rho G\longrightarrow N(\mathcal{P}G), \tag{5.17}$$

is plainly G-equivariant, and has inverse

$$\begin{gathered}N(\mathcal{P}G)\longrightarrow N(G)\times_\rho G\\ (y_0,\ldots,y_p)\longmapsto [y_0y_1^{-1}|y_1y_2^{-1}|\ldots|y_{p-1}y_p^{-1}]y_p,\ p\geq 0.\end{gathered} \tag{5.18}$$

Thus (5.17) yields an isomorphism of simplicial principal right G-bundles from (5.12) to (5.14).

Remark 5.1. For the bar resolution, it is common to refer to the $(p+1)$-tuples written above as $(y_0, y_1, \ldots, y_p) \in G^{p+1}$ as *homogeneous generators* and to those of the kind $[x_0|x_1|\ldots|x_{p-1}]x$ as *non-homogeneous generators* [41, IV.5 p. 119].

6. Simplicial Principal Bundles

Let B be a simplicial set and K a simplicial group. A *twisting function* $\rho\colon B \to K$ consists of a sequence $\rho_1, \rho_2, \ldots$ of maps $\rho_p\colon B_p \to K_{p-1}$ $(p \geq 1)$ subject to

$$\rho(\partial_p b)\partial_{p-1}(\rho(b)) = \rho(\partial_{p-1}(b)),\ b \in B_p,\ p \geq 2, \tag{6.1}$$

$$\rho(s_p(b)) = e \in K_p,\ b \in B_p,\ p \geq 1, \tag{6.2}$$

cf. [1,9], [19, p. 406], [36]. Similarly as before, the commutative diagram

$$\begin{array}{ccccccc} B & \xrightarrow{\Delta} & B \times B & \xrightarrow{(\partial^{\text{last}},\rho)} & B \times K & \xrightarrow{(\rho,\partial^{\text{last}})} & K \times K \\ \Big\downarrow{\scriptstyle \partial^{\text{next to last}}} & & & & & & \Big\downarrow{\scriptstyle \mu} \\ B & & & \xrightarrow[\rho]{\qquad\qquad\qquad} & & & K \end{array} \tag{6.3}$$

depicts property (6.1).

Remark 6.1. The terminology 'twisting function' is consistent with the terminology in Subsection 5.5 relative to an ordinary group via the assignment to an ordinary group of its associated trivially simplicial group.

Let $\rho\colon B \to K$ be a twisting function. The *twisted cartesian product* $B \times_\rho K$ is the simplicial K-set having $B \times K$ as underlying graded object and

$$\partial_j\colon B_p \times K_p \longrightarrow B_{p-1} \times K_{p-1},\ p \geq 1,$$

$$\partial_j(u,x) = \begin{cases} (\partial_j(u), \partial_j(x)), & 0 \leq j < p, \\ (\partial_p(u), \rho_p(u)\partial_p(x)), & j = p, \end{cases}$$

$$s_j\colon B_p \times K_p \longrightarrow B_{p+1} \times K_{p+1},\ p \geq 0,$$

$$s_j(u,x) = (s_j(u), s_j(x)),\ 0 \leq j \leq p,$$

[1,9], [19, p. 406], [36]. The twisted cartesian product $B \times_\rho K$ is the total space of the resulting simplicial principal K-bundle $\mathrm{pr}_B\colon B \times_\rho K \to B$. Every simplicial principal (right) K-bundle arises in this manner [1,9,36]; see also Remark 6.13 below.

Proposition 6.2. *For a simplicial set B and a simplicial group K, the assignment to a simplicial principal bundle structure on $B \times K$ of the degree -1 morphism*

$$\rho\colon B \stackrel{\mathrm{Id}\times\{e\}}{\longrightarrow} B \times K \stackrel{\partial^{\text{last}}}{\longrightarrow} B \times K \stackrel{\mathrm{pr}_K}{\longrightarrow} K \tag{6.4}$$

of the underlying graded objects establishes a bijection between twisting functions from B to K and simplicial principal bundle structures $B \times K$ in such a way that $B \times_\rho K$ recovers the simplicial principal bundle structure. Thus every simplicial principal bundle structure is of the kind $B \times_\rho K$, for some uniquely determined twisting function $\rho\colon B \to K$. □

Remark 6.3. As in Subsection 5.5 above, we here give preferred treatment to the *last* face operator, as in [21] and [36,37]. This procedure is appropriate for principal bundles with structure group acting on the total space from the *right* and simplifies comparison with the bar construction.

The degree zero morphisms $\mathrm{Mor}_0(B,K)$ of the underlying graded sets from B to K form a group under pointwise multiplication. Let $\mathrm{Mor}_{-1}(B,K)$ denote the degree -1 morphisms of the underlying graded sets from B to K. The association

$$\mathrm{Mor}(B,K) \times \mathrm{Mor}_{-1}(B,K) \longrightarrow \mathrm{Mor}_{-1}(B,K),\ (\vartheta,\rho) \mapsto \vartheta * \rho$$

$$\vartheta * \rho\colon\ B \xrightarrow{\ \ \Delta\ \ } B \times B \times B \xrightarrow{(\partial^{\mathrm{last}},\rho,\vartheta)} B \times K \times K \xrightarrow{(\vartheta,\mathrm{Id},\partial^{\mathrm{last}})} K \times K \times K \xrightarrow{(\mathrm{left},\mathrm{right}^{-1})} K, \tag{6.5}$$

in formulas,

$$(\vartheta * \rho)(b) = \vartheta(\partial_p(b))\rho(b)\vartheta(\partial_p(b))^{-1},\ b \in B_p, \tag{6.6}$$

yields an action of $\mathrm{Mor}(B,K)$ on $\mathrm{Mor}_{-1}(B,K)$.

Lemma 6.4. *Let $\rho_1\colon B \to K$ be a twisting function, let $\vartheta\colon B \to K$ be a degree zero morphism of the underlying graded sets, and let $\rho_2 = \vartheta * \rho_1\colon B \to K$. Then ρ_2 is a twisting function if and only if*

$$\partial^{\mathrm{last}}\vartheta\partial^{\mathrm{ntlast}} = \partial^{\mathrm{last}}\partial^{\mathrm{last}}\vartheta \tag{6.7}$$

or, equivalently,

$$\partial_{p-1}\vartheta(\partial_{p-1}(b)) = \partial_{p-1}(\partial_p(\vartheta(b))),\ b \in B_p,\ p \geq 1. \tag{6.8}$$

Proof. Let $b \in B$. Since $\rho_2 = \vartheta * \rho_1$, by definition,

$$\begin{aligned}
\rho_2(\partial_p b) &= \vartheta(\partial_{p-1}\partial_p b)\rho_1(\partial_p b)(\partial_{p-1}(\vartheta(\partial_p b)))^{-1}\\
\partial_{p-1}(\rho_2(b)(\partial_p(\vartheta b))) &= \partial_{p-1}(\vartheta(\partial_p b)\rho_1(b)) = [\partial_{p-1}(\vartheta(\partial_p b))]\partial_{p-1}(\rho_1(b))\\
\rho_2(\partial_p b)\partial_{p-1}(\rho_2(b))\partial_{p-1}(\partial_p(\vartheta b)) &= \vartheta(\partial_{p-1}\partial_p b)\rho_1(\partial_p b)\partial_{p-1}(\rho_1(b))\\
&= \vartheta(\partial_{p-1}\partial_p b)\rho_1(\partial_{p-1} b).
\end{aligned}$$

On the other hand, still by definition,

$$\rho_2(\partial_{p-1} b)\partial_{p-1}\vartheta(\partial_{p-1} b) = \vartheta(\partial_{p-1}(\partial_p b))\rho_1(\partial_{p-1} b).$$

Hence

$$\rho_2(\partial_p b)\partial_{p-1}(\rho_2(b)) = \rho_2(\partial_{p-1} b)$$

if and only if $\partial_{p-1}(\partial_p(\vartheta b)) = \partial_{p-1}\vartheta(\partial_{p-1} b)$. □

Let $\mathcal{T}(B,K) \subseteq \mathrm{Mor}_{-1}(B,K)$ denote the set of twisting functions from B to K, and let $\mathrm{Mor}_{\mathrm{tw}}(B,K) \subseteq \mathrm{Mor}_0(B,K)$ be the subset consisting of those ϑ that are subject to (6.7), necessarily a subgroup. The following is an immediate consequence of Lemma 6.4.

Proposition 6.5. *The action* (6.5) *restricts to an action of* $\mathrm{Mor}_{\mathrm{tw}}(B,K)$ *on* $\mathcal{T}(B,K)$. □

The assignment to a pair (B,K) consisting of a simplicial set B and a simplicial group K of the $\mathrm{Mor}_{\mathrm{tw}}(B,K)$-orbits

$$\mathcal{D}(B,K) = \mathcal{T}(B,K)/\mathrm{Mor}_{\mathrm{tw}}(B,K) \tag{6.9}$$

is a functor from the category of pairs of the kind (B,K) to the category of sets. This functor is the simplicial version of *Berikashvili*'s functor.

For a morphism $\phi\colon K_1 \to K_2$ of simplicial groups and a morphism $\chi\colon B_1 \to B_2$ of simplicial sets, a (ϕ,χ)-*morphism* from the simplicial principal K_1-bundle $E_1 \to B_1$ to the simplicial principal K_2-bundle $E_2 \to B_2$ is a commutative diagram

$$\begin{array}{ccc}
E_1 & \xrightarrow{\ \Psi\ } & E_2\\
\downarrow & & \downarrow\\
B_1 & \xrightarrow[\ \chi\ ]{} & B_2
\end{array}$$

in the category of simplicial sets having Ψ equivariant relative to K_1, the K_1-action on E_2 being via ϕ.

Consider a degree zero morphism $\vartheta\colon B \to K$ of the underlying graded sets. Define the map

$$\vartheta *\colon B \times K \stackrel{(\Delta,\mathrm{Id})}{\longrightarrow} B \times B \times K \stackrel{(\mathrm{Id},\vartheta,\mathrm{Id})}{\longrightarrow} B \times K \times K \stackrel{(\mathrm{Id},\mu)}{\longrightarrow} B \times K, \tag{6.10}$$

in formulas

$$\vartheta * (b,x) = (b,\vartheta(b)x). \tag{6.11}$$

For twisting functions ρ_1 and ρ_2 from B to K such that $\vartheta*\rho_1 = \rho_2$, the map $\vartheta*$ is an $(\mathrm{Id}, \mathrm{Id})$-morphism

$$\vartheta*\colon B \times_{\rho_1} K \longrightarrow B \times_{\rho_2} K \tag{6.12}$$

of simplicial principal bundles, necessarily an isomorphism. Every $(\mathrm{Id}, \mathrm{Id})$-morphism of simplicial principal K-bundles from $B \times_{\rho_1} K$ to $B \times_{\rho_2} K$ is necessarily of this kind, cf. [18, 6.2 p. 100] (for left principal bundles). The following is a consequence of Lemma 6.4.

Proposition 6.6. *For a simplicial set B, a simplicial group K, and twisting functions $\rho_1, \rho_2\colon B \to K$, the assignment to an $(\mathrm{Id}, \mathrm{Id})$-morphism $\Psi\colon B \times_{\rho_1} K \to B \times_{\rho_2} K$ of simplicial principal bundles of the morphism*

$$\vartheta_\Psi\colon B \xrightarrow{\mathrm{Id}\times\{e\}} B \times_{\rho_1} K \xrightarrow{\Psi} B \times_{\rho_2} K \xrightarrow{\mathrm{pr}_K} K \tag{6.13}$$

of simplicial sets establishes a bijection between morphisms $\vartheta\colon B \to K$ of simplicial sets such that $\vartheta * \rho_1 = \rho_2$ and $(\mathrm{Id}, \mathrm{Id})$-morphisms of simplicial principal bundles in such a way that

$$\Psi = \vartheta_\Psi*\colon B \times_{\rho_1} K \xrightarrow{\Psi} B \times_{\rho_2} K. \tag{6.14}$$

Thus any $(\mathrm{Id}, \mathrm{Id})$-morphism of simplicial principal bundles from $B \times_{\rho_1} K$ to $B \times_{\rho_2} K$ is of the kind $\vartheta*\colon B \times_{\rho_1} K \to B \times_{\rho_2} K$, for some uniquely determined degree zero morphism $\vartheta\colon B \to K$ of the underlying graded sets subject to (6.7). □

Proposition 6.7. *Let K_1, K_2 be simplicial groups, B_1, B_2 simplicial sets, $\rho_1\colon B_1 \to K_1$, $\rho_2\colon B_2 \to K_2$ twisting functions, $\phi\colon K_1 \to K_2$ a morphism of simplicial groups, $\chi\colon B_1 \to B_2$ a morphism of simplicial sets, and $\vartheta\colon B_1 \to K_2$ a degree zero morphism of the underlying graded sets such that*

$$\vartheta * (\phi\rho_1) = \rho_2\chi\colon B_1 \to K_2. \tag{6.15}$$

Then the composite

$$[\chi, \vartheta, \phi]\colon B_1 \times_{\rho_1} K_1 \xrightarrow{\mathrm{Id}\times\phi} B_1 \times_{\phi\rho_1} K_2 \xrightarrow{\vartheta*} B_1 \times_{\rho_2\chi} K_2 \xrightarrow{\chi\times\mathrm{Id}} B_2 \times_{\rho_2} K_2 \tag{6.16}$$

is a (ϕ, χ)-morphism of simplicial principal bundles, and every (ϕ, χ)-morphism of simplicial principal bundles from $B_1 \times_{\rho_1} K_1$ to $B_2 \times_{\rho_2} K_2$ arises in this manner from a suitable degree zero morphism $\vartheta\colon B_1 \to K_2$ of the underlying graded sets subject to (6.7) *and* (6.15).

Proof. The argument is similar to that for the proof of Proposition 3.5. We leave the details to the reader. □

Theorem 6.8. *For a simplicial set B and a simplicial group K, the assignment to a twisting function $\rho\colon B \to K$ of the K-principal twisted cartesian product $B \times_\rho K$ induces a bijection between $\mathcal{D}(B, K) = \mathcal{T}(B, K)/\mathrm{Mor}_{\mathrm{tw}}(B, K)$ and isomorphism classes of simplicial principal K-bundles on B. Thus the value $\mathcal{D}(B, K)$ of Berikashvili's functor $\mathcal{D}$ on (B, K) parametrizes isomorphism classes of simplicial principal K-bundles on B.*

Proof. Proposition 6.6 implies that twisting functions in the same $\mathrm{Mor}_{\mathrm{tw}}(B, K)$-orbit determine isomorphic simplicial principal (B, K)-bundles, that is, the map from $\mathcal{D}(B, K)$ to the set of isomorphism classes of simplicial principal K-bundles on B is well defined and that, furthermore, this map is injective. Proposition 6.2 implies that this map is surjective. □

Recall that $|R(B \times K)|$, $|RB|$ and $|RK|$ denote the normalized chain complexes, each one, endowed with the Alexander-Whitney diagonal, a differential graded coalgebra, and the latter, furthermore, an augmented differential graded algebra under the multiplication map which the group structure induces. Consider the Eilenberg–Zilber contraction

$$(|RB| \otimes |RK| \underset{g}{\overset{\iota}{\rightleftarrows}} |R(B \times K)|, h) \tag{6.17}$$

[13, Theorem 2.1 p. 51], written in [19, Section 4 p. 404], with K substituted for F, as

$$(B \otimes K \underset{f}{\overset{\nabla}{\rightleftarrows}} B \times K, \Phi). \tag{6.18}$$

Let $\rho\colon B \to K$ be twisting function. It determines the perturbation

$$\partial^\rho\colon |R(B\times K)| \longrightarrow |R(B\times K)|,$$
$$\partial^\rho(b,x) = (\partial_p(b), (\rho_p(b)-1)\partial_p(x)),\ (b,x)\in |R(B\times K)|_p,$$

of the differential d on the normalized chain complex $|R(B\times K)|$ of $B\times K$ so that $d_\rho = d+\partial^\rho$ is a differential on $|R(B\times K)|$, and we write the resulting chain complex as $|R(B\times_\rho K)|$. The perturbation lemma yields a perturbation $\mathcal{D}^\rho$ of the tensor product differential $d^\otimes$ on $|RB|\otimes|RK|$ together with a contraction

$$((|RB|\otimes|RK|, d^\otimes + \mathcal{D}^\rho) \underset{g_\rho}{\overset{\iota_\rho}{\rightleftarrows}} |R(B\times_\rho K)|, h_\rho), \tag{6.19}$$

and [19, 4.4 Lemma and 4.5 Lemma, Section 4] assert that $(|RB|\otimes|RK|, d^\otimes+\mathcal{D}^\rho)$ is a principal twisted object or, equivalently, a $(|RB|,|RK|)$-bundle. By Proposition 3.2, the composite

$$\tau^\rho\colon |RB| \xrightarrow{\mathrm{Id}\otimes\eta} |RB|\otimes|RK| \xrightarrow{\mathcal{D}^\rho} |RB|\otimes|RK| \xrightarrow{\varepsilon\otimes\mathrm{Id}} |RK| \tag{6.20}$$

is a twisting cochain, the differential $d^\otimes+\mathcal{D}^\rho$ is the twisted differential $d^{\tau^\rho} = d^\otimes + \tau^\rho\cap$, see also [19, p. 410], and the contraction (6.19) takes the form

$$(|RB|\otimes_{\tau^\rho}|RK| \underset{g_\rho}{\overset{\iota_\rho}{\rightleftarrows}} |R(B\times_\rho K)|, h_\rho). \tag{6.21}$$

By (2.8),

$$\mathcal{D}^\rho = \sum_{n\geq 1}\mathcal{D}^\rho_n,\ \mathcal{D}^\rho_j = g\partial^\rho(h\partial^\rho)^{j-1}\iota = g(\partial^\rho h)^j\partial^\rho\iota,\ \ j\geq 1.$$

Thus, with the notation

$$\tau^\rho_j\colon |RB| \xrightarrow{\mathrm{Id}\otimes\eta} |RB|\otimes|RK| \xrightarrow{\mathcal{D}^\rho_j} |RB|\otimes|RK| \xrightarrow{\varepsilon\otimes\mathrm{Id}} |RK|,\ \ j\geq 1,$$

the twisting cochain τ^ρ takes the form

$$\tau^\rho = \sum_{j\geq 1}\tau^\rho_j. \tag{6.22}$$

We now recall the "Serre filtrations": A member (b,x) of $B\times K$ has filtration $\leq p$ when b is the degeneration of a member of B_p, and

$$F_p(|RB|\otimes|RK|) = \sum_{0\leq i\leq p}|RB|_i\otimes|RK|.$$

In terms of the Moore complexes, with the notation $\mathrm{pr}\colon R(B\times_\rho K)\to RB$ for the projection,

$$F_p|RB| = F_p\mathrm{Mo}(RB) = \sum_{j\leq p}\mathrm{Mo}_j(RB)$$
$$F_p\mathrm{Mo}(R(B\times_\rho K)) = \mathrm{pr}^{-1}F_p\mathrm{Mo}(RB).$$

Theorem 6.9. *For a simplicial set B and a simplicial group K, the assignment to a twisting function $\rho\colon B\to K$ of the twisting cochain $\tau^\rho\colon |RB|\to|RK|$ induces a map*

$$\mathcal{T}(B,K)\longrightarrow \mathcal{T}_{\mathrm{aug}}(|RB|,|RK|) \tag{6.23}$$

that is natural in the data in the following sense: Let $\rho_1\colon B\to K$ be a twisting function, let $\vartheta\colon B\to K$ be a morphism of simplicial sets subject to (6.7), and let $\rho_2 = \vartheta * \rho_1$, necessarily a twisting function. Then ϑ induces an augmented homotopy

$$h^\vartheta\colon \tau^{\rho_1}\simeq\tau^{\rho_2}\colon |RB|\longrightarrow|RK|$$

of twisting cochains such that $h^\vartheta\cap\colon |RB|\otimes_{\tau_1}|RK|\to|RB|\otimes_{\tau_2}|RK|$ is an $(\mathrm{Id},\mathrm{Id})$-isomorphism of augmented $(|RB|,|RK|)$-bundles. Hence the map (6.23) passes to a map

$$\mathcal{D}(B,K)\longrightarrow\mathcal{D}_{\mathrm{aug}}(|RB|,|RK|) \tag{6.24}$$

and hence to a map from isomorphism classes of simplicial principal K-bundles on B to the isomorphism classes of augmented $(|RB|, |RK|)$-bundles.

Proof. The first claim is immediate.

Consider a twisting function $\rho_1\colon B \to K$, let $\vartheta\colon B \to K$ be a morphism of simplicial sets subject to (6.7), and let $\rho_2 = \vartheta * \rho_1$, necessarily a twisting function. To simplify the notation, let $\tau_1 = \tau^{\rho_1}$, $\tau_2 = \tau^{\rho_2}$, $\iota_1 = \iota_{\rho_1}$, $\iota_2 = \iota_{\rho_2}$, $g_1 = g_{\rho_1}$, $g_2 = g_{\rho_2}$, and let

$$\Psi^\vartheta = g_2|R(\vartheta *)|\iota_1\colon |RB| \otimes_{\tau_1} |RK| \longrightarrow |RB| \otimes_{\tau_2} |RK|,$$

necessarily a chain map. Since $g_2\iota_2 = |RB| \otimes_{\tau_2} |RK|$, this map renders the diagram

$$\begin{CD} |RB| \otimes_{\tau_1} |RK| @>{\iota_1}>> |R(B \times_{\rho_1} K)| \\ @V{\Psi^\vartheta}VV @VV{|R(\vartheta *)|}V \\ |RB| \otimes_{\tau_2} |RK| @>>{\iota_2}> |R(B \times_{\rho_2} K)| \end{CD}$$

commutative.

With $K \times K$ substituted for K, the Eilenberg-Zilber contraction (6.17) reads

$$(|RB| \otimes |RK| \otimes |RK| \underset{g_{K\times K}}{\overset{\iota_{K\times K}}{\rightleftarrows}} |R(B \times K \times K)|, h_{K\times K}) \tag{6.25}$$

and, applying the perturbation lemma yields, for $k = 1, 2$, the perturbed contraction

$$((|RB| \otimes_{\tau_k} |RK|) \otimes |RK| \underset{\widetilde{g}_k}{\overset{\widetilde{\iota}_k}{\rightleftarrows}} |R((B \times_{\rho_k} K) \times K)|, \widetilde{h}_k). \tag{6.26}$$

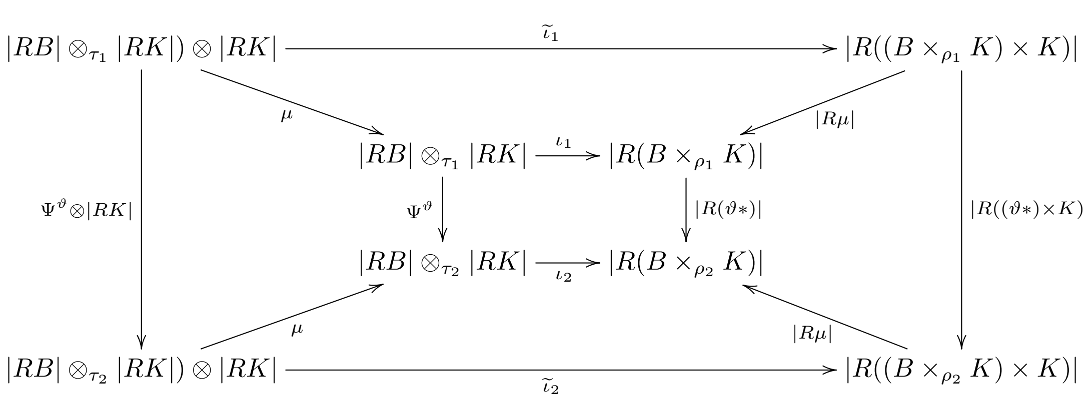
In the same vein, with $B \times B$ substituted for B, the Eilenberg-Zilber contraction (6.17) reads

$$(|RB| \otimes |RB| \otimes |RK| \underset{g_{B\times B}}{\overset{\iota_{B\times B}}{\rightleftarrows}} |R(B \times B \times K)|, h^{B\times B}) \tag{6.27}$$

and, applying the perturbation lemma yields, for $k = 1, 2$, the perturbed contraction

$$(|RB| \otimes (|RB| \otimes_{\tau_k} |RK|) \underset{\overline{g}_k}{\overset{\overline{\iota}_k}{\rightleftarrows}} |R(B \times (B \times_{\rho_k} K)|, \overline{h}_k). \tag{6.28}$$

Since $\vartheta *$ is an $(\mathrm{Id}, \mathrm{Id})$-(iso)morphism of simplicial principal bundles, the naturality of the constructions implies that Ψ^ϑ is an $(\mathrm{Id}, \mathrm{Id})$-(iso)morphism of augmented $(|RB|, |RK|)$-bundles. Indeed, the diagram



is commutative, and so is the diagram

$$
\begin{array}{ccccccc}
|RB|\otimes(|RB|\otimes_{\tau_1}|RK|) & & & \xrightarrow{\quad\bar\iota_1\quad} & & & |R(B\times(B\times_{\rho_1}K))| \\
 & \nwarrow^{\Delta} & & & & \nearrow^{|R\Delta|} & \\
\Big\downarrow{\scriptstyle |RB|\otimes\Psi^\vartheta} & & |RB|\otimes_{\tau_1}|RK| & \xrightarrow{\iota_1} & |R(B\times_{\rho_1}K)| & & \Big\downarrow{\scriptstyle |R(B\times(\vartheta *))|} \\
 & & \Big\downarrow{\scriptstyle \Psi^\vartheta} & & \Big\downarrow{\scriptstyle |R(\vartheta *)|} & & \\
 & & |RB|\otimes_{\tau_2}|RK| & \xrightarrow[\iota_2]{} & |R(B\times_{\rho_2}K)| & & \\
 & \swarrow_{\Delta} & & & & \searrow_{\Delta} & \\
|RB|\otimes(|RB|\otimes_{\tau_2}|RK|) & & & \xrightarrow[\quad\bar\iota_2\quad]{} & & & |R(B\times(B\times_{\rho_2}K))|.
\end{array}
$$

Since Ψ^ϑ is an $(\mathrm{Id},\mathrm{Id})$-morphism of augmented $(|RB|,|RK|)$-bundles, by Proposition 3.4, we conclude that the composite

$$h^\vartheta\colon |RB| \xrightarrow{|RB|\otimes\eta} |RB|\otimes|RK| \xrightarrow{\Psi^\vartheta} |RB|\otimes|RK| \xrightarrow{\varepsilon\otimes|RK|} |RK|$$

is an augmented homotopy

$$h^\vartheta\colon \tau^{\rho_1}\simeq\tau^{\rho_2}\colon |RB|\longrightarrow|RK|$$

of twisting cochains such that $\Psi^\vartheta = h^\vartheta\cap\colon |RB|\otimes_{\tau_1}|RK|\to|RB|\otimes_{\tau_2}|RK|$. □

Remark 6.10. The question emerges under which circumstances the map from isomorphism classes of simplicial principal K-bundles on B to the isomorphism classes of augmented $(|RB|,|RK|)$-bundles in Theorem 6.9 is an injection, surjection, or bijection.

Corollary 6.11. *Let $\chi\colon B_1\to B_2$ be a morphism of simplicial sets, $\phi\colon K_1\to K_2$ a morphism of simplicial groups, $\rho_1\colon B_1\to K_1$ and $\rho_2\colon B_2\to K_1$ twisting functions. Then a degree zero morphism $\vartheta\colon B_1\to K_2$ of the underlying graded sets subject to* (6.7) *and* (6.15) *induces an augmented homotopy*

$$h^\vartheta\colon \phi\tau^{\rho_1}\simeq\tau^{\rho_2}\chi\colon |RB_1|\longrightarrow|RK_2|$$

of twisting cochains such that, with the notation in (3.15), (6.16), *and* (6.21), *the diagram*

$$
\begin{array}{ccc}
|RB_1|\otimes_{\tau^{\rho_1}}|RK_1| & \xrightarrow{\iota_{\rho_1}} & |R(B_1\times_{\rho_1}K_1)| \\
\Big\downarrow{\scriptstyle [|R\chi|,h^\vartheta,|R\phi|]} & & \Big\downarrow{\scriptstyle |R[\chi,\vartheta,\phi]|} \\
|RB_2|\otimes_{\tau^{\rho_2}}|RK_2| & \xrightarrow[\iota_{\rho_2}]{} & |R(B_2\times_{\rho_2}K_2)|
\end{array}
$$

is commutative. □

Finally we consider simplicial principal bundles having structure group an ordinary group G, taken as a trivially simplicial group. In this case, for a simplicial set, the defining properties (6.1) and (6.2) of a twisting function $\rho\colon B\to G$ come down to (5.9) and (5.10), with B substituted for $N\mathcal{C}$ and G for $\mathrm{Mor}_{\mathcal{C}}$ and, G being viewed as an ordinary group, we use the notation $\mathcal{T}_{\mathrm{triv}}(B,G)$ for the set of maps $\rho\colon B\to G$ subject to (5.9) and (5.10), that is, for the set of twisting functions from B to G when G is taken as an ordinary group. A straightforward verification establishes the following.

Complement 6.12. For a simplicial set B and a group G, taken as a trivially simplicial group, the restriction $\mathcal{T}(B,G)\to\mathcal{T}_{\mathrm{triv}}(B,G)$ is a bijection and, furthermore, a member ϑ of $\mathrm{Mor}_{\mathrm{tw}}(B,G)$ satisfies the identity

$$\vartheta(b)=\vartheta(\partial_0\ldots\partial_{p-2}\partial_{p-1}(b)),\quad b\in B_p,\quad p\geq 1.$$

Hence the restriction

$$\mathrm{Mor}_{\mathrm{tw}}(B,G)\longrightarrow\mathrm{Map}(B_0,G) \tag{6.29}$$

is an isomorphism of groups and, in terms of the induced action of $\mathrm{Map}(B_0, G)$ on $\mathcal{T}_{\mathrm{triv}}(B, G)$,

$$\mathcal{D}(B, G) = \mathcal{T}_{\mathrm{triv}}(B, G)/\mathrm{Map}(B_0, G). \quad \square \tag{6.30}$$

Remark 6.13. Let K be a simplicial group. Recall that the W-construction yields the universal simplicial K-principal bundle [12, §17]: For $n \geq 0$,

$$(WK)_n = K_0 \times \cdots \times K_n, \tag{6.31}$$

with face and degeneracy operators given by the formulas

$$\begin{aligned} \partial_0(x_0, \ldots, x_n) &= (\partial_0 x_1, \ldots, \partial_0 x_n) \\ \partial_j(x_0, \ldots, x_n) &= (x_0, \ldots, x_{j-2}, x_{j-1}\partial_j x_j, \partial_j x_{j+1}, \ldots, \partial_j x_n), \quad 1 \leq j \leq n \\ s_j(x_0, \ldots, x_n) &= (x_0, \ldots, x_{j-1}, e, s_j x_j, s_j x_{j+1}, \ldots, s_j x_n), \quad 0 \leq j \leq n; \end{aligned} \tag{6.32}$$

further, $(\overline{W}K)_0 = \{e\}$ and, for $n \geq 1$

$$(\overline{W}K)_n = K_0 \times \cdots \times K_{n-1}, \tag{6.33}$$

with face and degeneracy operators given by the formulas

$$\begin{aligned} \partial_0(x_0, \ldots, x_{n-1}) &= (\partial_0 x_1, \ldots, \partial_0 x_{n-1}), \\ \partial_j(x_0, \ldots, x_{n-1}) &= (x_0, \ldots, x_{j-2}, x_{j-1}\partial_j x_j, \partial_j x_{j+1}, \ldots, \partial_j x_{n-1}), \\ &\qquad 1 \leq j \leq n-1, \\ \partial_n(x_0, \ldots, x_{n-1}) &= (x_0, \ldots, x_{n-2}), \\ s_0(e) &= e \in K_0, \\ s_j(x_0, \ldots, x_{n-1}) &= (x_0, \ldots, x_{j-1}, e, s_j x_j, s_j x_{j+1}, \ldots, s_j x_{n-1}), \\ &\qquad 0 \leq j \leq n. \end{aligned} \tag{6.34}$$

The formulas (6.32) and (6.34) are consistent with those in [21, A.14] for a simplicial algebra; they differ from those in [9] (pp. 136 and 161) where the constructions are carried out with structure group acting from the *left*. The maps

$$\rho_n = \mathrm{pr}_{K_{n-1}} \colon (\overline{W}K)_n = K_0 \times \cdots \times K_{n-1} \to K_{n-1}, \ \ n \geq 1, \tag{6.35}$$

assemble to a twisting function $\rho \colon \overline{W}K \to K$, and the canonical map $(\overline{W}K) \times_\rho K \to WK$ is an isomorphism of right K-simplicial sets. Any simplicial principal K-bundle $P \to B$ admits a classifying map $B \to \overline{W}K$ and can hence be written as a twisted Cartesian product of the base B with K. For a modern account of the W-construction, with structure group acting from the left, see [16, §V.4 p. 269].

7. Principal Bundles on the Nerve of a Category

Let G be a group (discrete, topological, Lie, according to the case considered), viewed as a category with a single object, and let $F \colon \mathcal{C} \to G$ be a functor. Thus F assigns to every object of $\mathcal{C}$ the identity element of G and to every morphism $f \colon x_0 \to x_1$ of $\mathcal{C}$ a group element $F(f) \in G$ such that, whenever $f_1 \colon x_0 \to x_1$ and $f_2 \colon x_1 \to x_2$,

$$F(f_1 f_2) = F(f_1)F(f_2).$$

Since F is a functor, this assignment extends to a morphism

$$NF \colon N\mathcal{C} \longrightarrow NG$$

of simplicial objects. The pullback diagram

$$\begin{array}{ccc} \mathcal{P}_{\mathcal{C},F} & \xrightarrow{F_G} & \mathcal{P}G \\ {\scriptstyle \Pi_F}\downarrow & & \downarrow{\scriptstyle \Pi} \\ \mathcal{C} & \xrightarrow[F]{} & G \end{array} \tag{7.1}$$

of categories characterizes the pullback category $\mathcal{P}_{\mathcal{C},F}$. By construction, the group G acts freely from the right on $\mathcal{P}_{\mathcal{C},F}$, and the canonical functor $\mathcal{P}_{\mathcal{C},F}/G \to \mathcal{C}$ is an isomorphism of categories. Hence the nerve construction yields the simplicial principal right G-bundle

$$N\Pi_F\colon N(\mathcal{P}_{\mathcal{C},F}) \longrightarrow N\mathcal{C}. \tag{7.2}$$

The composite $\rho(NF)\colon N\mathcal{C} \to G$ of the induced morphism $NF\colon N\mathcal{C} \to NG$ of simplicial sets with the universal twisting function $\rho\colon N(G) \to G$, see Subsection 5.5, yields the twisting function $N\mathcal{C} \to G$ which recovers the simplicial right G-set $N(\mathcal{P}_{\mathcal{C},F})$, the total object of (7.2), as the twisted cartesian product $N\mathcal{C} \times_{\rho(NF)} G$ in such a way that the bundle projection map $N\Pi_F\colon N\mathcal{C} \times_{\rho(NF)} G \to N\mathcal{C}$ amounts to the canonical projection to $N\mathcal{C}$.

The group $\mathrm{Map}(\mathrm{Ob}_{\mathcal{C}}, G)$ of maps from $\mathrm{Ob}_{\mathcal{C}}$ to G with pointwise multiplication acts on the set $\mathrm{Funct}(\mathcal{C}, G)$ of functors form $\mathcal{C}$ to G via natural transformations of functors: Consider two functors $F_1, F_2\colon \mathcal{C} \to G$ and a map $\Phi\colon \mathrm{Ob}_{\mathcal{C}} \to G$ such that, for two objects x and y of $\mathcal{C}$ and a morphism f from x to y, the diagram

$$\begin{array}{ccc} F_1(x) & \xrightarrow{F_1(f)} & F_1(y) \\ {\scriptstyle \Phi(x)}\big\downarrow & & \big\downarrow{\scriptstyle \Phi(y)} \\ F_2(x) & \xrightarrow[F_2(f)]{} & F_2(y) \end{array} \tag{7.3}$$

is commutative. Then $F_2 = \Phi(F_1)$ is the result of the action by $\Phi \in \mathrm{Map}(\mathrm{Ob}_{\mathcal{C}}, G)$ on F_1.

The assignment to a pair $(\mathcal{C}, G)$ consisting of a category $\mathcal{C}$ and a group G of the $\mathrm{Map}(\mathrm{Ob}_{\mathcal{C}}, G)$-orbits

$$\mathcal{D}(\mathcal{C}, G) = \mathrm{Funct}(\mathcal{C}, G)/\mathrm{Map}(\mathrm{Ob}_{\mathcal{C}}, G) \tag{7.4}$$

is a functor from the category of pairs of the kind $(\mathcal{C}, G)$ to the category of sets. This functor is the present version of *Berikashvili*'s functor.

Theorem 7.1. *For a category $\mathcal{C}$ and a group G, the assignment to a functor $F\colon \mathcal{C} \to G$ of the simplicial principal bundle $N\Pi_F$, see* (7.2)*, induces an injection from $\mathcal{D}(\mathcal{C}, G)$ to the set of isomorphism classes of simplicial principal G-bundles on $N\mathcal{C}$. Thus the value $\mathcal{D}(\mathcal{C}, G)$ of Berikashvili's functor $\mathcal{D}$ on $(\mathcal{C}, G)$ parametrizes the isomorphism classes of simplicial principal G-bundles on $N\mathcal{C}$ of the kind $N\Pi_F$, for some functor $F\colon \mathcal{C} \to G$.*

Proof. Consider two functors $F_1, F_2\colon \mathcal{C} \to G$ and a map $\Phi\colon \mathrm{Ob}_{\mathcal{C}} \to G$ such that $F_2 = \Phi(F_1)$. By Complement 6.12, the map Φ determines a unique map $\vartheta\colon N\mathcal{C} \to G$ subject to (6.7), and

$$\vartheta*\colon N\mathcal{C} \times_{\rho(NF_1)} G \longrightarrow N\mathcal{C} \times_{\rho(NF_2)} G \tag{7.5}$$

yields an isomorphism of simplicial principal G-bundles on $N\mathcal{C}$. By Proposition 6.6, every isomorphism $N\Pi_{F_1} \to N\Pi_{F_2}$ of simplicial principal G-bundles over the identity of $N\mathcal{C}$ arises in this manner. □

Example 7.2. Let B be a simplicial complex or, more generally, simplicial set, and let $\mathcal{C}_B$ be the category whose set of objects Ob_B is the set S of simplices in B, with one morphism from $x \in S$ to $y \in S$ whenever $x \le y$, that is, whenever x is a face of y. As a simplicial set, the nerve $N\mathcal{C}_B$ of $\mathcal{C}_B$ is the barycentric subdivision of B. Thus a functor $F\colon \mathcal{C}_B \to G$ assigns to every simplex of B, that is, to every vertex of the barycentric subdivision $N\mathcal{C}_B$ of B, the identity element of G, and to every oriented edge (1-simplex) $x_0 \subseteq x_1$ of $N\mathcal{C}_B$ a group element $F(x_0 \subseteq x_1) \in G$ such that, whenever $x_0 \subseteq x_1$ and $x_1 \subseteq x_2$,

$$F(x_0 \subseteq x_2) = F(x_0 \subseteq x_1)F(x_1 \subseteq x_2).$$

Since F is a functor, this assignment extends to a morphism

$$NF\colon N\mathcal{C}_B \longrightarrow NG$$

of simplicial objects. Since, as a simplicial set, the nerve $N\mathcal{C}_B$ of $\mathcal{C}_B$ is the barycentric subdivision of B, by Theorem 7.1, applying Berikashvili's functor to $(\mathcal{C}_B, G)$ yields a set $\mathcal{D}(\mathcal{C}_B, G)$ parametrizing the isomorphism classes of simplicial principal G-bundles on the barycentric subdivision $N\mathcal{C}_B$ of B of the kind $N\Pi_F$, for some functor $F\colon \mathcal{C}_B \to G$.

The lean geometric realization $||N\mathcal{C}_B||$ of the nerve $N\mathcal{C}_B$ of $\mathcal{C}_B$ is homeomorphic to the lean geometric realization $||B||$ of B. When B is a simplicial complex the homeomorphism is actually natural. The proof where B is a general simplicial set is in [14] together with the observation that the homeomorphism cannot be taken to be natural ("Korollar" p. 508). Given the functor $F\colon \mathcal{C}_B \to G$, geometric realization yields the principal G-bundle

$$||N\Pi_F||\colon ||N(\mathcal{P}_{\mathcal{C}_B,F})|| \longrightarrow ||N\mathcal{C}_B||. \tag{7.6}$$

Hence the value $\mathcal{D}(\mathcal{C}_B, G)$ of Berikashvili's functor on $(\mathcal{C}_B, G)$ parametrizes isomorphism classes of principal G-bundles of the kind (7.6) on the geometric realization of the barycentric subdivision of B.

Example 7.3. Let M be a (topological, smooth, analytic, algebraic, according to the case considered) manifold and $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ an open cover of M. Consider $\mathcal{U}$ as a partially ordered set, with order relation by inclusion. Let $\mathcal{C}_\mathcal{U}$ be the associated category. It has $\mathrm{Ob}_{\mathcal{C}_\mathcal{U}} = \mathcal{U}$ and, for two members U and V of $\mathcal{U}$ with $U \subseteq V$ a morphism from U to V.

For a subset σ of Λ, let $U_\sigma = \cap_{\alpha\in\sigma} U_\alpha$. The family $\{U_\sigma\}$ of the non-empty U_σ as σ ranges over finite subsets of Λ forms an open cover of M as well, and we denote this open cover by $B\mathcal{U}$. Here we use the letter B as a mnemonic for 'barycentric subdivision', see below. For $\tau \subseteq \sigma \subseteq \Lambda$, necessarily $U_\sigma \subseteq U_\tau$. The associated category $\mathcal{C}_{B\mathcal{U}}$ has $\mathrm{Ob}_{\mathcal{C}_{B\mathcal{U}}} = B\mathcal{U}$ and, for two objects U_σ and U_τ of $B\mathcal{U}$ with $\tau \subseteq \sigma \subseteq \Lambda$, a morphism from U_σ to U_τ.

Following [49, §4], we assign to $\mathcal{C}_{B\mathcal{U}}$ a (topological, smooth, analytic, algebraic, according to the case considered) category $\mathcal{M}_\mathcal{U}$ as follows (the notation in [49, §4] is X for M and $\mathbf{X_U}$ for $\mathcal{M}_\mathcal{U}$): Let $\mathrm{Ob}(\mathcal{M}_\mathcal{U}) = \coprod_{\sigma\in\Lambda} U_\sigma$, the disjoint union of the non-empty U_σ, for finite subsets σ of Λ. Thus the objects of $\mathcal{M}_\mathcal{U}$ are pairs (x, U_σ) with $x \in U_\sigma$, for finite subsets σ of Λ. Define a morphism $(x, U_\sigma) \to (y, U_\tau)$ of $\mathcal{M}_\mathcal{U}$ to be an inclusion $i\colon U_\sigma \to U_\tau$ with $i(x) = y$, for $\tau \subseteq \sigma$. Hence, for two finite subsets $\sigma_0 \subseteq \sigma_1$ of Λ, a morphism $(x_1, U_{\sigma_1}) \to (x_0, U_{\sigma_0})$ of $\mathcal{M}_\mathcal{U}$ is the inclusion $i\colon U_{\sigma_1} \to U_{\sigma_0}$ with $i(x_1) = x_0$. For an ascending sequence $\sigma_0 \subseteq \sigma_1 \subseteq \sigma_2$ and two morphisms $(x_2, U_{\sigma_2}) \to (x_1, U_{\sigma_1})$ and $(x_1, U_{\sigma_1}) \to (x_0, U_{\sigma_0})$ of $\mathcal{M}_\mathcal{U}$, the composite of $(x_2, U_{\sigma_2}) \to (x_1, U_{\sigma_1})$ and $(x_1, U_{\sigma_1}) \to (x_0, U_{\sigma_0})$ is the morphism

$$c((x_2, U_{\sigma_2}) \to (x_1, U_{\sigma_1}), (x_1, U_{\sigma_1}) \to (x_0, U_{\sigma_0})) = (x_2, U_{\sigma_2}) \to (x_1, U_{\sigma_1}) \to (x_0, U_{\sigma_0}). \tag{7.7}$$

Thus the disjoint union $\coprod_{[\sigma_0\subseteq\sigma_1]} U_{\sigma_1}$ of the non-empty U_{σ_1} over the inclusions $\sigma_0 \subseteq \sigma_1 \subseteq \Lambda$ of finite subsets of Λ parametrizes the space $\mathrm{Mor}(\mathcal{M}_\mathcal{U})$ of $\mathcal{M}_\mathcal{U}$. Consider an inclusion $\sigma_0 \subseteq \sigma_1 \subseteq \Lambda$ of finite subsets of Λ and let $U_{\sigma_0\subseteq\sigma_1}$ denote the constituent of $\mathrm{Mor}(\mathcal{M}_\mathcal{U})$ which that inclusion parametrizes. The following arrows characterize the so far missing pieces of structure that turn $\mathcal{M}_\mathcal{U}$ into a category:

$$\begin{aligned} s\colon U_{\sigma_0\subseteq\sigma_1} = U_{\sigma_1} &\xrightarrow{\text{incl}} U_{\sigma_0}, \\ t\colon U_{\sigma_0\subseteq\sigma_1} = U_{\sigma_1} &\xrightarrow{=} U_{\sigma_1}, \\ \mathrm{Id}\colon U_{\sigma_0} &\xrightarrow{=} U_{\sigma_0\subseteq\sigma_0}. \end{aligned}$$

The nerve $N(\mathcal{M}_\mathcal{U})$ of $\mathcal{M}_\mathcal{U}$ is the simplicial manifold having, for $p \geq 0$,

$$N(\mathcal{M}_\mathcal{U})_p = \coprod_{[\sigma_0\subseteq\ldots\subseteq\sigma_p]} U_{\sigma_p} \tag{7.8}$$

the disjoint union of the non-empty U_{σ_p}, parametrized by ascending sequences

$$\sigma_0 \subseteq \cdots \subseteq \sigma_p \subseteq \Lambda \tag{7.9}$$

of finite subsets of Λ. By construction, for an ascending sequence of subsets of Λ of the kind (7.9), necessarily

$$U_{\sigma_p} = \cap_{0\leq j\leq p} U_{\sigma_j}. \tag{7.10}$$

The face and degeneracy operators are the corresponding inclusions. See also [Section 2 p. 237] [11]. As a simplicial manifold, the nerve $N\mathcal{M}_\mathcal{U}$ of $\mathcal{M}_\mathcal{U}$ is the barycentric subdivision of the ordinary nerve $N\mathcal{U}$ of $\mathcal{U}$.

A functor $F\colon \mathcal{M}_\mathcal{U} \to G$ determines the corresponding simplicial principal G-bundle (7.2), viz.

$$N\Pi_F\colon N(\mathcal{P}_{\mathcal{M}_\mathcal{U},F}) \longrightarrow N\mathcal{M}_\mathcal{U}, \tag{7.11}$$

on $N\mathcal{M}_{\mathcal{U}}$. Thus, by Theorem 7.1, the value $\mathcal{D}(\mathcal{M}_{\mathcal{U}}, G)$ of Berikashvili's functor on $(\mathcal{M}_{\mathcal{U}}, G)$ parametrizes the isomorphism classes of simplicial principal G-bundles on $N\mathcal{M}_{\mathcal{U}}$ of the kind $N\Pi_F$, for some functor $F\colon \mathcal{M}_{\mathcal{U}} \to G$.

View the manifold M as a trivially simplicial manifold M. Then the canonical projection $\mathcal{M}_{\mathcal{U}} \to M$ is a morphism of simplicial manifolds. A partition of unity of M subordinate to $B\mathcal{U}$ induces a section $\iota\colon M \to ||N\mathcal{M}_{\mathcal{U}}||$, and ι and the projection $||N\mathcal{M}_{\mathcal{U}}|| \to M$ constitute a deformation retraction. Thus, when M is topological or smooth and paracompact, the value of Berikashvili's functor on $(\mathcal{M}_{\mathcal{U}}, G)$ parametrizes certain isomorphism classes of principal G-bundles on M; if, furthermore, every object of $\mathcal{M}_{\mathcal{U}}$ is contractible, every simplicial principal bundle on $N\mathcal{M}_{\mathcal{U}}$ arises from a functor from $\mathcal{M}_{\mathcal{U}}$ to G, and the value $\mathcal{D}(\mathcal{M}_{\mathcal{U}}, G)$ of Berikashvili's functor on $(\mathcal{M}_{\mathcal{U}}, G)$ parametrizes isomorphism classes of principal G-bundles on M. Below we make this observation more precise.

For a functor $F\colon \mathcal{M}_{\mathcal{U}} \to G$, by construction, applying geometric realization to (7.11) yields the ordinary principal G-bundle

$$||N\Pi_F||\colon ||N(\mathcal{P}_{\mathcal{M}_{\mathcal{U}},F})|| \longrightarrow ||N\mathcal{M}_{\mathcal{U}}||, \tag{7.12}$$

and the classifying map

$$||NF||\colon ||N\mathcal{M}_{\mathcal{U}}|| \longrightarrow ||NG|| \tag{7.13}$$

thereof arises as the geometric realization of the morphism $NF\colon N\mathcal{M}_{\mathcal{U}} \to NG$ of simplicial objects which the functor F induces. Thus, within the present framework, the functor F determines the associated bundle and classifying map for free. The composite with the section $\iota\colon M \to ||N\mathcal{M}_{\mathcal{U}}||$ arising from a partition of unity of M subordinate to $B\mathcal{U}$ then yields the classifying map for the resulting G-bundle on M for free. If every member of $\mathcal{U}$ is contractible, every principal G-bundle on M arises in this way.

Consider the special case where the open cover $\mathcal{U}$ of M has a single member, i.e., M itself. The nerve $N\mathcal{U}$ of $\mathcal{U}$ is the trivially simplicial space associated to M (having, for $p \geq 0$, as degree p constituent a copy of M and every arrow the identity). The category $\mathcal{M}_{\mathcal{U}}$ is the smooth category having M as its space of objects and the nerve $N\mathcal{M}_{\mathcal{U}}$ of $\mathcal{M}_{\mathcal{U}}$ is, likewise, the trivially simplicial space associated to M. (The barycentric subdivision of a point is still a point.) A functor $F\colon \mathcal{M}_{\mathcal{U}} \to G$ assigns to each object $x \in M$ of $\mathcal{M}_{\mathcal{U}}$ the identity e of M and to the single morphism Id_M of $\mathcal{M}_{\mathcal{U}}$ the identity element of G. Hence the resulting G-bundle on M is trivial.

The observation in [49, §4] that G-transition functions relative to the open cover $B\mathcal{U}$ of M amount to a functor $F\colon \mathcal{M}_{\mathcal{U}} \to G$ reconciles the present characterization of a principal G-bundle in terms of an open cover with the more classical one.

To put flesh on the bones of the last remark, recall a *system of G-valued transition functions on M relative to the open cover $\mathcal{U}$* consists of a family of maps $g_{\lambda,\mu}\colon U_\lambda \cap U_\mu \to G$ $(\lambda, \mu \in \Lambda)$ subject to (T1) below:

(T1) For $\lambda, \mu, \nu \in \Lambda$, the diagram

$$\begin{array}{ccccc} (U_\lambda \cap U_\mu \cap U_\nu) \times (U_\lambda \cap U_\mu \cap U_\nu) & \xrightarrow{(g_{\lambda,\mu}, g_{\mu,\nu})} & G \times G & \xrightarrow{\mu} & G \\ \uparrow{\scriptstyle\Delta} & & & \nearrow{\scriptstyle g_{\lambda,\nu}} & \\ U_\lambda \cap U_\mu \cap U_\nu & & & & \end{array} \tag{7.14}$$

is commutative; see, e.g., [34, 2.4 Definition Section 5.2 p. 63].

A system $\{g_{\lambda,\mu}\colon U_\lambda \cap U_\mu \to G\}$ of G-valued transition functions on M relative to $\mathcal{U}$ necessarily satisfies (T2) and (T3) below [34, Section 5.2 p. 63]:

(T2) For $\lambda \in \Lambda$, the map $g_{\lambda,\lambda}\colon U_\lambda \to G$ factors as

$$g_{\lambda,\lambda}\colon U_\lambda \to \{e\} \to G. \tag{7.15}$$

(T3) For $\lambda, \mu \in \Lambda$, the maps $g_{\lambda,\mu}, g_{\mu,\lambda}\colon U_\lambda \cap U_\mu \to G$ coincide.

Two systems $\{g_{\lambda,\mu}\colon U_\lambda \cap U_\mu \to G\}$ and $\{g'_{\lambda,\mu}\colon U_\lambda \cap U_\mu \to G\}$ of G-valued transition functions on M relative to $\mathcal{U}$ are *equivalent* [34, 2.6 Definition Section 5.2 p. 63] provided there exist maps $r_\lambda\colon U_\lambda \to G$ $(\lambda \in \Lambda)$ satisfying the relations (E) below:

(E) For $\lambda, \mu \in \Lambda$ and $y \in U_\lambda \cap U_\mu$,

$$g'_{\lambda,\mu}(y) = r_\lambda(y)^{-1} g_{\lambda,\mu}(y) r_\mu(y). \tag{7.16}$$

This relation is an equivalence relation among systems of G-transition functions on $\mathcal{U}$. The equivalence classes form the first non-abelian cohomology set $\mathrm{H}^1(\mathcal{U}, G)$ of $\mathcal{U}$ with coefficients in G; see, e.g., [23, I.3 p. 40].

Relative to the open cover $B\mathcal{U}$ of M, consider a system of G-valued transition functions. Such a system takes the form $\{g_{\sigma_0 \subseteq \sigma_1} \colon U_{\sigma_0 \subseteq \sigma_1} \to G\}_{\sigma_0 \subseteq \sigma_1}$, as $\sigma_0 \subseteq \sigma_1$ ranges over injections of finite subsets of Λ. The constraints (T1) – (T3) say that this system of transition functions defines the functor

$$\begin{aligned} F \colon \mathcal{M}_{\mathcal{U}} &\longrightarrow G \\ F\colon \operatorname{Ob}(\mathcal{M}_{\mathcal{U}}) &\longrightarrow \{e\} \subseteq G \\ F|_{U_{\sigma_0 \subseteq \sigma_1}} = g_{\sigma_0 \subseteq \sigma_1} \colon U_{\sigma_0 \subseteq \sigma_1} &\longrightarrow G, \end{aligned} \tag{7.17}$$

and every such functor determines a system of G-valued transition functions. Hence:

Theorem 7.4. *The assignment to a system of G-transition functions on $B\mathcal{U}$ of the G-valued functor F on $\mathcal{M}_{\mathcal{U}}$ which the assignment* (7.17) *characterizes induces a bijection*

$$\mathrm{H}^1(B\mathcal{U}, G) \longrightarrow \mathcal{D}(\mathcal{M}_{\mathcal{U}}, G) \tag{7.18}$$

onto the set $\mathcal{D}(\mathcal{M}_{\mathcal{U}}, G)$ (Berikashvili's functor evaluated on $(\mathcal{M}_{\mathcal{U}}, G)$), and this bijection is natural in the data. □

Suppose that M is smooth and that M admits a partition of unity subordinate to the open cover $B\mathcal{U}$ of M. When M is paracompact, such a partition of unity is available. A classical result says that two principal G-bundles on M arising from the same open cover are isomorphic if and only if the corresponding systems of G-transition functions are equivalent, cf. [34, 2.7 Theorem Section 5.2 p. 63]. Thus, when each member of the open cover $\mathcal{U}$ of M is contractible, the value $\mathcal{D}(\mathcal{M}_{\mathcal{U}}, G)$ of Berikashvili's functor on the pair $(\mathcal{M}_{\mathcal{U}}, G)$ parametrizes isomorphism classes of principal G-bundles on M.

Remark 7.5. Theorem 7.4 suggests one could view the value $\mathcal{D}(C, A)$ of Berikasvili's functor in Section 3 on the pair (C, A), the value $\mathcal{D}(B, K)$ of that functor in Section 6 on the pair (B, K), and the value $\mathcal{D}(\mathcal{C}, G)$ of that functor on the pair $(\mathcal{C}, G)$ in the present Section as a kind of first non-abelian cohomology set. This is consistent with the interpretation of a moduli space of flat connections as a first non-abelian cohomology space.

8. Twisting Cochains and Berikashvili's Functor in a Categorical Setting

Let $\mathcal{C}$ be a coaugmented dg R-cocategory, with object set $O_{\mathcal{C}}$ and coaugmentation $\eta \colon R[O_{\mathcal{C}}] \to \mathcal{C}$, and let $\mathcal{A}$ be an augmented dg R-category, with object set $O_{\mathcal{A}}$ and augmentation $\varepsilon \colon \mathcal{A} \to R[O_{\mathcal{A}}]$. A *twisting cochain* is a degree -1 morphism $t \colon \mathcal{C} \to \mathcal{A}$ of graded R-graphs, subject to the requirements

$$Dt = t \cup t, \ \ \varepsilon t = 0, \ \ t\eta = 0. \tag{8.1}$$

We will now unravel the meaning of this definition. In particular, when $O_{\mathcal{C}}$ and $O_{\mathcal{A}}$ consist of a single element, this notion of twisting cochain comes down to the standard one.

We return to the general case. The values of the diagonal Δ characterizing the cocategory structure of $\mathcal{C}$ lie in the appropriate *tensor square* of $\mathcal{C}$ with itself in the category of differential graded $R[O_{\mathcal{C}}]$-modules. This tensor square is the appropriate *coend* $\mathcal{C} \otimes_{R[O_{\mathcal{C}}]} \mathcal{C}$, that is, $\mathcal{C} \otimes_{R[O_{\mathcal{C}}]} \mathcal{C}$ is the differential graded $R[O_{\mathcal{C}}]$-module having the same set of objects $O_{\mathcal{C}}$ as $\mathcal{C}$ and, given the pair (x, y) of objects, the R-chain complex $(\mathcal{C} \otimes_{R[O_{\mathcal{C}}]} \mathcal{C})(x, y)$ of arrows from x to y is the sum

$$\bigoplus_{z \in O_{\mathcal{C}}} \mathcal{C}(x, z) \otimes \mathcal{C}(z, y),$$

the tensor product for each object z being the ordinary tensor product of R-chain complexes.

A *homogeneous morphism* $\alpha\colon \mathcal{C} \to \mathcal{A}$ of *graded R-graphs* of a fixed degree $k \in \mathbb{Z}$ consists of (i) a set map

$$\alpha\colon O_{\mathcal{C}} = \mathrm{Ob}(\mathcal{C}) \longrightarrow \mathrm{Ob}(\mathcal{A}) = O_{\mathcal{A}}$$

together with (ii), for each (ordered) pair (x, y) of objects of $\mathcal{C}$, a morphism

$$\alpha_{x,y}\colon \mathcal{C}(x, y) \longrightarrow \mathcal{A}(\alpha(x), \alpha(y))$$

of graded R-modules of degree k; we will denote the degree of such a homogeneous morphism α by $|\alpha|$. Given two homogeneous morphisms $\alpha, \beta\colon \mathcal{C} \to \mathcal{A}$ of graded R-graphs which coincide on objects, that is,

$$\alpha = \beta\colon \mathrm{Ob}(\mathcal{C}) \longrightarrow \mathrm{Ob}(\mathcal{A}), \tag{8.2}$$

their *cup product* $\alpha \cup \beta$ is the homogeneous morphism $\alpha \cup \beta\colon \mathcal{C} \to \mathcal{A}$ of graded R-graphs of degree $|\alpha| + |\beta|$ which, on objects, is given by (8.2) and which, for any (ordered) pair (x, y) of objects of $\mathcal{C}$, is given as the composite of the following three morphisms:

$$\begin{array}{ccc} \mathcal{C}(x,y) & \xrightarrow{\Delta} & \bigoplus_{z\in O_{\mathcal{C}}} \mathcal{C}(x,z)\otimes\mathcal{C}(z,y) \\ \bigoplus_{z\in O_{\mathcal{C}}} \mathcal{C}(x,z)\otimes\mathcal{C}(z,y) & \xrightarrow{\alpha_{x,z}\otimes\alpha_{z,x}} & \bigoplus_{z\in O_{\mathcal{C}}} \mathcal{A}(\alpha(x),\alpha(z))\otimes\mathcal{A}(\alpha(z),\alpha(y)) \\ \bigoplus_{z\in O_{\mathcal{C}}} \mathcal{A}(\alpha(x),\alpha(z))\otimes\mathcal{A}(\alpha(z),\alpha(y)) & \xrightarrow{c} & \mathcal{A}(\alpha(x),\alpha(y)) \end{array} \tag{8.3}$$

This composition is well defined since, given $f \in \mathcal{C}(x, y)$, the value

$$\Delta(f) \in \bigoplus_{z\in O_{\mathcal{C}}} \mathcal{C}(x, z) \otimes \mathcal{C}(z, y)$$

has at most finitely many non-zero components whence, even though the lower-most arrow in (8.3) is, perhaps, not well defined when $O_{\mathcal{C}}$ is infinite, the evaluation relative to the composition c is well defined.

With these preparations out of the way, a twisting cochain $t\colon \mathcal{C} \to \mathcal{A}$ consists of a set map

$$t\colon \mathrm{Ob}(\mathcal{C}) \longrightarrow \mathrm{Ob}(\mathcal{A})$$

together with, for each (ordered) pair (x, y) of objects of $\mathcal{C}$, a morphism

$$\alpha_{x,y}\colon \mathcal{C}(x, y) \longrightarrow \mathcal{A}(\alpha(x), \alpha(y))$$

of graded R-modules of degree -1; and t is required to satisfy the identities (8.1).

The following is an immediate generalization of the corresponding facts for differential graded algebras and coalgebras:

Proposition 8.1. *Let $\mathcal{C}$ be a coaugmented dg R-cocategory and $\mathcal{A}$ an augmented dg R-category. A twisting cochain $t\colon \mathcal{C} \to \mathcal{A}$ induces a morphism*

$$\bar{t}\colon \mathcal{C} \longrightarrow \mathcal{B}\mathcal{A} \tag{8.4}$$

of coaugmented dg cocategories and a morphism

$$\bar{t}\colon \Omega\mathcal{C} \longrightarrow \mathcal{A} \tag{8.5}$$

of augmented dg categories.

In this proposition, the notation $\bar{t}$ is slightly abused. Under these circumstances, we will refer to either $\bar{t}$ as the *adjoint* of t.

Under the present circumstances, Berikashvili's functor is still defined: As before, let $\mathcal{C}$ be a coaugmented dg R-cocategory, with object set $O_{\mathcal{C}}$ and with coaugmentation $\eta\colon R[O_{\mathcal{C}}] \to \mathcal{C}$, and let $\mathcal{A}$ be an augmented dg R-category, with object set $O_{\mathcal{A}}$ and augmentation $\varepsilon\colon \mathcal{A} \to R[O_{\mathcal{A}}]$. Let

$$\varphi\colon O_{\mathcal{C}} \longrightarrow O_{\mathcal{A}}$$

be a fixed map and let $\mathcal{A}_\varphi$ be the graded R-module of homogeneous morphisms $\mathcal{C} \to \mathcal{A}$ of graded R-graphs which, on objects, are given by φ. The standard Hom-differential D turns $\mathcal{A}_\varphi$ into an R-chain complex and the cup product turns $\mathcal{A}_\varphi$ into a differential graded algebra, with unit given by the composite

$$\mathcal{C} \xrightarrow{\ \varepsilon\ } R[O_{\mathcal{C}}] \xrightarrow{\ R\varphi\ } R[O_{\mathcal{A}}] \xrightarrow{\ \eta\ } \mathcal{A}.$$

Denote the set of twisting cochains in $\mathcal{A}_\varphi$ by $\mathcal{T}(\mathcal{A}_\varphi)$. Next let G be the group of invertible elements of $\mathcal{A}^0_\varphi$, and define the action

$$G \times \mathcal{T}(\mathcal{A}_\varphi) \longrightarrow \mathcal{T}(\mathcal{A}_\varphi),\ (x,y) \mapsto x * y,\ x \in G, y \in \mathcal{T}(\mathcal{A}_\varphi),$$

of G on $\mathcal{T}(\mathcal{A}_\varphi)$, by means of the formula

$$x * y = xyx^{-1} + (Dx)x^{-1}. \tag{8.6}$$

This is well defined, that is, given $x \in G$ and $y \in \mathcal{T}(\mathcal{A}_\varphi)$, the value $x * y$ satisfies the requirements (8.1) as well. This is readily seen by a straightforward calculation relying on the formulas

$$(Dx)x^{-1} + xDx^{-1} = 0, \quad (Dx^{-1})x + x^{-1}Dx = 0$$

which, in turn, follow from $xx^{-1} = 1$. We denote the set of orbits $(\mathcal{T}(\mathcal{A}_\varphi))/G$ by $\mathcal{D}(\mathcal{A}_\varphi)$. The assignment to $(\mathcal{C}, \mathcal{A}, \varphi)$ of

$$\mathcal{D}(\mathcal{C}, \mathcal{A}, \varphi) = \mathcal{D}(\mathcal{A}_\varphi) \tag{8.7}$$

is a functor from the category of triples of the kind $(\mathcal{C}, \mathcal{A}, \varphi)$ to the category of sets. This functor is the present version of *Berikashvili*'s functor. The set $\mathcal{D}(\mathcal{C}, \mathcal{A}, \varphi)$, i.e., Berikashvili's functor $\mathcal{D}$, evaluated at $(\mathcal{C}, \mathcal{A}, \varphi)$, still resembles a *moduli space of gauge equivalence classes of connections*.

All the previous examples for Berikashvili's functor can be subsumed under this general version of Berikashvili's functor. For example, return to the circumstances of Example 7.2. Let $R\mathcal{C}_B$ be the corresponding category enriched in the category of R-modules. Then $N(R\mathcal{C}_B)$ amounts to the induced simplicial $R[O]$-module $RN\mathcal{C}_B$, and the dg cocategory

$$\mathcal{B}R\mathcal{C}_B = |NR\mathcal{C}_B| = |RN\mathcal{C}_B| \tag{8.8}$$

(where as before $|\cdot|$ refers to the normalized chain complex functor) recovers the ordinary dg coalgebra of normalized R-chains on the simplicial set $N\mathcal{C}_B$ (the barycentric subdivision of B) in the following way: The R-chain complex which underlies the dg coalgebra $|N\mathcal{C}_B|$ (arising from the normalized R-chains on $N\mathcal{C}_B$) is the direct sum

$$|N\mathcal{C}_B| = \bigoplus_{x,y \in O} |RN\mathcal{C}_B|(x,y),$$

and the Alexander-Whitney diagonal

$$\Delta\colon |N\mathcal{C}_B| \longrightarrow |N\mathcal{C}_B| \otimes |N\mathcal{C}_B|$$

is induced by the cocategory diagonal of $\mathcal{B}R\mathcal{C}_B$. Given an ordinary augmented dg algebra A, viewed as an augmented dg category with a single object, a twisting cochain

$$t\colon \mathcal{B}R\mathcal{C}_B \longrightarrow A$$

in the sense of the present discussion comes down to an ordinary twisting cochain $|N\mathcal{C}_B| \to A$, defined on the ordinary coaugmented dg R-coalgebra $|N\mathcal{C}_B|$ of normalized R-chains on the barycentric subdivision $N\mathcal{C}_B$ of B.

The general version of Berikashvili's functor deserves further study. To this end, the appropriate framework is, perhaps, that of A_∞-categories due to Kontsevich and Fukaya: An A_∞-category is an "A_∞-algebra with more than one object". More precisely: A unital A_∞-*category* (over R) is a unital A_∞-algebra in the closed monoidal category $\mathrm{Chain}_{R\mathcal{G}}$ (category of *graphs* enriched in the closed monoidal category Chain_R of R-chain complexes) [15]; thus, a unital A_∞-*category* (over R) with object set O is a unital A_∞-algebra in the category of $R[O]$-chain complexes. Further, given two augmented dg categories $\mathcal{A}_1$ and $\mathcal{A}_2$, an A_∞-*functor* from $\mathcal{A}_1$ to $\mathcal{A}_2$ see, e.g., [10, p. 688] and the literature there, is a twisting cochain from $\mathcal{A}_1$ to $\mathcal{A}_2$ or, equivalently, an ordinary (dg) functor from $\Omega\mathcal{B}A_1$ to $\mathcal{A}_2$.

Also, the general version of Berikashvili's functor is, perhaps, relevant for the results and observations in [51] and [52].

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