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FK-SPACES CONTAINING ABSOLUTELY k-SUMMABLE SERIES SPACE AND MATRIX OPERATORS

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Abstract. In a later paper [34], the Banach space of all absolutely k-summable series has been introduced and studied by Sarıgöl and Agarwal. In the present paper, we introduce a paranormed space containing this space, following the Maddox papers [19–21], and show that it is an FK-space and isometrically isomorphic to the well-known space $l(\mu)$. We also establish the Schauder basis and α, β, γ duals of this space and qualify the related matrix operators. Our results include, as particular cases, a number of the well-known results in [3, 6–15, 24, 25, 28, 31, 33, 34].

1. Introduction

A vector subspace of the space ω, the set of all sequences of complex or real numbers, is called *a sequence space*. We write c to denote the set of all convergent sequences, and $e = (1, 1, \dots)$. Let Λ and Γ be two sequence spaces and $U = (u_{nv})$ be an infinite matrix of complex entries. The matrix U defines a matrix operator from Λ to Γ if, for every sequence $\lambda = (\lambda_v)$, the U-transform of λ, *i.e.*, $U(\lambda) = (U_n(\lambda))$ in Γ, where

$$U_n(\lambda) = \sum_{v=0}^{\infty} u_{nv}\lambda_v$$

provided that the series converges for $n \geq 0$. We mean the class of all such matrices by (Λ, Γ).

The sequence space

$$\Lambda_U = \{\lambda = (\lambda_n) \in \omega : U(\lambda) \in \Lambda\} \tag{1.1}$$

is called the matrix domain of an infinite matrix U in a sequence space Λ.

The $\alpha-, \beta-$ and $\gamma-$ duals of the space Λ are defined by

$$\Lambda^{\alpha} = \left\{\epsilon = (\epsilon_n) \in \omega : \sum_{n=0}^{\infty} |\epsilon_n\lambda_n| < \infty, \text{ for all } \lambda \in \Lambda\right\},$$

$$\Lambda^{\beta} = \left\{\epsilon = (\epsilon_n) \in \omega : \sum_{n=0}^{\infty} \epsilon_n\lambda_n \text{ converges for all } \lambda \in \Lambda\right\},$$

$$\Lambda^{\gamma} = \left\{\epsilon = (\epsilon_n) \in \omega : \sup_m \left|\sum_{n=0}^{m} \epsilon_n\lambda_n\right| < \infty, \text{ for all } \lambda \in \Lambda\right\},$$

respectively.

A subspace Λ is called an FK space if it is a Frechet space with continuous coordinate functionals $p_n : \Lambda \to \mathbb{C}$ given by $p_n(\lambda) = \lambda_n$ for all $\lambda \in \Lambda$ and for all $n \geq 0$, where $\mathbb{C}$ is the field of complex numbers; an FK space of which metric is defined by a norm is called a BK space. An FK space Λ, consisting of all finite sequences, has an AK property if

$$\lim_{m\to\infty} \lambda^{[m]} = \lim_{m\to\infty} \sum_{j=0}^{m} \lambda_j e^{(j)} = \lambda,$$

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for every sequence $\lambda \in \Lambda$, where $e^{(j)}$ is the sequence of which only the non-zero term is 1 in jth place for each j. For example, the Maddox space

$$l(\mu) = \left\{ \lambda = (\lambda_n) : \sum_{n=1}^{\infty} |\lambda_n|^{\mu_n} < \infty \right\}$$

is an FK space with AK under the natural paranorm

$$\|\lambda\|_{l(\mu)} = \left(\sum_{n=0}^{\infty} |\lambda_n|^{\mu_n} \right)^{1/M},$$

where $M = \max\{1, \sup_n \mu_n\}$ [19–21].

Throughout the paper, we suppose that T and R are the triangles, $\phi = (\phi_n)$ is a positive sequence and $\mu = (\mu_n)$ is a bounded sequence of strictly positive real numbers, unless otherwise stated. Also, μ_n^* is the conjugate of μ_n, i.e., $1/\mu_n + 1/\mu_n^* = 1$, $\mu_n > 1$, and $1/\mu_n^* = 0$ for $\mu_n = 1$.

We state the product of any infinite matrices A and B, denoted by AB, as

$$(ab)_{nr} = \sum_{v=0}^{\infty} a_{nv} b_{vr}.$$

We require the following conditions:

(i) $\lim_n u_{nv}$ exists for each v.

(ii) $\sup_{n,v} |u_{nv}|^{\mu_v} < \infty$.

(iii) There exists an integer $C > 1$ such that

$$\sup_n \sum_{v=0}^{\infty} \left| u_{nv} C^{-1} \right|^{\mu_v^*} < \infty.$$

(iv) There exists an integer $C > 1$ such that

$$\sup \left\{ \sum_{v=0}^{\infty} \left| \sum_{n \in N} u_{nv} C^{-1} \right|^{\mu_v^*} : N \subset \mathbb{N} \text{ finite} \right\} < \infty. \tag{1.2}$$

(v) There exists some C such that

$$\sup_v \sum_{n=0}^{\infty} \left| u_{nv} C^{-1/\mu_v} \right|^{\eta_n} < \infty.$$

Lemma 1.1 ([5]). *The following statements hold. For all $v \in \mathbb{N}$:*

(a) *If $\mu_v \leq 1$, then $U \in (l(\mu), c)$ iff* (i) *and* (ii) *hold.*

(b) *If $\mu_v > 1$, then $U \in (l(\mu), c)$ iff* (i) *and* (iii) *hold.*

(c) *If $\mu_v > 1$, then $U \in (l(\mu), l)$ iff* (iv) *holds, where $l = l(e)$.*

(d) *If $\mu_v \leq 1$ and $\eta_v \geq 1$, then $U \in (l(\mu), l(\eta))$ iff* (v) *holds.*

Note that the following lemma provides ease of applications and gives the equivalent condition to condition (1.2).

Lemma 1.2 ([32]). *Let*

$$U_\mu(U) = \sum_{v=0}^{\infty} \left(\sum_{n=0}^{\infty} |u_{nv}| \right)^{\mu_v}$$

and

$$L_\mu(U) = \sup \left\{ \sum_{v=0}^{\infty} \left| \sum_{n \in N} u_{nv} \right|^{\mu_v} : N \subset \mathbb{N} \text{ finite} \right\}.$$

Then

$$(2K)^{-2} U_\mu[U] \leq L_\mu[U] \leq U_\mu[U],$$

where $K = \max\{1, 2^{H-1}\}$ and $H = \sup_v \mu_v$.

Lemma 1.3 ([22]). *Let Λ be an FK space with AK, Γ be an arbitrary subset of ω, T be a triangle and S be its inverse. Then we have $U \in (\Lambda_T, \Gamma)$ if and only if $\widehat{U} \in (\Lambda, \Gamma)$ and $V^{(n)} \in (\Lambda, c)$ for all n, where*

$$\widehat{u}_{nj} = \sum_{i=j}^{\infty} u_{ni} s_{ij}; \quad n, j \geq 0,$$

$$v_{mj}^{(n)} = \begin{cases} \sum\limits_{i=j}^{m} s_{ij} u_{ni}, & 0 \leq j \leq m, \\ 0, & j > m. \end{cases}$$

The summability theory is a major field of mathematics, which has various applications in analysis, applied mathematics and engineering sciences, specially quantum mechanics, probability theory, Fourier analysis, approximation theory, fixed point theory, etc. It deals with the generalization of the concept of convergence of sequences and series, and purposes to assign a limit value for non-convergent sequences and series using an operator described by an infinite matrix. The reason why matrices are used for a general linear operator is that a linear operator from a sequence space to another can generally be given with an infinite matrix. This reveals the importance of sequence spaces and matrix operators in summability theory. In recent times, a large literature has appeared devoted to the characterization of all matrices operators that transform one given sequence space into another (see, [1, 3, 4, 6–16, 18, 23–29, 31, 33–35]).

In the recent paper, the Banach space $|T_\phi|_k$ of all absolutely k- summable series has been introduced and studied by Sarıgöl and Agarwal [34], which includes many known sequence spaces. In the present paper, we define a general paranormed space $|T_\phi|(\mu)$, following Maddox paper [19–21], and show that it is an FK-space and isometrically isomorphic to the well-known space $l(\mu)$. Also, we establish the Schauder basis and the α-, β- and γ-duals of the space, and qualify certain matrix operators related to this space. Finally, we have some important results as special cases.

2. FK Space $|T_\phi|(\mu)$

In this subsection, using the summability $|T, \phi|(\mu)$, we define the series space $|T_\phi|(\mu)$ following the Maddox paper [19–21].

Let $\Sigma\lambda_v$ be an infinite series with the sequence of n-th partial sum s_n. Also, let (ϕ_n) and (μ_n) be two sequences of positive numbers. Then the series $\Sigma\lambda_v$ is said to be summable $|T, \phi|(\mu)$ if [7]

$$\sum_{n=1}^{\infty} (\phi_n)^{\mu_n - 1} |T_n(s) - T_{n-1}(s)|^{\mu_n} < \infty.$$

The absolute summability $|T, \phi|(\mu)$ includes various summabilities which were intensively investigated. Some examples and applications can be seen from references [6–15, 24, 25, 29, 31, 33, 34].

Let T be an infinite triangle matrix, i.e., $t_{nr} = 0$ for $r > n$, and $t_{nn} \neq 0$ for $n \geq 0$. Then, we introduce the space $|T_\phi|(\mu)$ as the set of all series $\Sigma\lambda_v$ summable by $|T, \phi_n|(\mu)$, or equivalently,

$$|T_\phi|(\mu) = \left\{ \lambda = (\lambda_v) \in w : \sum_{n=0}^{\infty} \left| (\phi_n)^{1/\mu_n^*} \sum_{r=v}^{n} (t_{nr} - t_{n-1,r}) \lambda_r \right|^{\mu_n} < \infty \right\}.$$

If $\widetilde{T}, \overline{T}$ and $\widehat{T}$ are the triangular matrices defined by

$$\tilde{t}_{nv} = \phi_n^{1/\mu_n^*} \hat{t}_{nv}, \quad \hat{t}_{nv} = \bar{t}_{nv} - \bar{t}_{n-1,v}, \ (\bar{t}_{n,n-1} = 0), \quad \bar{t}_{nv} = \sum_{i=v}^{n} t_{ni}, \tag{2.1}$$

then, using the notation (1.1), it can be redefined by $|T_\phi|(\mu) = (l(\mu))_{\widetilde{T}}$, i.e.,

$$|T_\phi|(\mu) = \left\{ \lambda = (\lambda_n) \in w : \sum_{n=0}^{\infty} \left| \widetilde{T}_n(\lambda) \right|^{\mu_n} < \infty \right\}.$$

Note that the space $|T_\phi|(\mu)$ includes various known sequence spaces depending on a variable matrix T and the sequences μ, ϕ. For example, one can observe that when $\mu_n = k \geq 1$ for all $n \geq 0$, the space $|T_\phi|(\mu)$ reduces to the space $|T_\phi|_k$ studied by Sarıgöl and Agarwal [34]. For other references, we refer to the papers ([3, 6–15, 23, 25, 28, 30, 31, 33]).

Also, $|T_\phi|(e) = |T|$ for all sequence ϕ.

On the other hand, since each triangle matrix has a unique inverse, so the inverse matrices $\widetilde{T}^{-1}$, $\overline{T}^{-1}$ and $\hat{T}^{-1}$ exist and can be expressed as follows:

$$\widetilde{t}_{nv}^{-1} = \phi_v^{-1/\mu_v^*} \widehat{t}_{nv}^{-1}, \quad \overline{t}_{nv}^{-1} = t_{nv}^{-1} - t_{n-1,v}^{-1}, \quad t_{-1,0}^{-1} = 0,$$

and also $\widehat{T} = E\overline{T}$, or $\widehat{T}^{-1} = \overline{T}^{-1} E^{-1}$, *i.e.*,

$$\left(\widehat{t}\right)_{nv}^{-1} = \begin{cases} \sum_{i=v}^{n} (t_{ni}^{-1} - t_{n-1,i}^{-1}), & 0 \le v \le n, \\ 0, & v > n, \end{cases}$$

where the matrix $E = (e_{ni})$ is given by

$$e_{ni} = \begin{cases} -1, & i = n-1, \\ 1, & i = n, \\ 0, & \text{otherwise.} \end{cases}$$

We begin with the theorem on some topological structures of this space.

Theorem 2.1. *The set $|T_\phi|(\mu)$ is a linear space with the coordinatewise addition and scalar multiplication, and also it is an FK-space under the paranorm*

$$\|\lambda\|_{|T_\phi|(\mu)} = \left(\sum_{v=0}^{\infty} \left| \widetilde{T}_n(\lambda) \right|^{\mu_n} \right)^{1/M}, \tag{2.2}$$

where $M = \max\{1, \sup_n \mu_n\}$.

Proof. The proof of the first part is a routine verification, so it is omitted. On the other hand, since the matrix given by (2.1) is triangle and $l(\mu)$ is an FK-space, by Theorem 4.3.2 in [36], $|T_\phi|(\mu) = (l(\mu))_{\widetilde{T}}$ is also an FK-space. □

Theorem 2.2. *The space $|T_\phi|(\mu)$ is isometrically isomorphic to the Maddox space $l(\mu)$, i.e., $|T_\phi|(\mu) \cong l(\mu)$.*

Proof. Define the mapping $\widetilde{T} : |T_\phi|(\mu) \to l(\mu)$ by $\widetilde{T}(\lambda) = \left(\widetilde{T}_n(\lambda)\right)$, where the matrix $\widetilde{T}$ is given by (2.1). Clearly, $\widetilde{T}$ is a linear bijection. Also, by (2.2), $\|\lambda\|_{|T_\phi|(\mu)} = \left\|\widetilde{T}(\lambda)\right\|_{l(\mu)}$, so it preserves the paranorm. Thus $|T_\phi|(\mu) \cong l(\mu)$. □

Since $\widetilde{T} : |T_\phi|(\mu) \to l(\mu)$ is an isomorphism, so the inverse image of the basis $\left(e^{(j)}\right)$ of the space $l(\mu)$ constitutes the basis of the space $|T_\phi|(\mu)$. Therefore we get the following result.

Theorem 2.3. *Let $a_j = \widetilde{T}_j(x)$ for each $j \ge 0$. Define the sequence $b^{(j)} = (b_n^{(j)})$ of the elements of the space $|T_\phi|(\mu)$ for each fixed j by*

$$b_n^{(j)} = \begin{cases} \widetilde{t}_{nj}^{-1}, & 0 \le j \le n, \\ 0, & j > n. \end{cases}$$

Then the sequence $(b^{(j)})$ is a Schauder basis of the space $|T_\phi|(\mu)$ and every $x \in |T_\phi|(\mu)$ has a unique representation of the form

$$x = \sum_{j=0}^{\infty} a_j b^{(j)}.$$

Theorem 2.4. *The space $|T_\phi|(\mu)$ is separable.*

Proof. The proof is obtained by Theorem 2.1 and Theorem 2.3. □

Theorem 2.5. *Define the sets* $D^{(1)}, D^{(2)}, D^{(3)}, D^{(4)}$ *and* $D^{(5)}$ *by*

$$\begin{aligned}
D^{(1)} =& \left\{\epsilon : \exists C > 1,\ \sum_{v=0}^{\infty}\left(\sum_{r=v}^{m}\left|C^{-1}\epsilon_r \widetilde{t}_{rv}^{-1}\right|\right)^{\mu_v^*} < \infty\right\},\\
D^{(2)} =& \left\{\epsilon : \exists C > 1,\ \sup_{v} C^{1/\mu_v}\sum_{r=v}^{\infty}\left|\epsilon_r \widetilde{t}_{rv}^{-1}\right| < \infty\right\},\\
D^{(3)} =& \left\{\epsilon : \sum_{r=v}^{\infty}\epsilon_r \widetilde{t}_{rv}^{-1} \text{ converges for each } v\right\},\\
D^{(4)} =& \left\{\epsilon : \exists C > 1,\ \sup_{m}\sum_{v=0}^{\infty}\left|\sum_{r=v}^{m}C^{-1}\epsilon_r \widetilde{t}_{rv}^{-1}\right|^{\mu_v^*} < \infty\right\},\\
D^{(5)} =& \left\{\epsilon : \sup_{m,v}\left|\sum_{r=v}^{m}\epsilon_n \widetilde{t}_{rv}^{-1}\right|^{\mu_v} < \infty\right\}.
\end{aligned}$$

(i) *If* $\mu_v > 1$ *for all* v, *then*

$$\{|T_\phi|(\mu)\}^{\alpha} = D^{(1)}, \quad \{|T_\phi|(\mu)\}^{\beta} = D^{(4)} \cap D^{(3)}, \quad \{|T_\phi|(\mu)\}^{\gamma} = D^{(4)}.$$

(ii) *If* $\mu_v \le 1$ *for all* v, *then*

$$\{|T_\phi|(\mu)\}^{\alpha} = D^{(2)}, \quad \{|T_\phi|(\mu)\}^{\beta} = D^{(5)} \cap D^{(3)}, \quad \{|T_\phi|(\mu)\}^{\gamma} = D^{(5)}.$$

Proof. We prove only the $\beta-$ dual of the space since the other parts can be proved similarly. Now, $\epsilon \in \{|T_\phi|(\mu)\}^{\beta}$ if and only if $\left(\sum_{r=0}^{m}\epsilon_r\lambda_r\right) \in c$ for all $\lambda \in |T_\phi|(\mu)$. Let $y = \widetilde{T}(\lambda)$. Then $y \in l(\mu)$. So, it follows from the inverse of the matrix $\widetilde{T}$ that

$$\sum_{r=0}^{m}\epsilon_r\lambda_r = \sum_{v=0}^{m}\sum_{r=v}^{m}\epsilon_r\widetilde{t}_{rv}^{-1}y_v = \sum_{v=0}^{\infty}b_{mv}y_v,$$

where

$$b_{mv} = \begin{cases}\sum_{r=v}^{m}\epsilon_r\widetilde{t}_{rv}^{-1}, & 0 \le v \le m,\\ 0, & v > m.\end{cases}$$

Since $y = \widetilde{T}(\lambda) \in l(\mu)$ for every $\lambda \in |T_\phi|(\mu)$, $\epsilon \in \{|T_\phi|(\mu)\}^{\beta}$ if and only if $B = (b_{mv}) \in (l(\mu), c)$. So, it is deduced from Lemma 1.1 that $\epsilon \in D^{(4)} \cap D^{(3)}$ when $\mu_v > 1$ for all v, and $\epsilon \in D^{(5)} \cap D^{(3)}$ when $\mu_v \le 1$ for all v. □

3. Matrix Operators

In this subsection, we characterize some matrix operators on the space $|T_\phi|(\mu)$ and give their applications.

Theorem 3.1. *Assume that* $\mu = (\mu_n)$ *is a bounded sequence of positive numbers such that* $\mu_n > 1$ *for all* n. *Further, let* $U^{(1)} = U\widetilde{T}^{-1}$ *and* $B^{(1)} = \widehat{R}U^{(1)}$. *Then* $U \in (|T_\phi|(\mu), |R|)$ *if and only if there exists* $C > 1$ *such that for* $n = 0, 1, \ldots,$

$$u_{nv}^{(1)} = \sum_{r=v}^{\infty}u_{nr}\widetilde{t}_{rv}^{-1} \text{ converges for each } v, \tag{3.1}$$

$$\sup_{m}\sum_{v=0}^{m}\left|\sum_{r=v}^{m}C^{-1}u_{nr}\widetilde{t}_{rv}^{-1}\right|^{\mu_v^*} < \infty, \tag{3.2}$$

$$\sum_{v=0}^{\infty}\left(\sum_{n=0}^{\infty}\left|C^{-1}b_{nv}^{(1)}\right|\right)^{\mu_v^*} < \infty. \tag{3.3}$$

Moreover, each $U \in (|T_\phi|(\mu), |R|)$ *defines a bounded operator* L_U *such that* $L_U(\lambda) = U(\lambda)$ *for all* $\lambda \in |T_\phi|(\mu)$.

Proof. Let $\mu_n > 1$ for all n. Since $|T_\phi|(\mu) = (l(\mu))_{\widetilde{T}}$, it follows from Lemma 1.3 that $U \in (|T_\phi|(\mu), |R|)$ if and only if $U^{(1)} \in (l(\mu), |R|)$ and $V^{(n)} \in (l(\mu), c)$, where $V^{(n)}$ is given by

$$v_{mv}^{(n)} = \begin{cases} \sum_{j=v}^{m} \tilde{t}_{jv}^{-1} u_{nj}, & 0 \le v \le m, \\ 0, & v > m. \end{cases}$$

If we apply Lemma 1.1 (b) to the matrix $V^{(n)}$, then we get that $V^{(n)} \in (l(\mu), c)$ if and only if the conditions (3.1) and (3.2) hold. On the other hand, since $U^{(1)}(\lambda) \in |R|$ for all $\lambda \in l(\mu)$, therefore $B^{(1)}(\lambda) = \widehat{R}(U^{(1)}(\lambda)) \in l$, and it is seen that $U^{(1)} \in (l(\mu), |R|)$ if and only if $B^{(1)} \in (l(\mu), l)$. Now, using Lemma 1.1 (c) with the matrix $B^{(1)}$, we have $B^{(1)} \in (l(\mu), l)$ if and only if (3.3) holds, which completes the proof of the first part.

Further, since the spaces $|T_\phi|(\mu)$ and $|R|$ are the FK-spaces, the proof of the second part is immediately seen from Theorem 4.2.8 of Wilansky [36]. □

Theorem 3.2. *Let* (ϕ_n) *and* (ψ_n) *be two sequences of positive numbers. Let* $\mu = (\mu_n)$ *and* $\eta = (\eta_n)$ *be bounded sequences of positive numbers with* $\mu_n \le 1$ *and* $\eta_n \ge 1$ *for all* n. *Besides, let* $U^{(1)} = U\widetilde{T}^{-1}$ *and* $B^{(1)} = \widetilde{R}U^{(1)}$. *Then* $U \in (|T_\phi|(\mu), |R_\psi|(\eta))$ *if and only if there exists an integer* $C > 1$ *such that for* $n = 0, 1, \ldots,$

$$u_{nv}^{(1)} = \sum_{r=v}^{\infty} u_{nr} \widetilde{t}_{rv}^{-1} \text{ exists for all } v \tag{3.4}$$

$$\sup_{m,v} \left| \sum_{r=v}^{m} u_{nr} \widetilde{t}_{rv}^{-1} \right|^{\mu_v} < \infty \tag{3.5}$$

$$\sup_{v} \sum_{n=0}^{\infty} \left| C^{-1/\mu_v} b_{nv}^{(1)} \right|^{\eta_n} < \infty. \tag{3.6}$$

Moreover, each $U \in (|T_\phi|(\mu), |R_\psi(\eta)|)$ *defines a bounded operator* L_U *such that* $L_U(\lambda) = U(\lambda)$ *for all* $\lambda \in |T_\phi|(\mu)$.

Proof. Let $\mu_v \le 1$ and $\eta_v \ge 1$ for all v. Since $|T_\phi|(\mu) = (l(\mu))_{\widetilde{T}}$ and $|R_\psi|(\eta) = (l(\eta))_{\widetilde{R}}$, $U \in (|T_\phi|(\mu), |R_\psi(\eta)|)$ if and only if $B^{(1)} \in (l(\mu), l(\eta))$ and $V^{(n)} \in (l(\mu), c)$, where the matrices $U^{(1)}$ and $V^{(n)}$ are defined as in Theorem 3.1. Now, using Lemma 1.1 (b) and (d) with the matrices $V^{(n)}$ and $B^{(1)}$, respectively, we obtain $V^{(n)} \in (l(\mu), c)$ if and only if, for $n \ge 0$, the conditions (3.4) and (3.5) hold, and $B^{(1)} \in (l(\mu), l(\eta))$ if and only if the condition (3.6) holds, which concludes the first part of the proof.

The second part of the theorem follows immediately from Theorem 4.2.8 of Wilansky [36]. □

It may be noticed that Theorem 3.1 and Theorem 3.2 have many consequences and applications for some particular matrices. We give some of them as follows. If $\mu_n = k \ge 1$ for all $n \ge 0$, then $|T_\phi|(\mu) = |T_\phi|_k$. So, one has the following results of Sarıgöl and Agarwal [34] which also include some known results in [2, 11, 13, 15, 23–25, 27, 28, 31, 33, 35].

Corollary 3.1. *Let* $1 < k < \infty$ *and define* $U^{(1)} = U\widetilde{T}^{-1}$ *and* $B^{(1)} = \widehat{R}U^{(1)}$ *Then* $U \in \left(|T_\phi|_k, |R|\right)$ *if and only if*

$$\sum_{r=v}^{\infty} u_{nr} \widetilde{t}_{rv}^{-1} \text{ converges for each } v, n,$$

$$\sup_{m} \sum_{v=0}^{m} \left| \sum_{r=v}^{m} u_{nr} \widetilde{t}_{rv}^{-1} \right|^{k^*} < \infty,$$

$$\sum_{v=0}^{\infty}\left(\sum_{n=0}^{\infty}\left|b_{nv}^{(1)}\right|\right)^{k^*}<\infty.$$

Corollary 3.2. *Suppose that* $1\le k<\infty$ *and* (ψ_n) *are the sequences of positive numbers. Define* $U^{(1)}=U\widetilde{T}^{-1}$ *and* $B^{(1)}=\widetilde{R}U^{(1)}$. *Then* $U\in\left(|T|,|R_\psi|_k\right)$ *if and only if, for* $n=0,1,\dots,$

$$\sum_{r=v}^{\infty}u_{nr}\widetilde{t}_{rv}^{-1} \text{ exists for all } v,$$

$$\sup_{m,v}\left|\sum_{r=v}^{m}u_{nr}\widetilde{t}_{rv}^{-1}\right|<\infty,$$

$$\sup_{v}\sum_{n=0}^{\infty}\left|b_{nv}^{(1)}\right|^{k}<\infty.$$

Further, choose T and R as the weighted mean matrices, i.e., $t_{nj}=p_j/P_n$ and $r_{nj}=q_j/Q_n$ for all $n\in\mathbb{N}$, where $P_n=p_0+\cdots+p_n\to\infty$ and $Q_n=q_0+\cdots+q_n\to\infty$ as $n\to\infty$. Then, a few computations reveal that

$$\widetilde{t}_{nj}=\begin{cases}\phi_n^{1/\mu_n^*}\frac{P_jp_n}{P_nP_{n-1}}, & 1\le j\le n,\\ 0, & j>n,\end{cases}$$

$$\widetilde{r}_{nj}=\begin{cases}\psi_n^{1/\eta_n^*}\frac{Q_jq_n}{Q_nQ_{n-1}}, & 1\le j\le n,\\ 0, & j>n,\end{cases}$$

and

$$\widetilde{t}_{nj}^{-1}=\begin{cases}P_n/p_n, & j=n,\\ P_{n-2}/p_{n-1}, & j=n,\\ 0, & \text{otherwise.}\end{cases}$$

Also, $|T_\phi|(\mu)=\left|\overline{N}_p^{\phi}\right|(\mu)$ and $|R_\psi|(\eta)=\left|\overline{N}_q^{\psi}\right|(\eta)$. Thus Theorem 3.2 reduces to the following result of the authors [7].

Corollary 3.3. *Let* (ϕ_n) *and* (ψ_n) *be the sequences of positive numbers. Further, let* $\mu=(\mu_n)$ *and* $\eta=(\eta_n)$ *be the bounded sequences of positive numbers such that* $\mu_n\le 1$ *and* $\eta_n\ge 1$ *for all* n. *Then* $U\in\left(\left|\overline{N}_p^{\phi}\right|(\mu),\left|\overline{N}_q^{\psi}\right|(\eta)\right)$ *if and only if there exists an integer* $C>1$ *such that*

$$\sup_{m}\left|\frac{P_mu_{nm}}{\phi_m^{1/\mu_m^*}p_m}\right|<\infty,\quad n\ge 0,$$

$$\sup_{v}\left|\frac{P_v}{\phi_v^{1/\mu_v^*}p_v}\left(u_{nv}-\frac{P_{v-1}}{P_v}u_{n,v+1}\right)\right|^{\mu_v}<\infty,\quad n\ge 0,$$

$$\sup_{v}\sum_{n=1}^{\infty}\left|\frac{\psi_n^{1/\eta_n^*}q_nC^{-1/\mu_v}P_v}{Q_nQ_{n-1}\phi_v^{1/\mu_v^*}p_v}\sum_{j=1}^{n}Q_{j-1}\left(u_{jv}-\frac{P_{v-1}}{P_v}u_{j,v+1}\right)\right|^{\eta_n}<\infty.$$

Now, put T and R as the Nörlund mean matrices, i.e., $t_{nj}=p_{n-j}/P_n$ and $r_{nj}=q_{n-j}/Q_n$ for all $n\in\mathbb{N}$, where $p_0,q_0\neq 0$. Then it follows due to [10] that

$$\widetilde{t}_{nj}=\begin{cases}\phi_0^{1/\mu_0^*}, & j=n=0,\\ \phi_n^{1/\mu_n^*}\left(\frac{P_{n-j}}{P_n}-\frac{P_{n-j-1}}{P_{n-1}}\right), & 1\le j\le n,\\ 0, & j>n,\end{cases}$$

$$\widetilde{r}_{nj}=\begin{cases}\psi_0^{1/\eta_0^*}, & j=n=0,\\ \psi_n^{1/\eta_n^*}\left(\frac{Q_{n-j}}{Q_n}-\frac{Q_{n-j-1}}{Q_{n-1}}\right), & 1\leq j\leq n,\\ 0, & j>n,\end{cases}$$

and

$$\widetilde{t}_{nj}^{-1}=\begin{cases}\phi_j^{-1/\mu_j^*}\sum\limits_{i=j}^{n}C_{n-i}P_i, & 0\leq j\leq n,\\ 0, & j>n\end{cases}$$

where $C_0P_0=1$ and $\sum_{i=0}^{n}C_{n-i}P_i=0$ for $n\geq 1$. Also, $|T_\phi|(\mu)=\left|N_p^\phi\right|(\mu)$ and $|R_\psi|(\eta)=\left|N_q^\psi\right|(\eta)$. So, Theorem 4 reduces to the following result, which includes the result of [24] for $\mu=\eta=e$.

Corollary 3.4. *Let (ϕ_n) and (ψ_n) be the sequences of positive numbers. Further, let $\mu=(\mu_n)$ and $\eta=(\eta_n)$ be the bounded sequences of positive numbers such that $\mu_n\leq 1$ and $\eta_n\geq 1$ for all n. Then $U\in\left(\left|N_p^\phi\right|(\mu),\left|N_q^\psi\right|(\eta)\right)$ if and only if there exists an integer $C>1$ such that for $n\geq 0$,*

$$a_{nv}^{(1)}=\sum_{r=v}^{\infty}u_{nr}\widetilde{t}_{rv}^{-1}\ \textit{exists for all } v,$$

$$\sup_{m,v}\left|\sum_{r=v}^{m}u_{nr}\widetilde{t}_{rv}^{-1}\right|^{\mu_v}<\infty,$$

$$\sup_v\sum_{n=v}^{\infty}\left|C^{-1/\mu_v}\psi_n^{1/\eta_n^*}\sum_{i=v}^{n}\Big(\frac{Q_{n-i}}{Q_n}-\frac{Q_{n-1-i}}{Q_{n-1}}\Big)u_{iv}^{(1)}\right|^{\eta_n}<\infty.$$

Also, for the case $\mu=\eta=e$, $|T_\phi|(\mu)=|T|=l_{\widetilde{T}}$ and $|R_\psi|(\eta)=|R|=l_{\widetilde{R}}$. So, Theorem 3.1 immediately reduces to the following result due to Djolovič and Malkowsky [3].

Corollary 3.5. *Let $\widetilde{T}$ and $\widetilde{R}$ be the triangle matrices as in* (2.2)*. Then $U\in(l_{\widetilde{T}},l_{\widetilde{R}})$ if and only if*

$$\sup_{m,v}\left|\sum_{r=v}^{m}u_{nr}\widetilde{t}_{rv}^{-1}\right|^{\mu_v}<\infty$$

and

$$\sup_v\sum_{n=0}^{\infty}\left|\sum_{j=v}^{\infty}\widetilde{t}_{jv}^{-1}\sum_{i=0}^{n}\widetilde{r}_{ni}u_{ij}\right|<\infty.$$

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